Kripke Semantics for Basic Sequent Systems

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Abstract. We present a general method for providing Kripke semantics for the family of fully-structural multiple-conclusion propositional sequent systems. In particular, many well-known Kripke semantics for a variety of logics are easily obtained as special cases. This semantics is then used to obtain semantic characterizations of analytic sequent systems of this type, as well as of those admitting cut-admissibility. These characterizations serve as a uniform basis for semantic proofs of analyticity and cut-admissibility in such systems.

1 Introduction

This paper is a continuation of an on-going project aiming to get a unified semantic theory and understanding of analytic Gentzen-type systems and the phenomenon of strong cut-admissibility in them. In particular: we seek for general effective criteria that can tell in advance whether a given system is analytic, and whether the cut rule is (strongly) admissible in it (instead of proving these properties from scratch for every new system). The key idea of this project is to use semantic tools which are constructed in a *modular* way. For this it is essential to use *non-deterministic* semantics. This was first done in [6], where the family of propositional multiple-conclusion canonical systems was defined, and it was shown that the semantics of such systems is provided by two-valued non-deterministic matrices – a natural generalization of the classical truth-tables. The sequent systems of this family are fully-structural (i.e. include all standard structural rules), and their logical derivation rules are all of a certain "ideal" type. Then single-conclusion canonical systems were semantically characterized in [5], using non-deterministic intuitionistic Kripke frames. In both works the semantics was effectively used for the goals described above.

The goal of the present paper is to extend the framework, methods, and results of [6] and [5] to a much broader family of sequent systems: the family of what we call *basic systems*, which includes every multiple-conclusion propositional sequent system we know that has all of Gentzen's original structural rules. Thus this family includes the various standard sequent systems for modal logics, as well as the usual *multiple*-conclusion systems for intuitionistic logic, its dual, and bi-intuitionistic logic — none of which is canonical in the sense of [6, 5].

The structure of the paper is as follows. We begin by precisely defining the family of basic systems. We then generalize Kripke semantics, and present a

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general method for providing such semantics for any given basic system. This method is modular, as we separately investigate the semantic effect of every logical rule of a basic system (and in fact even of the main ingredients of such rules), and combine these effects to obtain the full semantics of the system. In a variety of important cases, this leads to the known semantics of the corresponding logic. In addition, this method can be applied to new basic systems, including basic systems with non-deterministic connectives. Based on this method, in sections 5 and 6 we present semantic characterizations of analyticity and cut-admissibility in basic systems. These characterizations pave the way to uniform semantic proofs of these properties.[1]

Two important notes before we start: first, we consider here derivations from a set of assumptions (or "non-logical axioms"), and so we deal with *strong* soundness and completeness, and *strong* cut-admissibility ([2]). Second, we only investigate here propositional systems and logics, leaving the more complicated first-order case to a future work.

Most of the proofs are omitted due to lack of space, and will appear in the full version of the paper.

2 Preliminaries

In what follows $\mathcal{L}$ is a propositional language, and $Frm_{\mathcal{L}}$ is its set of wffs. We assume that $p_1, p_2, \ldots$ are the atomic formulas of $\mathcal{L}$. Since we only deal with fully-structural systems, it is most convenient to define sequents using *sets*:

Definition 1. A *signed formula* is an expression of the form $f{:}\psi$ or $t{:}\psi$, where ψ is a formula. A *sequent* is a finite set of signed formulas.

We shall usually employ the usual sequent notation $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite sets of formulas. $\Gamma \Rightarrow \Delta$ is interpreted as $\{f{:}\psi \mid \psi \in \Gamma\} \cup \{t{:}\psi \mid \psi \in \Delta\}$. We also employ the standard abbreviations, e.g. Γ, φ instead of $\Gamma \cup \{\varphi\}$, and $\Gamma \Rightarrow$ instead of $\Gamma \Rightarrow \emptyset$.

Definition 2. An *$\mathcal{L}$-substitution* is a function $\sigma : Frm_{\mathcal{L}} \to Frm_{\mathcal{L}}$, such that $\sigma(\diamond(\psi_1, \ldots, \psi_n)) = \diamond(\sigma(\psi_1), \ldots, \sigma(\psi_n))$ for every n-ary connective $\diamond$ of $\mathcal{L}$. An $\mathcal{L}$-substitution is extended to signed formulas, sequents, etc. in the obvious way.

Given a set μ of signed formulas, we denote by $frm[\mu]$ the set of (ordinary) formulas appearing in μ, and by $sub[\mu]$ the set of subformulas of the formulas of $frm[\mu]$. *frm* and *sub* are extended to sets of sets of signed formulas in the obvious way. Given a set $\mathcal{E}$ of formulas, a formula ψ (resp. sequent s) is called an *$\mathcal{E}$-formula* (*$\mathcal{E}$-sequent*) if $\psi \in \mathcal{E}$ ($frm[s] \subseteq \mathcal{E}$).

[1] Many efforts have been devoted to characterize cut-free sequent systems. For example, a semantic characterization of cut-admissibility was the subject of [7]. There, however, the authors consider substructural single-conclusion systems, and use phase semantics, which is significantly more abstract and complex than Kripke semantics.

3 Basic Systems

In this section we precisely define the family of *basic systems*, and present some examples of them. For doing so, we define the general structure of derivation rules allowed to appear in basic systems. Rules of this structure will be called *basic rules*. A key idea here is to explicitly differentiate between a rule and its application. Roughly speaking, the rule itself is a schema that is used in proofs by applying some substitution and (optionally) adding context-formulas.

To explain the intuition behind the following definition of a basic rule, we begin with specific examples. Consider the following schemas for introducing a unary connective $\Box$ (used in usual systems for modal logics, see e.g. [17]):

$$(1)\ \frac{\Gamma, \psi \Rightarrow \Delta}{\Gamma, \Box\psi \Rightarrow \Delta} \qquad (2)\ \frac{\Box\Gamma \Rightarrow \psi}{\Box\Gamma \Rightarrow \Box\psi} \qquad (3)\ \frac{\Gamma_1, \Box\Gamma_2 \Rightarrow \psi}{\Box\Gamma_1, \Box\Gamma_2 \Rightarrow \Box\psi}$$

$\Box\Gamma$ here is an abbreviation for $\{\Box\varphi \mid \varphi \in \Gamma\}$. An obvious distinction in these schemas is the distinction between *context* formulas and *non-context* formulas (see e.g. [15]). Here Γ, Γ_1 and Γ_2 are sets of context formulas, and ψ and $\Box\psi$ are non-context formulas. While the exact number of non-context formulas is explicitly specified in the scheme, any number of context formulas is possible. These three schemas demonstrate three possibilities regarding context-formulas:

1. No constraint on context-formulas on either side of the sequent (as in (1)).
2. Limiting the allowed set of context-formulas (as in (2), where only $\Box$-formulas may appear on the left, and no context-formulas are allowed on the right).
3. Modifying some context-formulas in the rule application (as in (3), where Γ_1 in the premise becomes $\Box\Gamma_1$ in the conclusion).

To deal with the different options concerning the treatment of context formulas, we associate with each rule a set of *context-relations*. The context-relations determine the required relation between the context formulas of the premises of the rule and those of the corresponding conclusion.

Definition 3.

1. A *context-relation* is a finite binary relation between signed formulas. Given a context-relation π, we denote by $\bar{\pi}$ the binary relation between signed formulas $\bar{\pi} = \{\langle\sigma(x), \sigma(y)\rangle \mid \sigma \textit{ is an } \mathcal{L}\textit{-substitution, and } \langle x, y\rangle \in \pi\}$. A pair of sequents $\langle s_1, s_2\rangle$ is called a π*-instance* if there exist (not necessarily distinct) signed formulas $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ such that $s_1 = \{x_1, \ldots, x_n\}$, $s_2 = \{y_1, \ldots, y_n\}$, and $x_i \bar{\pi} y_i$ for every $1 \leq i \leq n$.
2. A *basic premise* is an ordered pair of the form $\langle s, \pi\rangle$, where s is a sequent and π is a context-relation.
3. A *basic rule* is an expression of the form S/C, where S is a finite set of basic premises, and C is a sequent.
4. An *application* of the basic rule $\{\langle s_1, \pi_1\rangle, \ldots, \langle s_n, \pi_n\rangle\}/C$ is any inference step of the following form:

$$\frac{\sigma(s_1) \cup c_1 \quad \ldots \quad \sigma(s_n) \cup c_n}{\sigma(C) \cup c'_1 \cup \ldots \cup c'_n}$$

where σ is an $\mathcal{L}$-substitution, and $\langle c_i, c_i' \rangle$ is a π_i-instance for every $1 \leq i \leq n$.

Example 1. Below we present well-known examples of basic rules and context relations used in them (note that the names given here to these context-relations will be used in the sequel):

Implication The usual rules for classical implication are the two basic rules $\{\langle \Rightarrow p_1, \pi_0 \rangle, \langle p_2 \Rightarrow, \pi_0 \rangle\}/p_1 \supset p_2 \Rightarrow$ and $\{\langle p_1 \Rightarrow p_2, \pi_0 \rangle\}/ \Rightarrow p_1 \supset p_2$, where $\pi_0 = \{\langle f{:}p_1, f{:}p_1 \rangle, \langle t{:}p_1, t{:}p_1 \rangle\}$. π_0 is the most simple context-relation, and it is used in all sequent systems. By definition, π_0-instances are the pairs of the form $\langle s, s \rangle$. Thus, applications of these rules have the form (respectively):

$$\frac{\Gamma_1 \Rightarrow \Delta_1, \psi \quad \Gamma_2, \varphi \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2, \psi \supset \varphi \Rightarrow \Delta_1, \Delta_2} \qquad \frac{\Gamma, \psi \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \psi \supset \varphi}$$

For intuitionistic implication, one replaces the second rule with the rule $\{\langle p_1 \Rightarrow p_2, \pi_{int} \rangle\}/ \Rightarrow p_1 \supset p_2$, where $\pi_{int} = \{\langle f{:}p_1, f{:}p_1 \rangle\}$. π_{int}-instances are all pairs of the form $\langle \Gamma \Rightarrow, \Gamma \Rightarrow \rangle$. Thus, applications of this rule allow to infer $\Gamma \Rightarrow \psi \supset \varphi$ from $\Gamma, \psi \Rightarrow \varphi$.

Exclusion The rules for dual-intuitionistic exclusion (in a multiple-conclusion sequent system) are the basic rules $\{\langle p_1 \Rightarrow p_2, \{\langle t{:}p_1, t{:}p_1 \rangle\}\rangle\}/p_1 \prec p_2 \Rightarrow$ and $\{\langle \Rightarrow p_1, \pi_0 \rangle, \langle p_2 \Rightarrow, \pi_0 \rangle\}/ \Rightarrow p_1 \prec p_2$ (see [8]). Applications of these rule have the form:

$$\frac{\psi \Rightarrow \Delta, \varphi}{\psi \prec \varphi \Rightarrow \Delta} \qquad \frac{\Gamma_1 \Rightarrow \Delta_1, \psi \quad \Gamma_2, \varphi \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2, \psi \prec \varphi}$$

Modal Necessity Different basic rules for introducing $\Box$ are used in different modal logics (see [17] for a survey; for *GL* see e.g. [13, 1]). For example, the systems **K**, **K4**, **GL**, **S4** and **S5** are obtained by adding the following rules to the standard sequent system for classical logic:

- **(*K*)** $\{\langle \Rightarrow p_1, \pi_K \rangle\}/ \Rightarrow \Box p_1$. where $\pi_K = \{\langle f{:}p_1, f{:}\Box p_1 \rangle\}$ (π_K-instances are all pairs of the form $\langle \Gamma \Rightarrow, \Box\Gamma \Rightarrow \rangle$).
- **(*K*4)** $\{\langle \Rightarrow p_1, \pi_{K4} \rangle\}/ \Rightarrow \Box p_1$, where $\pi_{K4} = \{\langle f{:}p_1, f{:}\Box p_1 \rangle, \langle f{:}\Box p_1, f{:}\Box p_1 \rangle\}$.
- **(*GL*)** $\{\langle \Box p_1 \Rightarrow p_1, \pi_{K4} \rangle\}/ \Rightarrow \Box p_1$.
- **(*S*4)** $\{\langle \Rightarrow p_1, \pi_{S4} \rangle\}/ \Rightarrow \Box p_1$ where $\pi_{S4} = \{\langle f{:}\Box p_1, f{:}\Box p_1 \rangle\}$.
- **(*S*5)** $\{\langle \Rightarrow p_1, \pi_{S5} \rangle\}/ \Rightarrow \Box p_1$ where $\pi_{S5} = \{\langle f{:}\Box p_1, f{:}\Box p_1 \rangle, \langle t{:}\Box p_1, t{:}\Box p_1 \rangle\}$.
- **($\Box \Rightarrow$)** In **S4** and **S5**, the following rule is also added: $\{\langle p_1 \Rightarrow, \pi_0 \rangle\}/\Box p_1 \Rightarrow$.

Applications of these rules have the form:

$$(K)\ \frac{\Gamma \Rightarrow \psi}{\Box\Gamma \Rightarrow \Box\psi} \qquad (K4)\ \frac{\Gamma_1, \Box\Gamma_2 \Rightarrow \psi}{\Box\Gamma_1, \Box\Gamma_2 \Rightarrow \Box\psi} \qquad (GL)\ \frac{\Gamma_1, \Box\Gamma_2, \Box\psi \Rightarrow \psi}{\Box\Gamma_1, \Box\Gamma_2 \Rightarrow \Box\psi}$$

$$(S4)\ \frac{\Box\Gamma \Rightarrow \psi}{\Box\Gamma \Rightarrow \Box\psi} \qquad (S5)\ \frac{\Box\Gamma \Rightarrow \Box\Delta, \psi}{\Box\Gamma \Rightarrow \Box\Delta, \Box\psi} \qquad (\Box \Rightarrow)\ \frac{\Gamma, \psi \Rightarrow \Delta}{\Gamma, \Box\psi \Rightarrow \Delta}$$

Finally, we define *basic systems*, and the consequence relations induced by them.

Definition 4.

– A *basic system* **G** consists of a finite set of basic rules, such that:
  1. The *identity axiom* is in **G**. The identity axiom is the basic rule $\emptyset / p_1 \Rightarrow p_1$. Applications of this rule provide all axioms of the form $\psi \Rightarrow \psi$.
  2. The *cut rule* is in **G**. Cut is the basic rule $\{\langle p_1 \Rightarrow\ , \pi_0\rangle, \langle \Rightarrow p_1, \pi_0\rangle\} /\ \Rightarrow$. Applications of this rule allow one to infer $\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2$ from $\Gamma_1, \psi \Rightarrow \Delta_1$ and $\Gamma_2 \Rightarrow \Delta_2, \psi$.
  3. The *weakening rules* are in **G**. These are the basic rules $\{\langle \Rightarrow\ , \pi_0\rangle\} / p_1 \Rightarrow$ and $\{\langle \Rightarrow\ , \pi_0\rangle\} /\ \Rightarrow p_1$. Applications of them allow one to infer $\Gamma, \psi \Rightarrow \Delta$ and $\Gamma \Rightarrow \Delta, \psi$ from $\Gamma \Rightarrow \Delta$.

  We denote by $\Pi_{\mathbf{G}}$ the set of context-relations appearing in the rules of **G**.
– A sequent s *follows* in basic system **G** from a set of sequents $\mathcal{S}$ ($\mathcal{S} \vdash_{\mathbf{G}} s$) if there exists a proof in **G** of s from $\mathcal{S}$.

Example 2. We list some known sequent systems, each of which is either a basic system, or it can easily be shown to be equivalent to a basic system:

– The family of canonical systems studied in [6] (which includes the propositional part of Gentzen's **LK** for classical logic).
– The propositional part of **LJ′** from [14] (the multiple-conclusion version of Gentzen's **LJ** for intuitionistic logic).
– The propositional part of $\mathbf{SLK^1}$ from [8] for bi-intuitionistic logic.
– All modal ordinary sequent systems described in [17], as well as that for **GL** described in Example 1.
– The (fully-structural) sequent systems for finite-valued logics in [3].
– All paraconsistent sequent systems investigated in [4].

4 Kripke Semantics

In this section we introduce a method for providing Kripke semantics for any given basic system. We provide a general definition of a (Kripke-) frame, and show that every basic system **G** induces a class of frames for which it is strongly sound and complete. Various fundamental soundness and completeness theorems for known logics are easily obtained as special cases.

Definition 5. A *frame* is a tuple $\langle W, \mathcal{R}, v\rangle$, where W is a set (of *worlds*), $\mathcal{R}$ is a finite set of binary relations on W (called *accessibility relations*), and $v : W \times Frm_{\mathcal{L}} \to \{\mathrm{T}, \mathrm{F}\}$ is a valuation function. Given a frame $\langle W, \mathcal{R}, v\rangle$, we say that a signed formula of the form $f{:}\psi$ (resp. $t{:}\psi$) is *true* in a world $a \in W$ if $v(a, \psi) = \mathrm{F}$ ($v(a, \psi) = \mathrm{T}$).

Three notions of truth of *sequents* are defined as follows:

Definition 6. Let $\mathcal{W} = \langle W, \mathcal{R}, v\rangle$ be a frame, and s be a sequent.

1. s is *true* in some $a \in W$ if there exists $x \in s$ such that x is true in a.
2. Let $R \in \mathcal{R}$. s is *R-true* in $a \in W$ if s is true in every $b \in W$ such that aRb.

3. $\mathcal{W}$ is a *model* of s if s is true in every $a \in W$. $\mathcal{W}$ is a model of a set of sequents $\mathcal{S}$ if it is a model of every $s \in S$.

Since we deal with arbitrary basic systems, no constraints on the set of relations and on the valuation function were imposed in the definition of a frame. These constraints are directly related to the context-relations and the basic rules of a specific basic system. The idea is that each context-relation and each basic rule imposes constraints on the set of frames. Next we describe these constraints.

Definition 7. Let $\mathbf{G}$ be a basic system. A frame $\mathcal{W} = \langle W, \mathcal{R}, v\rangle$ is $\mathbf{G}$-*legal* if the following conditions are met:

(1) $\mathcal{R}$ consists of a relation R_π for every context-relation $\pi \in \Pi_{\mathbf{G}}$, where R_{π_0} is the identity relation.
(2) For every $a, b \in W$, and $\pi \in \Pi_{\mathbf{G}}$, if $aR_\pi b$ then for every two signed formulas x, y such that $x\bar{\pi}y$, either x is not true in b or y is true in a.
(3) For every $a \in W$, $\mathcal{L}$-substitution σ, and $S/C \in \mathbf{G}$, if $\sigma(s)$ is R_π-true in a for every $\langle s, \pi\rangle \in S$, then $\sigma(C)$ is true in a.

Example 3. The constraints imposed by the context-relations π_{int} and π_K according to condition (2) of the previous definition are:

(1) Assume that $\pi_{int} \in \Pi_{\mathbf{G}}$. In $\mathbf{G}$-legal frames, if $aR_{\pi_{int}}b$, then for every formula ψ, either $f{:}\psi$ is not true in b or $f{:}\psi$ is true in a. Or equivalently, if $aR_{\pi_{int}}b$ and $v(a, \psi) = \text{T}$, then $v(b, \psi) = \text{T}$. Thus π_{int} imposes the usual persistence condition of intuitionistic logic (with respect to $R_{\pi_{int}}$).
(2) Assume that $\pi_K \in \Pi_{\mathbf{G}}$. In $\mathbf{G}$-legal frames, if $aR_{\pi_K}b$, then either $f{:}\psi$ is not true in b or $f{:}\Box\psi$ is true in a. Equivalently, if $aR_{\pi_K}b$ and $v(a, \Box\psi) = \text{T}$, then $v(b, \psi) = \text{T}$. Thus π_K imposes "one half" of the usual semantics of $\Box$.

Example 4. We present the constraints imposed by some basic rules according to condition (3) of the previous definition:

(1) Assume that $\mathbf{G}$ contains a rule of the form $\{\langle \Rightarrow p_1, \pi\rangle\}/ \Rightarrow \Box p_1$. In $\mathbf{G}$-legal frames, $v(a, \Box\psi) = \text{T}$ whenever $v(b, \psi) = \text{T}$ for every world b such that $aR_\pi b$. Thus this rule imposes the "other half" of the usual semantics of $\Box$.
(2) Assume that $\mathbf{G}$ contains a rule of the form $\{\langle \Rightarrow p_1, \pi\rangle, \langle p_2 \Rightarrow, \pi\rangle\}/p_1 \supset p_2 \Rightarrow$. In $\mathbf{G}$-legal frames, $v(a, \psi \supset \varphi) = \text{F}$ if $v(b, \psi) = \text{T}$ and $v(b, \varphi) = \text{F}$ for every world b such that $aR_\pi b$.
(3) Assume that $\mathbf{G}$ contains a rule of the form $\{\langle p_1 \Rightarrow p_2, \pi\rangle\}/ \Rightarrow p_1 \supset p_2$. In $\mathbf{G}$-legal frames, $v(a, \psi \supset \varphi) = \text{T}$ whenever for every world b such that $aR_\pi b$, either $v(b, \psi) = \text{F}$ or $v(b, \varphi) = \text{T}$.
(4) Assume $\mathbf{G}$ contains a rule of the form $\{\langle \Rightarrow, \pi\rangle\}/ \Rightarrow$ (application of this rule allow to infer s' from s where $\langle s, s'\rangle$ is a π-instance). In $\mathbf{G}$-legal frames, $(\Rightarrow)$ (the empty sequent) should be true in every world, in which it is $\mathcal{R}_\pi$-true. Since $(\Rightarrow)$ is not true in any world, this condition would hold iff for every world a there exists a world b such that $aR_\pi b$. In other words, if $\{\langle \Rightarrow, \pi\rangle\}/ \Rightarrow$ is in $\mathbf{G}$, then R_π should be a serial relation.

Example 5. Let **LK** be the usual basic system for classical logic. Here $\Pi_{\mathbf{LK}} = \{\pi_0\}$. In **LK**-legal frames, $\mathcal{R}$ consists of one relation R_{π_0} which is the identity relation. π_0 imposes a trivial condition, $v(a,\psi) = v(b,\psi)$ whenever $a = b$. The basic rules of **LK** impose the usual truth-tables in each world, e.g. $v(a, \psi \supset \varphi) = \text{T}$ iff either $v(a,\psi) = \text{F}$ or $v(a,\varphi) = \text{T}$.

Example 6. Assume that **G** contains the two standard rules for intuitionistic implication. Example 4 (3) and the combination of Example 3 (1) and Example 4 (2) together imply that in **G**-legal frames $v(a, \psi \supset \varphi) = \text{T}$ iff for every world b such that $aR_{\pi_{int}}b$, either $v(b,\psi) = \text{F}$ or $v(b,\varphi) = \text{T}$. Thus the two rules and π_{int} impose the usual Kripke semantics of intuitionistic implication.

We define the *semantic* consequence relation induced by a basic system **G**.

Definition 8. Let **G** be a basic system, and $\mathcal{S} \cup \{s\}$ be a set of sequents. $\mathcal{S} \vDash_{\mathbf{G}} s$ if every **G**-legal frame which is a model of $\mathcal{S}$ is also a model of s.

Remark 1. It can easily be seen that in any basic system **G** like **LK**, in which $\Pi_{\mathbf{G}} = \{\pi_0\}$, it suffices to consider only trivial Kripke frames which have a single world, and the corresponding accessibility relation is the identity relation.

Theorem 1 (Strong Soundness and Completeness). $\vdash_{\mathbf{G}} = \vDash_{\mathbf{G}}$ *for every basic system* **G**.

Theorem 1 generalizes several well-known completeness theorems for specific basic systems. For example:

Example 7 (KD). Let **KD** be the basic system for the modal logic KD (see [17]) obtained adding the rules $\{\langle \Rightarrow p_1, \pi_K\rangle\}/ \Rightarrow \Box p_1$ and $\{\langle \Rightarrow, \pi_K\rangle\}/ \Rightarrow$ to the usual system for classical logic. Applications of the latter allow to infer $\Box\Gamma \Rightarrow$ from $\Gamma \Rightarrow$. As $\Pi_{\mathbf{KD}} = \{\pi_0, \pi_K\}$, **KD**-legal frames include two relations, R_{π_0} (the identity relation) and R_{π_K}. Following Examples 3, 4 and 5, all connectives of **KD** have their usual semantics. By Example 4, in the presence of the second rule, R_{π_K} is a serial relation. Thus, we obtain the usual semantics of KD.

Theorem 1 is sometimes difficult to use directly. However, there is a subclass of **G**-legal frames that is still sufficient for completeness, and in many cases leads to simpler conditions on the accessibility relations.

Definition 9. Given a basic system **G**, a **G**-legal frame $\mathcal{W} = \langle W, \mathcal{R}, v\rangle$ is called *maximal* if for every $a, b \in W$, and $\pi \in \Pi_{\mathbf{G}}$, $aR_\pi b$ iff (*if* and only if) for every two signed formulas x, y such that $x\bar{\pi}y$, either x is not true in b or y is true in a.

Proposition 1. *Let* **G** *be a basic system, and* $\mathcal{W} = \langle W, \mathcal{R}, v\rangle$ *be a maximal* **G***-legal frame. The following hold for every* $\pi_1, \pi_2, \pi_3 \in \Pi_{\mathbf{G}}$*:*

(1) If $\pi_1 = \emptyset$*, then* R_{π_1} *is the full relation.*
(2) If $\bar{\pi_3} = \bar{\pi_1} \cup \bar{\pi_2}$*, then* $R_{\pi_3} = R_{\pi_2} \cap R_{\pi_1}$*. In particular:*
  (a) If $\bar{\pi_1} \subseteq \bar{\pi_2}$*, then* $R_{\pi_2} \subseteq R_{\pi_1}$*.*

(b) *If* $\bar{\pi_1} \subseteq \bar{\pi_0}$*, then* R_{π_1} *is a reflexive relation.*

(3) *If* $\bar{\pi_3} \subseteq \bar{\pi_1} \circ \bar{\pi_2}$*, then* $R_{\pi_2} \circ R_{\pi_1} \subseteq R_{\pi_3}$*. In particular, if* $\bar{\pi_1} \subseteq \bar{\pi_1} \circ \bar{\pi_1}$*, then* R_{π_1} *is a transitive relation.*

(4) *If* $x\bar{\pi_2}y$ *implies* $\overline{y}\bar{\pi_1}\overline{x}$ *(where* $\overline{f{:}\psi} \doteq t{:}\psi$ *and* $\overline{t{:}\psi} \doteq f{:}\psi$*), then* $R_{\pi_1} \subseteq R_{\pi_2}^{-1}$*. In particular, if* $x\bar{\pi_1}y$ *implies* $\overline{y}\bar{\pi_1}\overline{x}$*, then* R_{π_1} *is a symmetric relation.*

Example 8. Assume that $\pi_0, \pi_{int} \in \Pi_{\mathbf{G}}$ (as in the system for intuitionistic logic). Since $\bar{\pi_{int}} \circ \bar{\pi_{int}} = \bar{\pi_{int}}$, $R_{\pi_{int}}$ is a transitive relation in maximal **G**-legal frames. Since $\bar{\pi_{int}} \subseteq \bar{\pi_0}$, $R_{\pi_{int}}$ is reflexive. We obtain that in maximal **G**-legal frames $R_{\pi_{int}}$ is a preorder, as Kripke semantics for intuitionistic logic is usually defined.

Theorem 2. *Let* **G** *be a basic system, and* $\mathcal{S} \cup \{s\}$ *be a set of sequents. If* $\mathcal{S} \not\vdash_{\mathbf{G}} s$ *then there exists a* maximal **G***-legal frame which is a model of* $\mathcal{S}$*, but not of* s*.*

Taken together, Theorems 1 and 2 imply various well-known completeness theorems for specific basic systems. Indeed, with the exception of GL (which we discuss later), for every system in Example 2 our semantics is equivalent to the usual Kripke semantics of the corresponding logic. Here are some other examples:

Example 9 (KB). Let **KB** be the basic system for the modal logic KB ([17]), obtained from the system for classical logic by adding the rule $\{\langle \Rightarrow p_1, \pi\rangle\}/ \Rightarrow \Box p_1$, where $\pi = \{\langle f{:}p_1, f{:}\Box p_1\rangle, \langle t{:}\Box p_1, t{:}p_1\rangle\}$. Applications of this rule allow to infer $\Box\Gamma \Rightarrow \Delta, \Box\psi$ from $\Gamma \Rightarrow \Box\Delta, \psi$. **KB**-legal frames include two relations, R_{π_0} (the identity relation) and R_{π}. The rules of **KB** dictate the usual semantics of modal logic for every connective. By Proposition 1, in maximal **KB**-legal frames R_{π} is a symmetric relation. It follows that **KB** is sound and complete with respect to usual symmetric Kripke frames.

Example 10 (Intuitionistic $S5$*).* Consider the basic system $\mathbf{G_3}$ from [11] (obtained from the propositional part of $\mathbf{LJ}'$ by adding the usual $S5$ rules for $\Box$). Here $\Pi_{\mathbf{G_3}} = \{\pi_0, \pi_{int}, \pi_{S5}\}$. Maximal $\mathbf{G_3}$-legal frames include three relations, R_{π_0} (the identity relation), a preorder $R_{\pi_{int}}$, and an equivalence relation $R_{\pi_{S5}}$. π_{S5} and the rules for $\Box$ enforce the usual Kripke semantics of $\Box$ with respect to $R_{\pi_{S5}}$. π_{int} and the rules for the intuitionistic connective dictate the usual Kripke intuitionistic semantics with respect to $R_{\pi_{int}}$. Note that π_{int} also enforces persistence of $\Box$-formulas, i.e. if $aR_{\pi_{int}}b$ and $v(a, \Box\psi) = \text{T}$ then $v(b, \Box\psi) = \text{T}$. This condition is equivalent to the following one: if $aR_{\pi_{int}}b$ and $v(c, \psi) = \text{T}$ for every world c such that $aR_{\pi_{S5}}c$, then $v(d, \psi) = \text{T}$ for every world d such that $bR_{\pi_{S5}}d$. The Kripke semantics presented in [11] is not identical to the one obtained by our method. In particular, in our semantics $R_{\pi_{S5}}$ should be an equivalence relation, and no direct conditions bind $R_{\pi_{int}}$ and $R_{\pi_{S5}}$.

Example 11. Consider the basic system **G** obtained from the propositional part of $\mathbf{LJ}'$ by adding the rules: $\{\langle \Rightarrow \neg p_1, \pi_{S4}\rangle\}/ \Rightarrow \Box\neg p_1$ and $\{\langle p_1 \Rightarrow, \pi_0\rangle\}/\Box p_1 \Rightarrow$. Applications of the first rule allow to infer $\Box\Gamma \Rightarrow \Box\neg\psi$ from $\Box\Gamma \Rightarrow \neg\psi$. Maximal **G**-legal frames include the identity relation R_{π_0}, and two preorders $R_{\pi_{int}}$ and $R_{\pi_{S4}}$, such that $R_{\pi_{int}} \subseteq R_{\pi_{S4}}$. The rules of $\mathbf{LJ}'$ dictate the usual semantics of the intuitionistic connectives, The two other rules and π_{S4} impose the following three conditions: (1) if $v(b, \neg\psi) = \text{T}$ for every world b such that $aR_{\pi_{S4}}b$ then $v(a, \Box\neg\psi) = \text{T}$; (2) if $v(a, \psi) = \text{F}$ then $v(a, \Box\psi) = \text{F}$; (3) if $v(a, \Box\psi) = \text{T}$ then $v(b, \Box\psi) = \text{T}$ for every world b such that $aR_{\pi_{S4}}b$. As in Example 10, π_{int} also enforces persistence of $\Box$-formulas. But, since $R_{\pi_{int}} \subseteq R_{\pi_{S4}}$, this condition must hold if (3) holds. In this case we get non-deterministic semantics. To see this, note that if ψ is not of the form $\neg\varphi$ and $v(b, \psi) = \text{T}$ whenever $aR_{\pi_{S4}}b$, then $v(a, \Box\psi)$ can be freely chosen between T and F.

Remark 2. The last example provides a case in which the various constraints imposed by the rules (and context-relations) of a system do not uniquely determine the truth-value of a compound formula. Another, more natural, example is given by the system for *primal* intuitionistic logic from [10] (see also [5]). These examples demonstrate the need in general of *non-deterministic* semantics.

Example 12 (GL). Let **GL** be the basic system for the modal logic of provability *GL* (see Example 1). It is well-known that **GL** is sound and complete with respect to the set of modal Kripke frames whose accessibility relation is transitive and conversely well-founded. However, **GL** is not *strongly* complete with respect to this set of frames ([16]), and the compactness theorem fails for the logic induced by this semantics. Using our method, one obtains a different Kripke semantics for *GL* with an unusual interpretation of $\Box$. Indeed, maximal **GL**-legal frames include one (non-trivial) transitive relation, $R_{\pi_{K4}}$. The rules and context-relations of **GL** impose the usual truth-tables in every world for the classical connectives. Concerning $\Box$, a maximal **GL**-legal frame admits the usual semantics of $\Box$, and it should also satisfy the following condition: if $v(b, \psi) = \text{F}$ for some b such that $aR_{\pi_{K4}}b$, then there is some c such that $aR_{\pi_{K4}}c$, $v(c, \psi) = \text{F}$ and $v(c, \Box\psi) = \text{T}$. By Theorems 1 and 2, *GL* is *strongly* sound and complete with respect to this semantics (and so the compactness theorem does hold for this semantic consequence relation). It can easily be verified that every usual *GL*-frame is **GL**-legal. However, the converse is not true.

5 Semantic Characterization of Analyticity

In this section we investigate the crucial property of *analyticity* in the framework of basic systems. Roughly speaking, a sequent system is (strongly) *analytic* if whenever some sequent is provable in it (from a set of assumptions), then this sequent can be proven using only the syntactic material available within (the assumptions and) the proven sequent. For the formal definition, we use the following relation:

Definition 10. Let **G** be a basic system, $\mathcal{S} \cup \{s\}$ be a set of sequents, and $\mathcal{E}$ be a set of formulas. $\mathcal{S} \vdash^{\mathcal{E}}_{\mathbf{G}} s$ if there exists a proof in **G** of s from $\mathcal{S}$, containing only formulas from $\mathcal{E}$.

Definition 11. A basic system **G** is *analytic* if $\mathcal{S} \vdash_{\mathbf{G}} s$ implies $\mathcal{S} \vdash^{sub[\mathcal{S}\cup\{s\}]}_{\mathbf{G}} s$.

The following are two major consequences of analyticity.

Proposition 2 (Consistency). *Let* **G** *be an analytic basic system. Assume that the basic rule* $\emptyset / \Rightarrow$ *is not in* **G**. *Then,* $\nvdash_{\mathbf{G}} \Rightarrow$.

Proof. Assume that $\vdash_{\mathbf{G}} \Rightarrow$. Since **G** is analytic, $\vdash^{\emptyset}_{\mathbf{G}} \Rightarrow$. The only way one can prove $\Rightarrow$ without using any formulas, is using a rule of the form $\emptyset / \Rightarrow$. □

Proposition 3 (Decidability). *Let* **G** *be an analytic basic system. Given a finite set* $\mathcal{S}$ *of sequents and a sequent* s*, it is decidable whether* $\mathcal{S} \vdash_{\mathbf{G}} s$ *or not.*

Proof. Let $\mathcal{S}'$ be the set of sequents consisting of formulas from $sub[\mathcal{S} \cup \{s\}]$, and let $n = |\mathcal{S}'|$. Since **G** is analytic, if $\mathcal{S} \vdash_{\mathbf{G}} s$ then there exists a proof of s from $\mathcal{S}$ in **G** having length $\leq n$ (viewing a proof as a sequence), and consisting only of sequents from $\mathcal{S}'$. Thus an exhaustive proof-search is possible. □

We shall obtain a characterization of analyticity by identifying a *semantic* consequence relation that corresponds to $\vdash^{\mathcal{E}}_{\mathbf{G}}$. For this purpose, we define *semiframes*.

Definition 12. Let $\mathcal{E}$ be a set of formulas.

1. An $\mathcal{E}$*-semiframe* is a tuple $\langle W, \mathcal{R}, v \rangle$, where W and $\mathcal{R}$ are as in Definition 5, and $v : W \times \mathcal{E} \to \{\text{T}, \text{F}\}$ is a valuation function.
2. Given an $\mathcal{E}$-semiframe $\langle W, \mathcal{R}, v \rangle$, a signed formula of the form $f{:}\psi$ (resp. $t{:}\psi$) is *true* in some world $a \in W$ if $\psi \in \mathcal{E}$ and $v(a, \psi) = \text{F}$ ($v(a, \psi) = \text{T}$).
3. A frame $\langle W, \mathcal{R}, v' \rangle$ (see Definition 5) *extends* an $\mathcal{E}$-semiframe $\langle W, \mathcal{R}, v \rangle$ if $v'(a, \psi) = v(a, \psi)$ whenever $\psi \in \mathcal{E}$.

Note that a frame (Definition 5) is obtained as a special case, when $\mathcal{E} = Frm_{\mathcal{L}}$.

Definition 13. Given an $\mathcal{E}$-semiframe $\mathcal{W} = \langle W, \mathcal{R}, v \rangle$ and a sequent s:

1. s is *true* in some $a \in W$ if s is an $\mathcal{E}$-sequent and there exists $x \in s$ such that x is true in a.
2. Let $R \in \mathcal{R}$. s is R*-true* in $a \in W$ if s is an $\mathcal{E}$-sequent and s is true in every $b \in W$ such that aRb.
3. $\mathcal{W}$ is a *model* of a sequent s if s is true in every $a \in W$. $\mathcal{W}$ is a model of a set of sequents $\mathcal{S}$ if it is a model of every $s \in S$.

(Maximal) **G**-legal semiframes are defined as follows:

Definition 14. Let **G** be a basic system, and $\mathcal{E}$ be a set of formulas.

- An $\mathcal{E}$-semiframe $\mathcal{W} = \langle W, \mathcal{R}, v \rangle$ is **G**-*legal* if the following hold:
  (1) $\mathcal{R}$ consists of a relation R_π for every context-relation $\pi \in \Pi_{\mathbf{G}}$, where R_{π_0} is the identity relation.
  (2) For every $a, b \in W$, and $\pi \in \Pi_{\mathbf{G}}$, if $aR_\pi b$ then for every two signed $\mathcal{E}$-formulas x, y such that $x\bar{\pi}y$, either x is not true in b, or y is true in a.
  (3) For every $a \in W$, $\mathcal{L}$-substitution σ, and $S/C \in \mathbf{G}$, if $frm[\sigma(C)] \subseteq \mathcal{E}$, and $\sigma(s)$ is R_π-true in a for every $\langle s, \pi \rangle \in S$, then $\sigma(C)$ is true in a.
- A **G**-legal $\mathcal{E}$-semiframe $\mathcal{W} = \langle W, \mathcal{R}, v \rangle$ is called *maximal* if for every $a, b \in W$ and $\pi \in \Pi_{\mathbf{G}}$, the converse of the condition in (2) holds as well.

Proposition 4. *Let* $\mathbf{G}$ *be a basic system, and* $\mathcal{W} = \langle W, \mathcal{R}, v\rangle$ *be a maximal* $\mathbf{G}$*-legal* $\mathcal{E}$*-semiframe.*

1. $(1), (2)$ *and* (4) *from Proposition 1 hold without any changes.*
2. *If for every two signed* $\mathcal{E}$*-formulas* x, y*,* $x\bar{\pi}_3 y$ *implies that there exists a signed* $\mathcal{E}$*-formula* z *such that* $x\bar{\pi}_1 z$ *and* $z\bar{\pi}_2 y$*, then* $R_{\pi_2} \circ R_{\pi_1} \subseteq R_{\pi_3}$*.*
3. $|W| \leq 2^{|\mathcal{E}|}$*.*

We now define the semantic consequence relation $\vDash^{\mathcal{E}}_{\mathbf{G}}$, and prove a stronger soundness and completeness theorem.

Definition 15. Let $\mathbf{G}$ be a basic system, $\mathcal{S} \cup \{s\}$ be a set of sequents, and $\mathcal{E}$ be a set of formulas. $\mathcal{S} \vDash^{\mathcal{E}}_{\mathbf{G}} s$ if every $\mathbf{G}$-legal $\mathcal{E}$-semiframe which is a model of every $\mathcal{E}$-sequent $s' \in \mathcal{S}$, is also a model of s.

Theorem 3. *Let* $\mathbf{G}$ *be a basic system, and* $\mathcal{E}$ *be a set of formulas.*

1. $\vdash^{\mathcal{E}}_{\mathbf{G}} = \vDash^{\mathcal{E}}_{\mathbf{G}}$*.*
2. *If* $\mathcal{S} \nvDash^{\mathcal{E}}_{\mathbf{G}} s$ *then there exists a* maximal $\mathbf{G}$*-legal* $\mathcal{E}$*-semiframe which is a model of every* $\mathcal{E}$*-sequent* $s' \in \mathcal{S}$*, but not a model of* s*.*

Theorems 1 and 2 are derived now as corollaries, by choosing $\mathcal{E} = Frm_{\mathcal{L}}$. Together with Proposition 4, Theorem 3 makes it possible to have a semantic decision procedure for $\vdash_{\mathbf{G}}$ in case $\mathbf{G}$ is analytic (compare with the syntactic one in Proposition 3). Indeed, let $\mathcal{E} = sub[\mathcal{S} \cup \{s\}]$. To decide whether $\mathcal{S} \vdash_{\mathbf{G}} s$, it suffices to check triples of the form $\langle W, \mathcal{R}, v\rangle$ where $|W| \leq 2^{|\mathcal{E}|}$, $|\mathcal{R}| = |\Pi_{\mathbf{G}}|$, and $v \in W \times \mathcal{E} \to \{\mathrm{T}, \mathrm{F}\}$. $\mathcal{S} \nvDash^{\mathcal{E}}_{\mathbf{G}} s$ iff one of these semiframes is a $\mathbf{G}$-legal model of $\mathcal{S}$, which is not a model of s. If $\mathbf{G}$ is analytic, then $\mathcal{S} \vdash^{\mathcal{E}}_{\mathbf{G}} s$ iff $\mathcal{S} \vdash_{\mathbf{G}} s$. In this case the semantics is effective, leading to a counter-model search procedure. Another corollary is the following characterization of analyticity.

Corollary 1 (Semantic Characterization of Analyticity). *A basic system* $\mathbf{G}$ *is analytic iff for every* $\mathcal{S}$ *and* s*,* $\mathcal{S} \vDash_{\mathbf{G}} s$ *implies* $\mathcal{S} \vDash^{sub[\mathcal{S}\cup\{s\}]}_{\mathbf{G}} s$*.*

The above characterization might be quite complicated to be used in practice. We present a simpler *sufficient* criterion:

Corollary 2. *Let* $\mathbf{G}$ *be a basic system. If every maximal* $\mathbf{G}$*-legal* $\mathcal{E}$*-semiframe can be extended to a* $\mathbf{G}$*-legal frame for every finite set* $\mathcal{E}$ *of formulas closed under subformulas, then* $\mathbf{G}$ *is analytic.*

The last corollary provides a uniform and simple method of proving that a specific basic system is analytic. Indeed, the required "extension property" can very easily be proved for the Kripke semantics of various basic systems mentioned above. This includes (the propositional parts of): **LK**, **LJ**$'$, **SLK**$^{\mathbf{1}}$ from [8], various systems for modal logics from [17] (including those presented in Examples 7 and 9), the family of coherent canonical systems from [6], and many more. Hence all of them are analytic.[2]

[2] Concerning **SLK**$^{\mathbf{1}}$, it was shown in [9] and [12] that it does not enjoy cut-admissibility. From our results it follows that it is nevertheless analytic. This answers a question raised in [12].

Remark 3. The criterion given in Corollary 2 is *not necessary for analyticity*. For example, the system **GL** (see Example 12) is analytic (and even enjoys strong cut-admissibility, as can be shown by a straightforward generalization of the proof of cut-admissibility for it given in [1]), yet it does not meet the semantic condition of Corollary 2. To see this, let $\mathcal{E} = \{p_1, \Box p_1\}$, and let $\mathcal{W} = \langle W, \{R_{\pi_{K4}}\}, v\rangle$ be an $\mathcal{E}$-semiframe, where $W = \{a, b\}$, $R_{\pi_{K4}} = \{\langle a, a\rangle, \langle a, b\rangle\}$, $v(a, p_1) = v(b, p_1) = v(a, \Box p_1) = \text{F}$ and $v(b, \Box p_1) = \text{T}$. W is a maximal **GL**-legal $\mathcal{E}$-semiframe, but it cannot be extended to a **GL**-legal frame: there is no way to assign a truth-value to $\Box\Box p_1$. This phenomenon might be connected with the fact that the natural first-order extension of **GL** does *not* enjoy cut-admissibility ([1]). Further research is needed to clarify this issue (and hopefully to find an effective semantic criterion which is both sufficient and necessary).

Remark 4. While analyticity is defined using the *subformula* relation, it is also possible to study more general notions of analyticity. Indeed, let $\preceq$ be any partial order on $Frm_{\mathcal{L}}$, such that $\{\psi \mid \psi \preceq \varphi\}$ is finite and computable for every φ. For every set $\mathcal{S}$ of sequents, let $\preceq[\mathcal{S}] = \{\psi \mid \exists\varphi \in frm[\mathcal{S}].\psi \preceq \varphi\}$. It is now possible to define $\preceq$-analyticity as in Definition 11 with $\preceq$ instead of *sub*. Consistency and decidability of a basic system follow from its $\preceq$-analyticity. By a straightforward generalization of Corollary 1, we obtain that a basic system **G** is $\preceq$-analytic iff $\mathcal{S} \vDash_{\mathbf{G}} s$ implies $\mathcal{S} \vDash_{\mathbf{G}}^{\preceq[\mathcal{S}\cup\{s\}]} s$. This can be used for basic systems which are not strictly analytic, but are nevertheless $\preceq$-analytic for some well-founded partial order on $Frm_{\mathcal{L}}$. For example, this is the case with some systems of the family $\mathbf{LJ}(\mathbf{S})$ in [4], which extend $\mathbf{LJ}'$ with different rules for negation. For these systems, it can be proven that whenever $\mathcal{S} \vdash s$, then there also exists a proof involving only subformulas of $S \cup \{s\}$ and their negations.

6 Semantic Characterization of Strong Cut-Admissibility

While analyticity of a proof system suffices for many desirable properties, cut-admissibility is traditionally preferred (especially if all other rules enjoy the subformula property, in which case cut-admissibility implies analyticity). Since in this work we deal with proofs from arbitrary sets of assumptions (not necessarily the empty one), we again consider a stronger property: the one which was called *strong* cut-admissibility in [2]. In this section we provide a semantic characterization of the basic sequent systems which enjoy this property. This characterization can serve as a uniform basis for semantic proofs of many (strong) cut-admissibility theorems in various basic systems.

Definition 16. Let **G** be a basic system, $\mathcal{S} \cup \{s\}$ be a set of sequents, and $\mathcal{E}$ be a set of formulas. $\mathcal{S} \vdash_{\mathbf{G}}^{\mathcal{E}_{cuts}} s$ if there exists a proof in **G** of s from $\mathcal{S}$, in which the cut formula of every application of the cut rule is in $\mathcal{E}$.

Definition 17. A basic system **G** enjoys *strong cut-admissibility* if $\mathcal{S} \vdash_{\mathbf{G}} s$ implies $\mathcal{S} \vdash_{\mathbf{G}}^{frm[\mathcal{S}]_{cuts}} s$.

As for analyticity, the semantic characterization of cut-admissibility is obtained by identifying a semantic consequence relation that corresponds to $\vdash_{\mathbf{G}}^{\mathcal{E}_{cuts}}$. For that purpose, we define another generalization of frames, called *quasiframes*.

Definition 18. Let $\mathcal{E}$ be a set of formulas.

1. An $\mathcal{E}$-*quasiframe* is a tuple $\langle W, \mathcal{R}, v\rangle$, where W and $\mathcal{R}$ are as in Definition 5, and $v : W \times Frm_{\mathcal{L}} \to \{\text{T}, \text{F}, \text{I}\}$ is a valuation function, such that $v(w, \psi) \in \{\text{T}, \text{F}\}$ for every $\psi \in \mathcal{E}$.
2. Given an $\mathcal{E}$-quasiframe $\langle W, \mathcal{R}, v\rangle$, a signed formula of the form $f{:}\psi$ (resp. $t{:}\psi$) is *true* in some world $a \in W$ if $v(a, \psi) \in \{\text{F}, \text{I}\}$ $(v(a, \psi) = \{\text{T}, \text{I}\})$.
3. A frame $\langle W, \mathcal{R}, v'\rangle$ (see Definition 5) *refines* an $\mathcal{E}$-quasiframe $\langle W, \mathcal{R}, v\rangle$ if $v'(a, \psi) \geq_k v(a, \psi)$ for every $a \in W$ and $\psi \in Frm_{\mathcal{L}}$, where the partial order $\geq_k$ on $\{\text{T}, \text{F}, \text{I}\}$ is defined by: $x \geq_k x$,$\text{T} \geq_k \text{I}$, and $\text{F} \geq_k \text{I}$.

The third truth-value I is used in quasiframes to distinguish the formulas that belong to $\mathcal{E}$ (on which cut is allowed) from those that do not. Note that if $\psi \notin \mathcal{E}$ then $v(a, \psi)$ can be I, making both $f{:}\psi$ and $t{:}\psi$ true in a. Note also that a frame (Definition 5) is again obtained as a special case: it is an $Frm_{\mathcal{L}}$-quasiframe. Definition 6 is extended to quasiframes without any changes. (Maximal) **G**-legal quasiframes are defined as follows:

Definition 19. Let **G** be a basic system, and $\mathcal{E}$ be a set of formulas.

1. An $\mathcal{E}$-quasiframe $\mathcal{W} = \langle W, \mathcal{R}, v\rangle$ is **G**-*legal* if the conditions formulated in Definition 7 hold for $\mathcal{W}$, except for the third condition, which should apply to all basic rules *except for the cut-rule.*
2. A **G**-legal $\mathcal{E}$-quasiframe $\mathcal{W}$ is called *maximal* if the condition in Definition 9 holds for $\mathcal{W}$.

Proposition 5. *Properties* (1) – (3) *from Proposition 1 hold for maximal* **G**-*legal quasiframes.*

We now define the relation $\vDash_{\mathbf{G}}^{\mathcal{E}_{cuts}}$, and strengthen Theorems 1 and 2.

Definition 20. Let **G** be a basic system, $\mathcal{S} \cup \{s\}$ be a set of sequents, and $\mathcal{E}$ be a set of formulas. $\mathcal{S} \vDash_{\mathbf{G}}^{\mathcal{E}_{cuts}} s$ if every **G**-legal $\mathcal{E}$-quasiframe which is a model of $\mathcal{S}$, is also a model of s.

Theorem 4. *Let* **G** *be a basic system, and* $\mathcal{E}$ *be a set of formulas.*

1. $\vdash_{\mathbf{G}}^{\mathcal{E}_{cuts}} = \vDash_{\mathbf{G}}^{\mathcal{E}_{cuts}}$.
2. *If* $\mathcal{S} \nvDash_{\mathbf{G}}^{\mathcal{E}_{cuts}} s$ *then there exists a* maximal **G**-*legal* $\mathcal{E}$-*quasiframe which is a model of* $\mathcal{S}$, *but not a model of* s.

The following characterization of strong cut-admissibility is a simple corollary.

Corollary 3 (Semantic Characterization of Strong Cut-Admissibility). *A basic system* **G** *enjoys strong cut-admissibility iff* $\mathcal{S} \vDash_{\mathbf{G}} s$ *implies* $\mathcal{S} \vDash_{\mathbf{G}}^{\mathcal{E}_{cuts}} s$ *where* $\mathcal{E} = frm[\mathcal{S}]$.

Example 13. It is well-known that the system **S5** (see Example 1) does not admit strong cut-admissibility. We demonstrate this fact using our semantic characterization. Let s be the sequent $\Rightarrow p_1, \Box\neg\Box p_1$. It is easy to see that s is provable in **S5** (using a cut on $\Box p_1$), and so (by soundness) $\vDash_{\mathbf{S5}} s$. Let $\mathcal{E} = \{p_1, \Box\neg\Box p_1\}$. We show that $\nvDash^{\mathcal{E}_{cuts}}_{\mathbf{S5}} s$ by constructing a **S5**-legal $\mathcal{E}$-quasiframe $\mathcal{W} = \langle W, \mathcal{R}, v\rangle$ which is not a model of s. Let $W = \{a_0, b_0\}$, and $\mathcal{R} = \{R_{\pi_0}, R_{\pi_{S5}}\}$, where R_{π_0} is the identity relation, and $R_{\pi_{S5}} = \{\langle a_0, a_0\rangle, \langle b_0, b_0\rangle, \langle a_0, b_0\rangle\}$. Define $v(a_0, p_1) = v(a_0, \Box\neg\Box p_1) = \text{F}$ and $v(a_0, \psi) = \text{I}$ for other formulas, $v(b_0, p_1) = v(b_0, \Box p_1) = \text{T}$, $v(b_0, \neg\Box p_1) = v(b_0, \Box\neg\Box p_1) = \text{F}$, and $v(b_0, \psi) = \text{I}$ for other formulas. One can now verify that $\mathcal{W}$ is not a model of s and that it a **S5**-legal $\mathcal{E}$-quasiframe. Indeed, all conditions from Definition 7 are met. For example:[3]

- The conditions imposed by π_{S5} are: (1) if $aR_{\pi_{S5}}b$ and $v(a, \Box\psi) = \text{T}$ then $v(b, \Box\psi) = \text{T}$; (2) if $aR_{\pi_{S5}}b$ and $v(a, \Box\psi) = \text{F}$ then $v(b, \Box\psi) = \text{F}$. Both (1) and (2) hold for $\mathcal{W}$.
- The rules for $\Box$ impose the following conditions: (1) if $v(b, \psi) \in \{\text{T}, \text{I}\}$ for every b such that $aR_{\pi_{S5}}b$, then $v(a, \Box\psi) \in \{\text{T}, \text{I}\}$; (2) if $v(a, \psi) \in \{\text{F}, \text{I}\}$ then $v(a, \Box\psi) \in \{\text{F}, \text{I}\}$. Again, both (1) and (2) hold for $\mathcal{W}$.

Corollary 3 implies that **S5** does not admit strong cut-admissibility.

We present a simpler *sufficient* criterion for strong cut-admissibility.[4]

Corollary 4. *Let* **G** *be a basic system. If every maximal* **G***-legal* $\mathcal{E}$*-quasiframe can be refined to a* **G***-legal frame, then* **G** *enjoys (strong) cut-admissibility.*

Example 14. It is easy to verify that the criterion given in Corollary 4 holds for the systems **LK**, **LJ′**, **K**,**K4**, and **S4** from Example 1, and for **KD** from Example 7. Hence all these systems enjoy strong cut-admissibility.

Example 15 (Intuitionistic S4). Consider the basic system $\mathbf{G_0}$ from [11], obtained from the propositional part of **LJ′** by adding the usual $S4$ rules for $\Box$ (see Example 1). Maximal $\mathbf{G_0}$-legal quasiframes include R_{π_0} (the identity relation), and two preorders $R_{\pi_{int}}$ and $R_{\pi_{S4}}$, such that $R_{\pi_{int}} \subseteq R_{\pi_{S4}}$. The context-relations and the rules of this system dictate the usual Kripke semantics of $\Box$ with respect to $R_{\pi_{S4}}$, and of the intuitionistic connectives with respect to $R_{\pi_{int}}$. It is straightforward to verify that every maximal $\mathbf{G_0}$-legal quasiframe can be refined to a $\mathbf{G_0}$-legal frame. It follows that $\mathbf{G_0}$ enjoys (strong) cut-admissibility.

7 Further Research Topics

The examples we have given are somewhat limited in comparison to the generality of the framework we have presented. For example, the conclusions of the

[3] The condition of Definition 9 also holds for $\mathcal{W}$, and so $\mathcal{W}$ is a *maximal* **S5**-legal quasiframe. Recall that property (4) from Proposition 1 does not necessarily hold for quasiframes, and so $R_{\pi_{S5}}$ can be non-symmetric.

[4] Here too the example of **GL** shows that this is not a necessary criterion.

basic rules were all either singletons or empty, and only one context-relation was involved in every basic rule. We leave it as a further research to exploit the full power of this framework. In addition, the following extensions of the framework will be investigated in the future: single-conclusion systems, hypersequential systems, systems employing more than two signs, substructural systems, first order logics and beyond.

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