Persistent Owicki-Gries Reasoning

A Program Logic for Reasoning about Persistent Programs on Intel-x86

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The advent of non-volatile memory (NVM) technologies is expected to transform how software systems are structured fundamentally, making the task of *correct* programming significantly harder. This is because ensuring that memory stores persist in the correct order is challenging, and requires low-level programming to flush the cache at appropriate points. This has in turn resulted in a noticeable *verification gap*.

To address this, we study the verification of NVM programs, and present *Persistent Owicki-Gries* (POG), the first program logic for reasoning about such programs. We prove the soundness of POG over the recent Intel-x86 model, which formalises the out-of-order persistence of memory stores and the semantics of the Intel cache line flush instructions. We then use POG to verify several programs that interact with NVM.

CCS Concepts: • **Theory of computation → Concurrency**; **Semantics and reasoning**; • **Software and its engineering → General programming languages**.

Additional Key Words and Phrases: non-volatile memory, program logic, x86-TSO, consistency, persistency

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1 INTRODUCTION

The emergence of *non-volatile memory* (NVM) technologies [Kawahara et al. 2012; Lee et al. 2009; Strukov et al. 2008] is expected to revamp the structure of modern software: NVM provides storage persistency across power failures with performance close to that of traditional (volatile) memory. As such, programs that require persistency of their data (e.g. databases) can achieve orders of magnitude lower latency by storing their data on NVM rather than on hard disks. It is therefore believed that NVM (a.k.a. persistent memory) will supplant RAM in the near future, thanks to its durable yet competitive performance. This belief is backed by widespread industrial support. Specifically, the two major architectures, ARMv8 and Intel-x86 which together account for almost 100% of the desktop and mobile market, have extended their official specifications to support persistent programming [Arm 2018; Intel 2019]. Intel has further (1) manufactured its own line of NVM, Optane technology [Intel 2019], with an extended academic study evaluating its performance [Izraelevitz et al. 2019]; and (2) released open-source NVM libraries such as PMDK [Intel 2015].

To describe the behaviour of programs under NVM, Intel has introduced a *persistency model* for their x86 architecture [Intel 2019], describing the order in which memory stores may persist to

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NVM. This model was formalised by Raad et al. [2020], wherein they extended the x86-TSO model [Sewell et al. 2010] (with thread-local buffers to model the delayed propagation of writes to other threads) with a global persistent buffer to model the out-of-order propagation of stores to NVM.

Although NVM research has grown rapidly in recent years in both persistency semantics [Condit et al. 2009; Gogte et al. 2018; Izraelevitz et al. 2016; Joshi et al. 2015; Kolli et al. 2017, 2016; Raad and Vafeiadis 2018; Raad et al. 2020, 2019] and algorithms/libraries that exploit NVM [Friedman et al. 2018; Nawab et al. 2017; Zuriel et al. 2019], there has been little work on *verifying* such artifacts. To our knowledge, the existing work [Friedman et al. 2018; Nawab et al. 2017; Raad and Vafeiadis 2018; Raad et al. 2020, 2019; Zuriel et al. 2019] offer low-level proofs about the correctness of small persistent algorithms and often make simplifying assumptions. In particular, they all work at the level of *traces* rather than at the level of *program syntax*, while [Friedman et al. 2018; Nawab et al. 2017; Zuriel et al. 2019] further assume sequential consistency as their concurrency model.

This is a significant omission because developing correct persistent data structures is rather error-prone. Since memory stores are typically persisted out of order, one has to use special low-level instructions for flushing the cache in order to ensure a correct persist ordering. Moreover, such persistent implementations are virtually impossible to test and debug, as one would have to use custom hardware to simulate crashes and check correct recovery from them.

To close this gap, we consider the *formal verification* of (multi-threaded) programs running over NVM. To this end, we adapt the well-known Owicki-Gries (OG) proof system [Owicki and Gries 1976], and develop *POG* (Persistent OG), the *first* program logic for reasoning about NVM programs. We show that the POG proof system is *sound* with respect to the Intel-x86 persistency semantics.

We develop POG over Intel-x86 for several reasons. First, Intel-x86 is a ubiquitous architecture with a formally-defined persistency model [Raad et al. 2020]. Indeed, excluding academic models proposed as proofs of concept, the only real-world persistency models currently available are those of low-level architectures, i.e. those of ARMv8 and Intel-x86 [Raad et al. 2020, 2019], and no existing mainstream programming language such as C/C++ currently has a formal persistency model. Second, existing research on language-level persistency models (e.g. [Gogte et al. 2020; Kolli et al. 2017]) suggests that similar persistency primitives to those of Intel-x86 are considered at the language level, and will likely be lifted to higher-levels. As such, the reasoning principles of POG will be useful in the future higher-level persistency models. Lastly, POG presents the first formal framework for reasoning about persistency primitives and their behaviour *abstractly*, i.e. at the program syntax level, rather than delving into all possible program executions and reasoning at the trace level. Given the complexity of the Intel-x86 persistency model, even verifying simple examples is non-trivial, especially when done at the trace level, and we believe that our syntax-level proof rules in POG help simplify such proofs significantly.

Challenges. Developing the reasoning principles of POG over the Intel-x86 persistency model involves two main challenges: (1) dealing with weak memory consistency, i.e. with the thread-local FIFO buffers of Intel-x86; and (2) dealing with weak persistency, i.e. the persistent buffer of Intel-x86, which allows for stores to persist *asynchronously* and *out of order*.

To address the first challenge, we base our program logic on a variant of OG, named OGRA [Lahav and Vafeiadis 2015], proved sound under release-acquire consistency (a memory model weaker than x86-TSO) and thus also sound under the Intel-x86 model as far as consistency is concerned.

To address the second challenge, we develop an *intermediate* operational model of Intel-x86 persistency, Ix86_{sim}, which forgoes the persistent buffer altogether and operates on the original x86-TSO model (i.e. with only the thread-local buffers). We show that our Ix86_{sim} model correctly captures the effect of the Intel-x86's persistency buffer. That is, the possible outcomes of a program under the Intel-x86 persistency model with two types of buffers are the same as those under Ix86_{sim}.

Our next challenge in designing the POG reasoning principles is modelling the behaviour of explicit persist instructions on Intel-x86, **flush** and $\mathbf{flush}_{\text{opt}}$, which when executed persist all pending writes on a given cache line to the memory. As we describe shortly in §2, $\mathbf{flush}_{\text{opt}}$ instructions offer weaker ordering constraints and may be reordered with respect to other instructions more freely. As such, their effect may not take place at the intended program point, making it more difficult to reason about their persistency behaviour. To keep the POG reasoning principles simple, we devise POG to focus only on the stronger **flush** instructions. We then provide a mechanism to extend our POG reasoning to weaker $\mathbf{flush}_{\text{opt}}$ instructions. More concretely, we present a transformation that allows us in most cases to rewrite a program using $\mathbf{flush}_{\text{opt}}$ to an *equivalent* program using **flush**. We can then use POG to reason about such programs by first using our transformation to replace $\mathbf{flush}_{\text{opt}}$ with **flush**, and then using POG to verify the transformed program.

Although our main contribution is POG, we remark that our Ix86_{sim} model and our transformation are also valuable contributions *per se*. Specifically, Ix86_{sim} may serve as the input to automated verification tools for concurrency, e.g. model checkers, especially those that already support x86-TSO [Abdulla et al. 2015; Clarke et al. 2004; Huang and Huang 2016]. Our persistency-preserving program transformation can be used to optimise code (e.g. at compile time) to replace **flush** with $\mathbf{flush}_{\text{opt}}$.

Contributions and Outline. Our contributions (detailed in §2) are as follows: (1) in §3 we present POG, the *first* program logic for verifying persistency guarantees; (2) in §4 we use POG to verify several examples; (3) in §5 we present our Ix86_{sim} model and show that it faithfully captures Intel-x86 persistency; (4) in §6 we show the POG is sound with respect to the Ix86_{sim} model; (5) in §7 we present our transformation for rewriting programs with $\mathbf{flush}_{\text{opt}}$ to equivalent ones with **flush**, and show that this transformation is sound. We discuss related and future work in §8.

Additional Material. The proofs of all theorems stated in this article are given in full in the accompanying technical appendix available at http://plv.mpi-sws.org/pog/.

2 OVERVIEW

2.1 Px86_{sim} at a Glance

Memory *consistency* models typically describe the permitted behaviours of programs by constraining the *volatile memory order*, i.e. the order in which memory writes are made visible to other threads. Analogously, memory *persistency* models [Pelley et al. 2014] describe the permitted behaviours of programs upon recovering from a crash (e.g. a power failure) by defining a *persistent memory order*, i.e. the order in which writes are committed to persistent memory. To distinguish between the two memory orders, memory *stores* are differentiated from memory *persists*. The former denotes the process of making a write visible to other threads, whilst the latter denotes the process of committing writes durably to persistent memory.

Raad et al. [2020] recently developed the Px86 ('persistent x86') models, formalising the persistency semantics of the Intel-x86 architecture. As they noted, the Intel manual [Intel 2019] is ambiguous at times and allows for weaker behaviours than originally intended. They thus formulated two persistency models: (1) Px86_{man}, which reflects the behaviour outlined in the manual; and (2) Px86_{sim}, which is a strengthening of Px86_{man} and captures the architectural intent. As Px86_{sim} reflects the architectural intent, in this article we focus on the Px86_{sim} model.

The Px86_{sim} model follows a *buffered*, *relaxed* persistency model. Under a buffered model, memory persists occur *asynchronously* [Condit et al. 2009; Izraelevitz et al. 2016; Joshi et al. 2015]: they are buffered in a queue to be committed to persistent memory at a future time. This way, persists occur after their corresponding stores and as prescribed by the persistent memory order, while allowing the execution to proceed ahead of persists. As such, after recovering from a crash, only a *prefix*

$x := 1;$ $y := 1;$	$x := 1;$ **flush** $x';$ $y := 1;$	$x := 1;$ **flush**$_{\text{opt}}$ $x';$ $y := 1;$	$x := 1;$ **flush**$_{\text{opt}}$ $x';$ **sfence**; $y := 1;$	$x := 1;$ $\Vert$ $\mathsf{a} := y;$ **flush** $x';$ $\Vert$ **if** $(\mathsf{a}{=}1)$ $y := 1;$ $\Vert$ $z := 1;$
(a)	(b)	(c)	(d)	(e)
$\lightning : x, y \in \{0,1\}$	$\lightning : y{=}1 \Rightarrow x{=}1$	$\lightning : x, y \in \{0,1\}$	$\lightning : y{=}1 \Rightarrow x{=}1$	$\lightning : z{=}1 \Rightarrow x{=}1$

Fig. 1. Example Px86$_{\text{sim}}$ programs and possible values upon recovery; in all examples x, y, z are locations in persistent memory, a is a (local) register, $x, x' \in X$ (x, x' are in cache line X), $y, z \notin X$, and initially $x{=}y{=}z{=}0$. Replacing the **sfence** instruction in (d) with **mfence** or an atomic RMW yields the same result. Similarly, replacing **flush** x' in (e) with **flush**$_{\text{opt}}$ x'; c yields the same result when c is an **sfence**/**mfence** or an RMW.

of the persistent memory order may have successfully persisted. Under relaxed persistency, the volatile and persistent memory orders may disagree: the order in which the writes are made visible to other threads may differ from the order in which they are persisted.

The relaxed and buffered persistency of Px86$_{\text{sim}}$ is demonstrated in Fig. 1a. If a crash occurs during (or after) the execution of this program, at crash time either write may or may not have persisted and thus $x, y \in \{0,1\}$ upon recovery. The relaxed nature of Px86$_{\text{sim}}$ allows for surprising behaviours that are not possible during normal (non-crashing) executions. Specifically, the two writes cannot be reordered under Intel-x86 and thus at no point during the normal execution of this program $x{=}0$, $y{=}1$ is observable. Nevertheless, in case of a crash it is possible under Px86$_{\text{sim}}$ to observe $x{=}0$, $y{=}1$ after recovery. This is due to the relaxed persistency of Px86$_{\text{sim}}$: the store order (x before y) is separate from the persist order (y before x). Under the Px86$_{\text{sim}}$ model the writes may persist (1) in any order, when they are on distinct locations; or (2) in the volatile memory order, when they are on the same location. That is, for each location, its store and persist orders coincide.

Intel-x86 provides explicit *persist* instructions, **flush** x, **flush**$_{\text{opt}}$ x and **wb** x, in order to afford more control over when pending writes are persisted. When executed, these instructions *asynchronously* persist all pending writes on all locations in the cache line of x [Intel 2019]. That is, when location x is in cache line X, written $x \in X$, an explicit persist on x persists all pending writes on all locations $x' \in X$. As noted by Raad et al. [2020], **flush** instructions are the strongest of the three in terms of their constraints on instruction reordering, whereas **flush**$_{\text{opt}}$ and **wb** are equally weak and have the same specification, with **wb** providing better performance than **flush**$_{\text{opt}}$. That is, **flush**$_{\text{opt}}$ and **wb** are indistinguishable under Px86$_{\text{sim}}$. As such, in the remainder of our discussion we focus on **flush** and **flush**$_{\text{opt}}$ instructions and describe their behaviour via several examples.

The persistency behaviour of **flush** is illustrated in Fig. 1b: given $x, x' \in X$, executing **flush** x' persists the earlier write on X (i.e. $x := 1$). As such, if a crash occurs during the execution of this program and $y{=}1$ upon recovery, then $x{=}1$. That is, if $y := 1$ has executed and persisted before the crash, then so must the earlier $x := 1$; **flush** x'. This is guaranteed by the ordering constraints on **flush**: **flush** instructions are ordered with respect to both earlier (in program order) and later writes. Hence, **flush** x' in Fig. 1b cannot be reordered with respect to $x := 1$ or $y := 1$. As such, upon recovery $y{=}1 \Rightarrow x{=}1$. Note that **flush** x persists X *asynchronously*: its execution does not block until X is persisted; rather, the execution proceeds and X is made persistent at a future point.

In contrast to **flush**, **flush**$_{\text{opt}}$ instructions provide weaker ordering guarantees in relation to writes, in that they are only ordered with respect to *earlier writes* on the *same* cache line. This is illustrated in the example of Fig. 1c obtained from that in Fig. 1b by replacing **flush** x' with **flush**$_{\text{opt}}$ x'. Unlike in Fig. 1b, the **flush**$_{\text{opt}}$ x' instruction is not ordered with respect to the later write ($y := 1$) and may thus be reordered after it. As such, **flush**$_{\text{opt}}$ x' may not execute at the intended program point (after $x := 1$ and before $y := 1$) and thus may not guarantee the intended persist

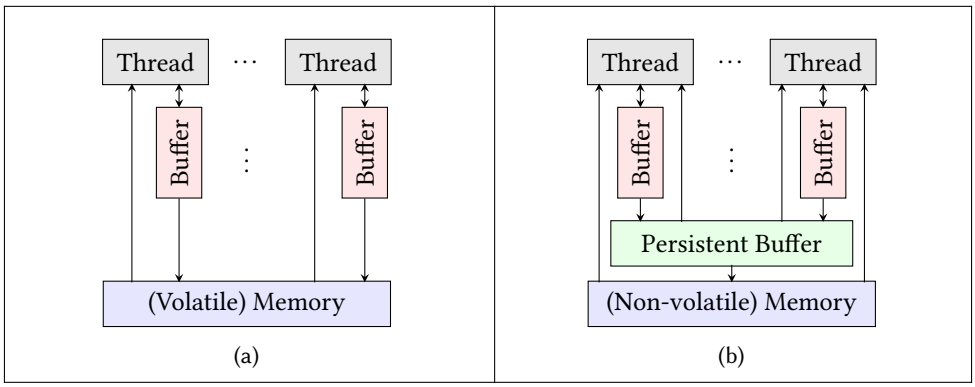

Fig. 2. Storage subsystems of x86-TSO (a) and Px86_{sim} (b) as depicted in [Raad et al. 2020]

ordering. That is, unlike in Fig. 1b, the $\textbf{flush}_{\text{opt}}\ x'$ instructions does not guarantee that the $x := 1$ write persists before $y := 1$, and is thus possible to observe $x{=}0 \wedge y{=}1$ upon recovery.

In order to prevent such reorderings and to strengthen the ordering constraints between $\textbf{flush}_{\text{opt}}$ and later instructions, one can use either *fence* instructions, namely **sfence** (store fence) and **mfence** (memory fence), or atomic *read-modify-write* (RMW) instructions such as compare-and-set (**CAS**) and fetch-and-add (**FAA**). More concretely, **sfence**, **mfence** and RMW instructions are ordered with respect to all (both earlier and later) $\textbf{flush}_{\text{opt}}$, **flush** and write instructions, and can be used to prevent reorderings such as that in Fig. 1c. This is illustrated in the example of Fig. 1d obtained from Fig. 1c by inserting an **sfence** after $\textbf{flush}_{\text{opt}}$. Unlike in Fig. 1c, the intervening **sfence** ensures that $\textbf{flush}_{\text{opt}}$ in Fig. 1d is ordered with respect to $y := 1$ and cannot be reordered after it, thus ensuring that $x := 1$ persists before $y := 1$ (i.e. $y{=}1 \Rightarrow x{=}1$ upon recovery), as in Fig. 1b.

The example in Fig. 1e illustrates how persist orderings can be imposed on the writes of different threads using *message passing*. Note that the program in the left thread of Fig. 1e is that of Fig. 1b. A message is passed from thread τ_1 to τ_2 when τ_2 reads a value written by τ_1. For instance, if the right thread in Fig. 1e reads 1 from y (written by the left thread), then the left thread passes a message to the right thread. Under Intel-x86 message passing ensures that the instruction writing the message and all those ordered before it (e.g. $x := 1; \textbf{flush}\ x; y := 1$) are executed (ordered) before the instruction reading it (e.g. $\mathsf{a} := y$). As such, since $x := 1; \textbf{flush}\ x'$ is executed before $\mathsf{a} := y$, and $z := 1$ is executed after $\mathsf{a} := y$ when $\mathsf{a}{=}1$, we know $x := 1; \textbf{flush}\ x'$ is executed before $z := 1$. Consequently, if upon recovery $z{=}1$ (i.e. $z := 1$ has persisted before the crash), then $x{=}1$ ($x := 1; \textbf{flush}\ x'$ must have also persisted before the crash). As before, replacing **flush** x' in Fig. 1e with $\textbf{flush}_{\text{opt}}\ x'; \mathsf{c}$ yields the same result upon recovery when c is an **sfence**/**mfence** or an RMW.

Lastly, observe that $\textbf{flush}/\textbf{flush}_{\text{opt}}$ instructions impose a particular *persist ordering*. Given $x \in X$, all writes on X ordered before **flush** x/$\textbf{flush}_{\text{opt}}\ x$ persist before all instructions (regardless of their cache line) ordered after **flush** x. For instance, since $x := 1$ in Fig. 1b is ordered before **flush** x', and **flush** x' is ordered before $y := 1$, the $x := 1$ write is guaranteed to persist before $y := 1$. Similarly, as $x := 1$ in Fig. 1d is ordered before $\textbf{flush}_{\text{opt}}\ x'$ which is in turn ordered before $y := 1$ (thanks to the intervening **sfence**), the $x := 1$ write is guaranteed to persist before $y := 1$.

The Operational Px86_{sim} Model. Raad et al. [2020] developed their operational Px86_{sim} model as an extension of the x86-TSO model by Sewell et al. [2010]. As illustrated in Fig. 2a, each thread in x86-TSO is connected to the (volatile) memory via a FIFO buffer. When a thread writes value v to location x, it records it in its buffer as the $\langle x, v \rangle$ entry. When a thread reads from x, it first consults

its own buffer. If it contains buffered writes for x, the thread reads the last such buffered write; otherwise, it consults the memory. Threads can debuffer their writes by propagating them (in FIFO order) to the memory at non-deterministic times. Additionally, the execution of a memory fence **mfence** drains the buffer of the executing thread.

To model the buffered persists of Px86_{sim}, Raad et al. [2020] extended the x86-TSO storage subsystem with a *persistent buffer* as depicted in Fig. 2b, containing those writes that are pending to be persisted to the (non-volatile) memory. As with the memory, the persistent buffer is accessible by all threads. However, while the memory is non-volatile, the persistent buffer is volatile and its contents are lost upon a crash. When writes in the thread-local buffer are debuffered, they are propagated to the persistent buffer, denoting the store of the write (i.e. when the write becomes visible to other threads). Pending writes in the persistent buffer are in turn debuffered and propagated to the memory at non-deterministic times, denoting the persist of the write (i.e. when the write is committed durably to memory). The execution of reads accordingly traverses this hierarchy: when reading from x, the thread first inspects its own local buffer for the last write to x when such a write exists; otherwise, it consults the persistent buffer for the last store to x if such a store exists; otherwise, it reads x from the memory. Recall that the writes on distinct locations may persist in any order, whereas the writes on the same location persist in the store order. To capture this, the persistent buffer is modelled as a queue, where the pending writes on each location are propagated in the FIFO queue order, while those on different locations are propagated in an arbitrary order.

2.2 Eliminating $\text{flush}_{\text{opt}}$ Instructions via Program Transformation

As discussed above, the $\textbf{flush}_{\text{opt}}$ instructions provide weaker ordering constraints and can be reordered e.g. with respect to writes on different cache lines. While this flexibility may in certain cases lead to better performance by affording the compiler more optimisation opportunities through reordering, it significantly complicates the task of reasoning about persistency behaviours. In particular, the weak ordering constraints on $\textbf{flush}_{\text{opt}}$ can lead to unintended persistency behaviours such as that in Fig. 1c (where it is possible to observe $x{=}0 \wedge y{=}1$ upon recovery), and to ensure correct persistency one must thus account for all possible such reorderings. It is therefore simpler to limit our reasoning to programs that solely use the stronger **flush** instructions.

As such, in order to support reasoning about the weaker $\textbf{flush}_{\text{opt}}$ instructions while simultaneously keeping our reasoning principles simple, we first (1) devise a mechanism that in most cases allows us to transform programs using $\textbf{flush}_{\text{opt}}$ to *equivalent* programs that *only* use **flush**; and then (2) design our POG reasoning principles for programs that only use **flush** instructions.

More concretely, for step (1) we note that the main use-case of $\textbf{flush}_{\text{opt}}$ (in which using $\textbf{flush}_{\text{opt}}$ rather than **flush** may prove advantageous for performance) prescribes a particular programming *pattern*. We then show that given a program C that uses $\textbf{flush}_{\text{opt}}$ in this pattern, one can transform C to a program C′ that uses *only* **flush** instructions, such that C and C′ have *equivalent persistency behaviours*, in that they yield the same values for all memory locations upon crash recovery.

Note that our intent through this transformation is not to forgo $\textbf{flush}_{\text{opt}}$ instructions altogether; rather, this transformation merely allows us to extend our reasoning to programs that use $\textbf{flush}_{\text{opt}}$ by considering equivalent programs using **flush**, while keeping our reasoning principles simple.

We present the formal details of this transformation in §7; in the remainder of this section and in §3-§6 we thus focus on programs using only **flush** (and not $\textbf{flush}_{\text{opt}}$) instructions.

2.3 An Intermediate Operational Model for Px86_{sim}

Our goal in this paper is to devise a program logic for reasoning about the persistency behaviour of programs under Px86_{sim}. Although program logics are typically built over operational models that manipulate the underlying state, such operational models often operate on states that comprise the

memory alone, without intermediate caches such as those of thread-local and persistent buffers in Px86_{sim}. This is because a large number of such logics operate under *sequential consistency* (SC) [Lamport 1979], while the presence of such caches introduces weak behaviours absent under SC. To remedy this, recent research [Lahav and Vafeiadis 2015; Sieczkowski et al. 2015; Svendsen et al. 2018; Turon et al. 2014; Vafeiadis and Narayan 2013] demonstrates how to reason about the weak behaviours introduced by e.g. thread-local buffers (see §2.4). However, no existing work currently supports the challenging task of reasoning about the persistency behaviour of programs. The difficulty of such reasoning is further compounded when considering the buffered behaviour of persists due to e.g. the persistent buffer of Px86_{sim}.

To streamline the task of devising a program logic for Px86_{sim}, we first (1) develop an *intermediate* operational semantics, Ix86_{sim}, that forgoes the persistent buffer, while emulating all valid Px86_{sim} behaviours; and then (2) devise a program logic for persistency reasoning over Ix86_{sim}. We proceed with an intuitive account of our Ix86_{sim} model; we briefly describe our program logic later in §2.4.

As discussed above, in our Ix86_{sim} model we forgo the persistent buffer altogether, thus operating on the x86-TSO storage system in Fig. 2a, with the volatile memory replaced with a non-volatile one. For simplicity, let us begin by assuming that **flush** instructions are executed *synchronously*. We later lift this assumption and describe how we handle the asynchronous behaviour of **flush**.

Recall that under Px86_{sim} the store and persist orders may disagree; i.e. the order in which writes in thread buffers are debuffered may differ from the order in which they are debuffered from the persistent buffer. As such, when forgoing the persistent buffer, additional care is required to preserve such weak behaviours. To see this, let us return to Fig. 1a, where $x := 1$ is always store-ordered before $y := 1$, while $y := 1$ may be persist-ordered before $x := 1$. We can then model the store order as in x86-TSO: upon executing each write the thread adds it to its local buffer, and non-deterministically debuffers its entries in the FIFO order, thus ensuring $x := 1$ is store-ordered before $y := 1$. However, without the additional persistent buffer, we can no longer model the out-of-order persists.

To remedy this, for each location x we record two versions: (1) the '*volatile*' version, written x_{v}, tracking the latest observable value of x; and (2) the '*synchronously-persisted*' ('synchronous') version, written x_{s}, tracking the latest persisted value of x provided that **flush** instructions are executed synchronously. Memory instructions (e.g. writes) are then carried out on volatile versions, leaving the synchronous versions untouched. Moreover, the volatile versions may non-deterministically propagate to the corresponding synchronous versions, modelling the notion that writes may be committed to memory at non-deterministic times. Similarly, since we assume **flush** instructions execute synchronously, given $x \in X$, executing **flush** x copies x'_{v} to x'_{s}, for all $x' \in X$.

Let us return to Fig. 1a and write $x{=}v$ to denote that the latest value observable for x is v; i.e. either the thread buffer contains no x entries and the value of x in memory is v, or the latest x entry in the thread buffer is $\langle x, v\rangle$. Similarly, let us write $x \in \{v_1, v_2\}$ for $x{=}v_1 \vee x{=}v_2$. Therefore:

(i) we assume that initially $x_{\mathsf{v}}{=}x_{\mathsf{s}}{=}y_{\mathsf{v}}{=}y_{\mathsf{s}}{=}0$;
(ii) after executing $x := 1$ we have: $x_{\mathsf{v}}{=}1 \wedge x_{\mathsf{s}} \in \{0,1\} \wedge y_{\mathsf{v}}{=}y_{\mathsf{s}}{=}0$;
(iii) upon subsequently executing $y := 1$ we have: $x_{\mathsf{v}}{=}1 \wedge x_{\mathsf{s}} \in \{0,1\} \wedge y_{\mathsf{v}}{=}1 \wedge y_{\mathsf{s}} \in \{0,1\}$.

Note that since initially $x_{\mathsf{s}}{=}0$ and the value of x_{v} may be copied to x_{s} non-deterministically, we must account for this propagation in (ii) and thus we have $x_{\mathsf{s}} \in \{0,1\}$; similarly for y_{s} in (iii). As such, at all program points ((i)–(iii)) we have $x_{\mathsf{s}}, y_{\mathsf{s}} \in \{0,1\}$. That is, if a crash occurs at any point, upon recovery we have $x_{\mathsf{s}}, y_{\mathsf{s}} \in \{0,1\}$, thus emulating the desired behaviour in Fig. 1a.

We can analogously emulate the behaviour of Fig. 1b:

(a) we assume that initially $x_{\mathsf{v}}{=}x_{\mathsf{s}}{=}y_{\mathsf{v}}{=}y_{\mathsf{s}}{=}0$;
(b) after executing $x := 1$ we have: $x_{\mathsf{v}}{=}1 \wedge x_{\mathsf{s}} \in \{0,1\} \wedge y_{\mathsf{v}}{=}y_{\mathsf{s}}{=}0$;

(c) after executing **flush** x we have: $x_{\mathsf{v}}{=}x_{\mathsf{s}}{=}1 \wedge y_{\mathsf{v}}{=}y_{\mathsf{s}}{=}0$;
(d) upon subsequently executing $y := 1$ we have: $x_{\mathsf{v}}{=}x_{\mathsf{s}}{=}1 \wedge y_{\mathsf{v}}{=}1 \wedge y_{\mathsf{s}} \in \{0,1\}$.

That is, executing **flush** x (c) copies x_{v} to x_{s}, and thus we have $y_{\mathsf{s}}{=}1 \Rightarrow x_{\mathsf{s}}{=}1$ at all program points.

***Modelling the Asynchronous Behaviour of* flush.** As we demonstrated above, tracking the synchronous version of locations (e.g. x_{s}) allows us to capture the necessary persist orderings (e.g. that x_{s} persists before y_{s}). However, as **flush** instructions execute asynchronously, synchronous versions do not accurately capture the memory state upon a crash. For instance, if a crash occurs after **flush** x' is executed (but not yet fully completed) in Fig. 1b, unlike what we wrote in (c) and (d) above, x may not necessarily contain 1. To address this, for each location x we record a third, *persisted* version, written x_{p}, denoting the latest persisted value of x (without assuming **flush** instructions are executed synchronously).

Intuitively, there is a chronological order on different versions of a location x: $x_{\mathsf{p}} \rightarrow x_{\mathsf{s}} \rightarrow x_{\mathsf{v}}$, in that while x_{p} reflects the last persisted value of x in memory, x_{s} and x_{v} denote later updates on x that are yet to be persisted, with x_{v} describing the latest such update. As discussed, x_{v} may non-deterministically be copied to x_{s}, allowing x_{s} to catch up with x_{v}. Intuitively, this amounts to a pending write on x (in the persistent buffer) being committed to memory. Analogously, the effect of asynchronous **flush** instructions may take effect at non-deterministic times. We may thus be inclined to copy x_{s} to x_{p} non-deterministically, allowing x_{p} to catch up with x_{s}. However, this is too weak. To see this, let us extend the (a)–(d) steps of Fig. 1b with x_{p} and y_{p} constraints (highlighted):

(a') $x_{\mathsf{v}}{=}x_{\mathsf{s}}{=}x_{\mathsf{p}}{=}y_{\mathsf{v}}{=}y_{\mathsf{s}}{=}y_{\mathsf{p}}{=}0$
(b') $x_{\mathsf{v}}{=}1 \wedge x_{\mathsf{s}}, x_{\mathsf{p}} \in \{0,1\} \wedge (x_{\mathsf{p}}{=}1 \Rightarrow x_{\mathsf{s}}{=}1) \wedge y_{\mathsf{v}}{=}y_{\mathsf{s}}{=}y_{\mathsf{p}}{=}0$
(c') $x_{\mathsf{v}}{=}x_{\mathsf{s}}{=}1 \wedge x_{\mathsf{p}} \in \{0,1\} \wedge y_{\mathsf{v}}{=}y_{\mathsf{s}}{=}y_{\mathsf{p}}{=}0$
(d') $x_{\mathsf{v}}{=}x_{\mathsf{s}}{=}1 \wedge y_{\mathsf{v}}{=}1 \wedge x_{\mathsf{p}}, y_{\mathsf{s}}, y_{\mathsf{p}} \in \{0,1\} \wedge (y_{\mathsf{p}}{=}1 \Rightarrow y_{\mathsf{s}}{=}1)$

First, note that $(x_{\mathsf{p}}{=}1 \Rightarrow x_{\mathsf{s}}{=}1)$ in (b') captures the chronological order between x_{s} and x_{p}: x_{s} may be copied to x_{p} and thus if $x_{\mathsf{p}}{=}1$ then $x_{\mathsf{s}}{=}1$; similarly for $(y_{\mathsf{p}}{=}1 \Rightarrow y_{\mathsf{s}}{=}1)$ in (d'). Second, note that x_{p} and y_{p} are our main interest as they denote the latest persisted values of x and y. We are hence interested in establishing $y_{\mathsf{p}}{=}1 \Rightarrow x_{\mathsf{p}}{=}1$ at all program points, thus modelling the desired behaviour in Fig. 1b. This is, however, not the case as (d') allows $y_{\mathsf{p}}{=}1 \wedge x_{\mathsf{p}}{=}0$.

To remedy this, we require that the non-deterministic copying of synchronous values to persisted ones be carried out *for all locations* at once, and not a single location. In doing so, we ensure that $y_{\mathsf{p}}{=}1 \Rightarrow x_{\mathsf{p}}{=}1$ holds at (d'), as desired. Intuitively, this captures the global persist orderings imposed by **flush**. In particular, recall that when $x \in X$, all X writes ordered before **flush** x persist before all instructions ordered after **flush** x. As such, upon propagating a write to the persistent memory, i.e. copying some y_{s} to y_{p}, we must ensure that the effects of each prior **flush** x is completed in that its preceding writes on X have also reached the memory. This amounts to (simultaneous) copying of x_{s} to x_{p} for $x \in X$ (since each prior **flush** on X has copied x_{v} to x_{s} for $x \in X$).

In summary, in our Ix86$_{\text{sim}}$ model: (1) memory operations on x (reads/writes) manipulate x_{v}; (2) when $x \in X$, **flush** x copies x'_{v} to x'_{s} for every $x' \in X$; (3) x_{v} may be copied to x_{s} non-deterministically; and (4) $\overline{x_{\mathsf{s}}}$ may be point-wise copied to $\overline{x_{\mathsf{p}}}$ non-deterministically, where $\overline{x_{\mathsf{s}}}$ denotes *all* synchronous locations and $\overline{x_{\mathsf{p}}}$ denotes the corresponding persistent ones.

2.4 POG: Persistent Owicki-Gries Reasoning

Having forgone the need for persistent buffers through our Ix86$_{\text{sim}}$ operational model, our next goal is to devise a program logic for persistency reasoning over Ix86$_{\text{sim}}$ which operates on the x86-TSO storage system in Fig. 2a. As mentioned in §2.3, our next challenge is accounting for the weak behaviours caused by the thread-local buffers in x86-TSO. Fortunately, existing work [Lahav and Vafeiadis 2015; Sieczkowski et al. 2015; Svendsen et al. 2018; Turon et al. 2014; Vafeiadis and

Narayan 2013] demonstrate how existing program logics for SC can be adapted to reason about such weak behaviours. Here, we follow the simple approach of [Lahav and Vafeiadis 2015], which demonstrated how Owicki-Gries (OG) reasoning [Owicki and Gries 1976] can be adapted to reason about the release-acquire fragment of the C11 memory model [Lahav et al. 2017].

OG Reasoning. As in Hoare logic, the basic constructs in OG are *Hoare triples* of the form $\{P\}\ \mathsf{C}\ \{Q\}$, where P and Q are assertions (sets of states) describing the pre- and post-condition of program C. OG reasoning extends the proof rules of Hoare logic with a rule to reason about concurrent programs of the form $\mathsf{C}_1 \,||\, \mathsf{C}_2$, which allows one to compose the verified programs C_1 and C_2 into a verified concurrent program, provided that the two proofs are *non-interfering*:

$$\frac{\{P_1\}\ \mathsf{C}_1\ \{Q_1\} \quad \{P_2\}\ \mathsf{C}_2\ \{Q_2\} \quad \text{two proofs are non-interfering}}{\{P_1 \wedge P_2\}\ \mathsf{C}_1 \,||\, \mathsf{C}_2\ \{Q_1 \wedge Q_2\}}$$

As such, OG is often deemed non-compositional as it refers to non-interference of proof outlines that cannot be checked based solely on the two input triples. However, as demonstrated in [Lahav and Vafeiadis 2015], presenting OG in the *rely-guarantee* (RG) [Jones 1983] style allows compositional reasoning. In this presentation, Hoare triples are interpreted under an *RG context*, $\langle \mathcal{R}; \mathcal{G} \rangle$. The *rely* component, $\mathcal{R}$, comprises a set of assertions assumed to be *stable* under memory updates carried out by the environment (i.e. other threads). The *guarantee* component, $\mathcal{G}$, in turn comprises a set of *guarded updates* that the thread may perform. A guarded update is of the form $\langle x, \mathsf{e}, P \rangle$, stating that when the program state satisfies the guard P, then the thread may update x to e.

An RG-style OG triple is of the form $\langle \mathcal{R}; \mathcal{G} \rangle \vdash \{P\}\ \mathsf{C}\ \{Q\}$, stating that: (1) every terminating run of C from a state in P results in a state in Q; (2) C updates the state in accordance with $\mathcal{G}$ while satisfying the prescribed guards; and (3) the same holds when C is run in parallel with any program C' whose updates preserve the assertions in $\mathcal{R}$. We can then rewrite the parallel composition rule in the RG-style as shown in Fig. 3 (PAR), requiring that the RG contexts of the proofs be non-interfering. That is, every update of C_1 in $\mathcal{G}_1$ preserves every assertion in $\mathcal{R}_2$, and vice versa.

Invariant-based Reasoning. A main advantage of Hoare logic and its descendants such as OG is their support for *compositional* reasoning: once we verify the behaviour of a small program C, we can use its specification to verify larger programs that use C. For instance, having established $\{P\}\ \mathsf{C}\ \{Q\}$, we can then use it to verify the larger program $\mathsf{C}' \triangleq \mathsf{C}; \mathsf{C}''$. That is, it suffices to find Q' such that $\{Q\}\ \mathsf{C}''\ \{Q'\}$, as we can then compose the two triples to derive $\{P\}\ \mathsf{C}'\ \{Q'\}$. In other words, this allows us to treat C as a black box, jumping over its body and assuming Q at the end.

However, this is no longer the case in the persistent setting: as the execution of C may crash, we cannot assume that its execution completed successfully and that Q describes the state on its completion. Rather, we must account for the possibility of C crashing at any point during its execution. As such, the state upon returning from C (either due to a crash or after successful completion) is that described by the *disjunction* (union) of states at each program point. To keep our presentation simple, we typically define an *invariant*, I, that holds at all program points, and could simply be defined as the disjunction of all states. Intuitively, I corresponds to the persistency behaviour we seek to establish, e.g. $y{=}1 \Rightarrow x{=}1$ in Fig. 1b. At each program point we can still strengthen I by adding additional conjuncts. However, when C is used in the larger context of C', we can simply treat its specification as $\{I\}\ \mathsf{C}\ \{I\}$. Note that this is a generalisation of the approach in [Chen et al. 2015], where Hoare triples are of the form $\{P\}\ \mathsf{C}\ \{Q\}\{I\}$, with Q denoting the non-crashing postcondition that holds once C executes successfully, and I denoting the crashing postcondition that holds in case of a crash, and is itself typically defined as a disjunction of postconditions at each point. As such, the overall postcondition of C is described by $Q \vee I$.

Stability. Recall that under Ix86_{sim}, x_{v} may be non-determi-nistically copied to x_{s} for each location, while x_{s} may in turn be non-deterministically copied to x_{p} for all locations at once. As such, we stipulate that the assertions used in our proofs be *stable* with respect to these propagations. In particular, we require that for all x: $P \Rightarrow P[x_{\mathsf{v}}/x_{\mathsf{s}}]$; i.e. if P holds beforehand, it should still hold after x_{v} is propagated to x_{s}. Analogously, we require that for all x: $P \Rightarrow P[\overline{x_{\mathsf{s}}/x_{\mathsf{p}}}]$.

POG Reasoning. We develop the POG ('persistent Owicki-Gries') logic for reasoning about persistency behaviours of programs under Px86_{sim}. We formulate POG as an extension of OG, and build it over our Ix86_{sim} model. We present the formal details of POG in §3. Here we introduce POG by verifying the example in Fig. 1b. Later in §4 we verify several other examples in POG.

Recall that our goal is to devise an invariant I that holds at all program points in Fig. 1b. In particular, we are interested in establishing $I \triangleq y_{\mathsf{p}}{=}1 \Rightarrow x_{\mathsf{p}}{=}1$, as shown in Fig. 1b. As discussed, memory operations (e.g. writes) on x in Ix86_{sim} manipulate the x_{v} location. This is reflected in the (WRITE) rule in Fig. 3, stating that when executing $x := \mathsf{e}$, the postcondition Q is obtained from the precondition P by substituting the new value e for x_{v}. Analogously, the (FLUSH) proof rule in Fig. 3 states that when executing **flush** x, the postcondition Q is obtained from the precondition P by copying x'_{v} to x'_{s} for every x' in the cache line of x. As such, assuming that initially all locations hold 0 and that x and x' are in the same cache line, we can verify the program in Fig. 1b as follows:

$$\begin{gathered}
\{I \wedge x_{\mathsf{v}}{=}x_{\mathsf{s}}{=}x_{\mathsf{p}}{=}0 \wedge y_{\mathsf{v}}{=}y_{\mathsf{s}}{=}y_{\mathsf{p}}{=}0\} \\
x := 1; \\
\{I \wedge x_{\mathsf{v}}{=}1 \wedge x_{\mathsf{s}}, x_{\mathsf{p}} \in \{0,1\} \wedge y_{\mathsf{v}}{=}y_{\mathsf{s}}{=}y_{\mathsf{p}}{=}0\} \\
\textbf{flush}\ x'; \\
\{I \wedge x_{\mathsf{v}}{=}x_{\mathsf{s}}{=}1 \wedge x_{\mathsf{p}} \in \{0,1\} \wedge y_{\mathsf{v}}{=}y_{\mathsf{s}}{=}y_{\mathsf{p}}{=}0\} \\
y := 1; \\
\{I \wedge x_{\mathsf{v}}{=}x_{\mathsf{s}}{=}1 \wedge y_{\mathsf{v}}{=}1 \wedge x_{\mathsf{p}}, y_{\mathsf{s}}, y_{\mathsf{p}} \in \{0,1\}\}
\end{gathered}$$

Note that as the program in Fig. 1b is sequential, we define the RG context as $\langle \top; \top \rangle$; i.e. the environment must preserve all assertions ($\mathcal{R}{=}\top$), and the program may perform any assignment ($\mathcal{G}{=}\top$). For simplicity, we have elided the RG context above. Observe that although $x := 1$ solely manipulates x_{v} and in its precondition we have $x_{\mathsf{s}}{=}x_{\mathsf{p}}{=}0$, after its execution we weakened the postcondition to $x_{\mathsf{s}}, x_{\mathsf{p}} \in \{0,1\}$. This is to ensure that the postcondition is *stable* since the value of x_{v} (i.e. 1) may non-deterministically propagate to x_{s} (and subsequently from x_{s} to x_{p}), as discussed above. Moreover, when analogously stabilising the post-condition of $y := 1$, we must propagate both x_{s} to x_{p} and y_{s} to y_{p} *simultaneously*, thus ruling out $x_{\mathsf{p}}{=}0$, $y_{\mathsf{p}}{=}1$ (as $x_{\mathsf{s}}{=}1$) and establishing I.

3 THE POG PROGRAM LOGIC

POG Language. The POG language given below is that of Px86_{sim} in [Raad et al. 2020] excluding $\textbf{flush}_{\text{opt}}$. We assume a finite set LOC of memory locations; a finite set REG of (local) registers; a finite set VAL of values; a finite set TID of thread identifiers; any standard interpreted language for expressions containing registers and values; and a finite set CL of cache lines partitioning locations ($\text{LOC}{=} \uplus \text{CL}$). We use x, y, z as metavariables for locations; $\mathsf{a}, \mathsf{b}, \mathsf{c}$ for registers; v for values; τ for thread identifiers; e for expressions; and X for cache lines.

$$\begin{aligned}
\text{SCOM} \ni \mathsf{c} ::= {} & \textbf{skip} \mid \textbf{while}(\mathsf{e})\ \mathsf{c} \mid \textbf{if}\ (\mathsf{e})\ \textbf{then}\ \mathsf{c}_1\ \textbf{else}\ \mathsf{c}_2 \mid \mathsf{c}_1; \mathsf{c}_2 \mid \mathsf{a} := \mathsf{e} \mid \mathsf{a} := x \mid x := \mathsf{e} \\
& \mid \textbf{flush}\ x \mid \textbf{sfence} \mid \textbf{mfence} \mid \mathsf{a} := \textbf{CAS}(x, \mathsf{e}_1, \mathsf{e}_2) \mid \mathsf{a} := \textbf{FAA}(x, \mathsf{e}) \\
\text{COM} \ni \mathsf{C} \triangleq {} & \text{TID} \xrightarrow{\text{fin}} \text{SCOM}
\end{aligned}$$

$$\frac{}{\langle\{P\};\emptyset\rangle \vdash \{P\}\ \mathbf{skip}\ \{P\}}\ (\textsc{skip}) \qquad \frac{P \Rightarrow Q[\mathsf{e}/\mathsf{a}]}{\langle\{P,Q\},\emptyset\rangle \vdash \{P\}\ \mathsf{a}:=\mathsf{e}\ \{Q\}}\ (\textsc{assign}) \qquad \frac{P \Rightarrow Q[x_{\mathsf{v}}/\mathsf{a}]}{\langle\{P,Q\},\emptyset\rangle \vdash \{P\}\ \mathsf{a}:=x\ \{Q\}}\ (\textsc{read})$$

$$\frac{P \Rightarrow Q[\mathsf{e}/x_{\mathsf{v}}]}{\langle\{P,Q\},\{(x_{\mathsf{v}},\mathsf{e},P)\}\rangle \vdash \{P\}\ x:=\mathsf{e}\ \{Q\}}\ (\textsc{write}) \qquad \frac{P \Rightarrow Q[x_{\mathsf{v}}/\mathsf{a}][x_{\mathsf{v}}+\mathsf{e}/x_{\mathsf{v}}]}{\langle\{P,Q\},\{(x_{\mathsf{v}},x_{\mathsf{v}}+\mathsf{e},P)\}\rangle \vdash \{P\}\ \mathsf{a}:=\mathbf{FAA}(x,\mathsf{e})\ \{Q\}}\ (\textsc{faa})$$

$$\frac{\begin{array}{c}P \wedge x_{\mathsf{v}}{\neq}\mathsf{e}_1 \Rightarrow Q[x_{\mathsf{v}}/\mathsf{a}] \\ P \wedge x_{\mathsf{v}}{=}\mathsf{e}_1 \Rightarrow Q[\mathsf{e}_1/\mathsf{a}][\mathsf{e}_2/x_{\mathsf{v}}]\end{array}}{\langle\{P,Q\},\{(x_{\mathsf{v}},\mathsf{e}_2,P\wedge x_{\mathsf{v}}{=}\mathsf{e}_1)\}\rangle \vdash \{P\}\ \mathsf{a}:=\mathbf{CAS}(x,\mathsf{e}_1,\mathsf{e}_2)\ \{Q\}}\ (\textsc{cas}) \qquad \frac{\begin{array}{c}x\in X \qquad X=\{x^1\ldots x^n\} \\ P \Rightarrow Q[x_{\mathsf{v}}^1/x_{\mathsf{s}}^1\ldots x_{\mathsf{v}}^n/x_{\mathsf{s}}^n]\end{array}}{\langle\{P,Q\},\emptyset\rangle \vdash \{P\}\mathbf{flush}\ x\{Q\}}\ (\textsc{flush})$$

$$\frac{\begin{array}{c}\langle\mathcal{R}_1;\mathcal{G}_1\rangle \vdash \{P\}\ \mathsf{c}_1\ \{R\} \\ \langle\mathcal{R}_2;\mathcal{G}_2\rangle \vdash \{R\}\ \mathsf{c}_2\ \{Q\}\end{array}}{\langle\mathcal{R}_1\cup\mathcal{R}_2;\mathcal{G}_1\cup\mathcal{G}_2\rangle \vdash \{P\}\ \mathsf{c}_1;\mathsf{c}_2\ \{Q\}}\ (\textsc{seq}) \qquad \frac{\begin{array}{c}\langle\mathcal{R};\mathcal{G}\rangle \vdash \{P\wedge \mathsf{e}{\neq}0\}\ \mathsf{c}_1\ \{Q\} \\ \langle\mathcal{R};\mathcal{G}\rangle \vdash \{P\wedge \mathsf{e}{=}0\}\ \mathsf{c}_2\ \{Q\}\end{array}}{\langle\mathcal{R}\cup\{P\};\mathcal{G}\rangle \vdash \{P\}\ \mathbf{if}\ (\mathsf{e})\ \mathbf{then}\ \mathsf{c}_1\ \mathbf{else}\ \mathsf{c}_2\ \{Q\}}\ (\textsc{ite})$$

$$\frac{\begin{array}{c}P\wedge \mathsf{e}{=}0 \Rightarrow Q \\ \langle\mathcal{R};\mathcal{G}\rangle \vdash \{P\wedge \mathsf{e}{\neq}0\}\ \mathsf{c}\ \{P\}\end{array}}{\langle\mathcal{R}\cup\{Q\};\mathcal{G}\rangle \vdash \{P\}\ \mathbf{while}(\mathsf{e})\ \mathsf{c}\ \{Q\}}\ (\textsc{while}) \qquad \frac{\begin{array}{c}P\Rightarrow P' \quad \mathcal{R}'\subseteq\mathcal{R} \quad \mathcal{G}'\subseteq\mathcal{G} \quad Q'\Rightarrow Q \\ \langle\mathcal{R}';\mathcal{G}'\rangle \vdash \{P'\}\ \mathsf{C}\ \{Q'\}\end{array}}{\langle\mathcal{R}\cup\{P,Q\};\mathcal{G}\rangle \vdash \{P\}\ \mathsf{C}\ \{Q\}}\ (\textsc{conseq})$$

$$\frac{\begin{array}{ccc}\langle\mathcal{R}_1;\mathcal{G}_1\rangle \vdash \{P_1\}\mathsf{C}_1\{Q_1\} & \langle\mathcal{R}_2;\mathcal{G}_2\rangle \vdash \{P_2\}\mathsf{C}_2\{Q_2\} & \langle\mathcal{R}_1;\mathcal{G}_1\rangle \text{ and } \langle\mathcal{R}_2;\mathcal{G}_2\rangle \text{ are non-interfering} \\ Q_1\wedge Q_2 \Rightarrow Q & \mathtt{fr}(\mathcal{R}_1,\mathsf{C}_1)\cap\mathtt{wr}(\mathsf{C}_2){=}\emptyset & \mathtt{fr}(\mathcal{R}_2,\mathsf{C}_2)\cap\mathtt{wr}(\mathsf{C}_1){=}\emptyset\end{array}}{\langle\mathcal{R}_1\cup\mathcal{R}_2\cup\{Q\};\mathcal{G}_1\cup\mathcal{G}_2\rangle \vdash \{P_1\wedge P_2\}\ \mathsf{C}_1\,||\,\mathsf{C}_2\ \{Q\}}\ (\textsc{par})$$

$$\frac{P\Rightarrow P' \quad \langle\top;\top\rangle \vdash \{P'\}\ \mathsf{C}\ \{Q'\} \quad \langle\top;\top\rangle \vdash \{R\}\ \mathsf{C}_{\mathsf{rec}}\ \{Q'\} \quad Q'\Rightarrow Q \quad Q'\Rightarrow \exists\overline{v}.R[\overline{v/\mathsf{a}}][\overline{x_{\mathsf{p}}/x_{\mathsf{s}}}][\overline{x_{\mathsf{p}}/x_{\mathsf{v}}}]}{\mathsf{C}_{\mathsf{rec}} \vdash \{P\}\ \mathsf{C}\ \{Q\}}\ (\textsc{rec})$$

Fig. 3. POG proof rules with the implicit assumption that the pre- and post-conditions are stable (Def. 2).

The sequential fragment of the language is given by the c grammar and includes the standard constructs of **skip**, loops, conditionals and sequential composition, as well as local variable assignment ($\mathsf{a}:=\mathsf{e}$), memory read from location x ($\mathsf{a}:=x$), memory write to x ($x:=\mathsf{e}$), memory persist (**flush** x), store fence (**sfence**), memory fence (**mfence**) and atomic RMW (read-modify-write) instructions. The RMW instruction $\mathsf{a}:=\mathbf{CAS}(x,\mathsf{e}_1,\mathsf{e}_2)$ denotes the atomic 'compare-and-swap', where the value of x is compared against e_1: if the values match then x is set to e_2 and 1 is returned in a; otherwise x is left unchanged and 0 is returned in a. Analogously, $\mathsf{a}:=\mathbf{FAA}(x,\mathsf{e})$ denotes the atomic 'fetch-and-add', where x is incremented by e and its old value is returned in a. Lastly, we model a multi-threaded program C as a function mapping each thread to its (sequential) program. We write $\mathsf{C}{=}\mathsf{c}_1\,||\cdots||\,\mathsf{c}_n$ when $dom(\mathsf{C}){=}\{\tau_1\cdots\tau_n\}$ and $\mathsf{C}(\tau_i){=}\mathsf{c}_i$, and write $\mathsf{C}_1\,||\,\mathsf{C}_2$ for $\mathsf{C}_1\uplus\mathsf{C}_2$. We lift a sequential program c to a program in Com and simply write c for $\mathsf{C}\triangleq[\tau\mapsto\mathsf{c}]$.

Instrumented Locations. Recall from §2 that in order to reason about the persistency behaviours of programs, we instrument the memory to track three separate versions for each memory location. To this end, we define the set of *instrumented locations* as $\textsc{iLoc}\triangleq\{x_{\mathsf{v}},x_{\mathsf{s}},x_{\mathsf{p}}\mid x\in\textsc{Loc}\}$, and define an instrumented memory, $\mathsf{IM}:\textsc{IMem}$, as a finite map from instrumented locations to values: $\textsc{IMem}:\textsc{ILoc}\xrightarrow{\text{fin}}\textsc{Val}$. As discussed in §2, for each memory location x:

(1) $\mathsf{IM}(x_{\mathsf{v}})$ denotes the *volatile* value of x, i.e. the value observed for x during the execution;
(2) $\mathsf{IM}(x_{\mathsf{s}})$ denotes the *synchronously persisted* value of x, i.e. the value observed for x after a crash, had the **flush** instructions executed synchronously; and

(3) $\mathsf{IM}(x_{\mathsf{p}})$ denotes the *persistent* value of x, i.e. the value observed for x after a crash.

Intuitively, $\mathsf{IM}(x_{\mathsf{p}})$ reflects the current value of x in memory, while $\mathsf{IM}(x_{\mathsf{v}})$ and $\mathsf{IM}(x_{\mathsf{s}})$ record additional information that enables persistent reasoning, as discussed in §2.

Assertions. The POG assertions represent sets of states in our Ix86_{sim} semantics. Our assertion language is that of first order logic with *equality* and i) three constant symbols x_{v}, x_{s}, x_{p} for each location x; ii) a constant symbol a for each register; and iii) a constant symbol v for each value.

3.1 The POG Proof System

The *POG triples* are of the form: $\langle \mathcal{R}; \mathcal{G} \rangle \vdash \{P\}\ \mathsf{C}\ \{Q\}$, stating that: (1) the persistent parts of Q (describing persistent versions of variables, e.g. x_{p}) are *invariant* throughout the execution of C; (2) every *terminating* run of C from a state in P results in a state in Q; (3) C updates the state in accordance with $\mathcal{G}$ while satisfying the prescribed guards; and (4) the above holds when C is run in parallel with any program C', provided that the C' updates preserve the assertions in $\mathcal{R}$. That is, while the persistent parts of Q hold at *all points* during the execution of C, those parts describing register values and volatile/synchronous versions hold only when C terminates successfully.

POG Proof Rules. We present the *POG proof rules* in Fig. 3. Ignoring the (REC) rule at the bottom, most rules remain largely unchanged from their OG counterparts and are merely adapted to RG-style as in [Lahav and Vafeiadis 2015]. Intuitively, the pre- and post-conditions of triples are accumulated in the rely component $\mathcal{R}$ to ensure they remain stable (invariant) under the updates performed by other threads. Conversely, each time a thread updates a location x via (WRITE), (CAS) and (FAA), this is recorded in its guarantee component $\mathcal{G}$ with the corresponding precondition. Moreover, the (READ), (WRITE), (CAS) and (FAA) rules accessing memory location x have been accordingly adjusted to access the latest value of x, namely that in x_{v}.

As discussed in §2.3, when $x \in X$, **flush** x propagates the latest persist-pending value of each $x' \in X$ to memory. This is reflected in the $P \Rightarrow Q[x_{\mathsf{v}}^1/x_{\mathsf{s}}^1 \ldots x_{\mathsf{v}}^n/x_{\mathsf{s}}^n]$ premise of the (FLUSH) rule.

The (PAR) rule describes the concurrent execution $\mathsf{C}_1 \,||\, \mathsf{C}_2$, where the non-interference premise ensures that C_1 and C_2 do not interfere with one another. Intuitively, C_1 and C_2 are non-interfering iff each update performed by C_1 preserves the state assumptions of C_2, and vice versa. Put formally, when C_1 and C_2 are respectively run under contexts $\langle \mathcal{R}_1; \mathcal{G}_1 \rangle$ and $\langle \mathcal{R}_2; \mathcal{G}_2 \rangle$, then every update $\langle x, \mathsf{e}, P \rangle$ of C in $\mathcal{G}_1$ must preserve every assertion R in $\mathcal{R}_2$, i.e. $P \wedge R \Rightarrow R[\mathsf{e}/x_{\mathsf{v}}]$; and vice versa.

Definition 1. The tuples $\langle \mathcal{R}_1; \mathcal{G}_1 \rangle$ and $\langle \mathcal{R}_2; \mathcal{G}_2 \rangle$ are *non-interfering* iff:

- for all $\langle x_{\mathsf{v}}, \mathsf{e}, P \rangle \in \mathcal{G}_1$ and all $R \in \mathcal{R}_2$: $R \wedge P \Rightarrow R[\mathsf{e}/x_{\mathsf{v}}]$
- for all $\langle x_{\mathsf{v}}, \mathsf{e}, P \rangle \in \mathcal{G}_2$ and all $R \in \mathcal{R}_1$: $R \wedge P \Rightarrow R[\mathsf{e}/x_{\mathsf{v}}]$

Stability. Recall from §2.4 that we require assertions to be stable against volatile-to-synchronous and synchronous-to-persistent version propagations. This is formalised in Def. 2.

Definition 2. An assertion P is *stable*, written $\mathsf{stable}(P)$, iff $P \Rightarrow P[\overline{x_{\mathsf{s}}/x_{\mathsf{p}}}]$ and $\forall x_{\mathsf{s}}.\ P \Rightarrow P[x_{\mathsf{v}}/x_{\mathsf{s}}]$.

Reasoning about Crash Recovery. The POG triples discussed thus far are of the form $\langle \mathcal{R}; \mathcal{G} \rangle \vdash \{P\}\mathsf{C}\{Q\}$, where Q describes the state once C has *terminated* (i.e. without crashing). However, as discussed in §2, the execution of C may crash (e.g. due to power loss), at which point the volatile and synchronous versions (e.g. x_{v} and x_{s}) are lost while the persistent versions (e.g. x_{p}) are preserved, and the execution is resumed by running a *recovery program*. As such, in anticipation of a possible crash at any point, we require the persistent parts of Q to be *invariant* throughout the execution.

In order to reason about programs in the presence of crashes, we present POG *recovery* triples of the form $\mathsf{C}_{\mathsf{rec}} \vdash \{P\}\mathsf{C}\{Q\}$, stating that every run of C from a state in P either: (1) terminates

successfully (without crashing) in a state in Q; or (2) crashes and its execution is resumed by repeatedly running the recovery program $\mathsf{C}_{\mathsf{rec}}$ until $\mathsf{C}_{\mathsf{rec}}$ terminates successfully in a state in Q. That is, the execution of $\mathsf{C}_{\mathsf{rec}}$ may itself crash and is rerun repeatedly until it terminates successfully. In other words, the volatile (and synchronous) values in the postcondition of POG triples describe the post-states *only if* C does not crash, whereas the volatile (and synchronous) values in the postcondition of recovery triples always describe the post-states after termination (either after C terminates successfully, or after C crashes and $\mathsf{C}_{\mathsf{rec}}$ is run repeatedly until it terminates successfully).

The recovery rule (REC) is given at the bottom of Fig. 3. The $\langle \top; \top \rangle \vdash \{P'\}\mathsf{C}\{Q'\}$ premise ensures that executing C from a state in P' either terminates successfully in a state in Q', or it crashes and the recovery $\mathsf{C}_{\mathsf{rec}}$ is run thereafter from a state in R, obtained from Q' by resetting register values and replacing volatile/synchronous versions with persistent ones ($Q' \Rightarrow \exists \overline{v}.\, R[\overline{v/\mathsf{a}}][\overline{x_{\mathsf{p}}/x_{\mathsf{s}}}][\overline{x_{\mathsf{p}}/x_{\mathsf{v}}}]$), as they are lost upon a crash. The existential quantification of $\overline{v}$ assigns (havocs) arbitrary values to local registers $\overline{a}$ upon recovery, ensuring that R makes no assumptions about the post-crash values of registers. The $\langle \top; \top \rangle \vdash \{R\}\mathsf{C}_{\mathsf{rec}}\{Q'\}$ in turn ensures that executing $\mathsf{C}_{\mathsf{rec}}$ from R either terminates successfully in Q', or it crashes and is rerun from R. Put together, as $P \Rightarrow P'$ and $Q' \Rightarrow Q$, this ensures that executing C (under $\mathsf{C}_{\mathsf{rec}}$) from P eventually terminates successfully in Q. Lastly, the RG contexts $\langle \top; \top \rangle$ ensure that C and $\mathsf{C}_{\mathsf{rec}}$ are run as closed programs (i.e. not in parallel with another program). In §4 we present an example of using (REC) to reason about recovery.

Fence Proof Rules. Note that our proof system in Fig. 3 does not include rules for **sfence** and **mfence**. For **sfence**, we can simply extend our rules with: $\langle \{P\}; \emptyset \rangle \vdash \{P\}$ **sfence** $\{P\}$. That is, **sfence** acts as a no-op in our Ix86_{sim} model and has the same specification as (SKIP). This is caused by two factors. First recall that as discussed in §2.3, under Ix86_{sim} a **flush** x instruction with $x \in X$ copies x'_{v} to x'_{s} for each $x' \in X$. That is, Ix86_{sim} *eliminates* each **flush** instruction on $x \in X = \{x^1 \cdots x^n\}$, and simply treats it as a series of writes on $x^1_{\mathsf{s}} \cdots x^n_{\mathsf{s}}$. Second, as noted by Raad et al. [2020], in the absence of **flush**/$\textbf{flush}_{\text{opt}}$ instructions, **sfence** instructions behave as no-ops (**skip**) and impose no additional ordering constraints. As such, since Ix86_{sim} eliminates **flush** instructions (and treats them as writes) and excludes $\textbf{flush}_{\text{opt}}$ instructions by design, **sfence** is a no-op under Ix86_{sim}. Nevertheless, as discussed in §2.1 and detailed later in in §7, **sfence** instructions can be used to enforce additional ordering constraints on $\textbf{flush}_{\text{opt}}$ instructions, allowing us in most cases to transform programs using $\textbf{flush}_{\text{opt}}$ to those using **flush** instead. We therefore opt to include **sfence** in the POG programming language to facilitate such transformations.

For **mfence**, we can derive reasoning principles by treating them as RMW instructions (as in Fig. 7 of [Lahav and Vafeiadis 2015]). More concretely, we can treat **mfence** instructions as RMWs (e.g. **FAA**) on a designated location f, which enforces a global order on all **mfence** instructions.

4 EXAMPLES

We use the POG proof rules to verify several representative examples.

Example 1. We begin with the concurrent example in Fig. 1e, with its proof sketch given in Fig. 4a. The RG contexts of the left and right threads are given respectively by $\langle \mathcal{R}_1; \mathcal{G}_1 \rangle$ and $\langle \mathcal{R}_2; \mathcal{G}_2 \rangle$, defined in Fig. 4a. As discussed in §3, the rely of each thread contains the assertions used in its proof outline (e.g. P_1 in $\mathcal{R}_1$), while the guarantee contains a guarded assignment for each write performed by the thread (e.g. $\langle P_2, y_{\mathsf{v}}, 1 \rangle$ in $\mathcal{G}_1$). The invariant we are interested in establishing is given by I. The first three conjuncts capture the chronological order between the different versions of x and z. The penultimate conjunct states that when z_{v}=1 (i.e. once the second thread executes $z := 1$ after having read 1 for y_{v} in a), then x_{s}=1 (i.e. **flush** x' must have already executed). The last conjunct is that of main interest, corresponding to the desired property in Fig. 1e. Note that the assertions at each program point are stable, and that $\langle \mathcal{R}_1; \mathcal{G}_1 \rangle$ and $\langle \mathcal{R}_2; \mathcal{G}_2 \rangle$ are non-interfering.

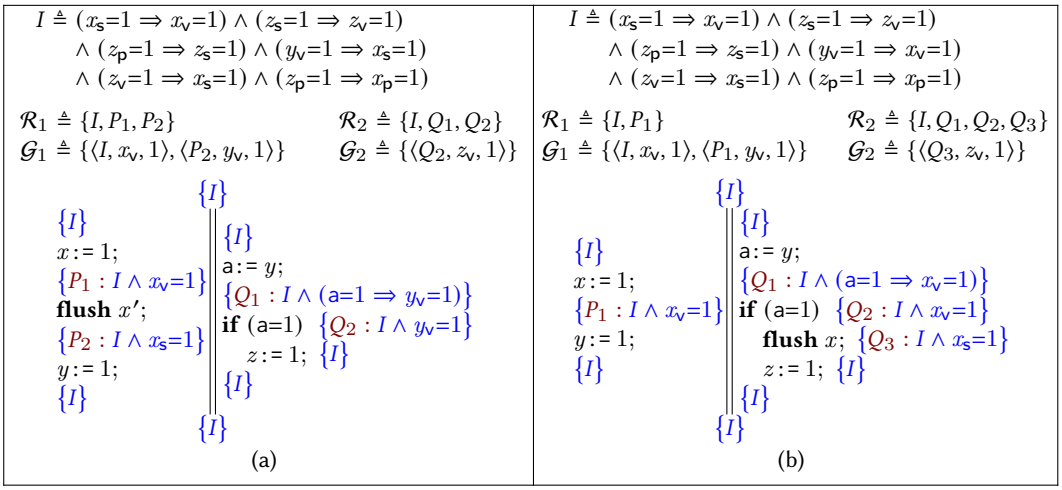

Fig. 4. Proof sketches of Example 1 (a) and Example 2 (b)

Example 2. We proceed with the example in Fig. 4b which is an adaptation of Fig. 4a with the **flush** moved to the right thread after reading from y. As such, the invariant I is similar to that of Fig. 4a, with the main difference lying in the fourth conjunct. In particular, when the left thread executes $y := 1$ yielding $y_v{=}1$, the earlier $x := 1$ has already executed, i.e. $x_v{=}1$. However, due to the absence of an intervening **flush** between the two writes, unlike in Fig. 4a we cannot assert $x_s{=}1$.

Example 3 (Atomic persists)**.** Our next example in Fig. 5a is inspired by the *persistent transactions* of Raad et al. [2019], where the authors showed how to ensure multiple stores on different locations (appear to) persist atomically before subsequent stores. Let us write τ_1 and τ_2 for the left and right threads, respectively. As shown in Fig. 5a , τ_1 writes 1 to x and y, and τ_2 writes 1 to z only if x contains 1, i.e. only if τ_1 has already executed $x := 1$. Our goal in this example is to ensure that the writes on x and y (by τ_1) both persist before the write on z (by τ_2). That is, we must ensure that the **flush** instructions of τ_1 are executed before τ_2 executes $z := 1$.

To this end, since τ_2 writes to z only after reading 1 from x (written by τ_1), we use a *lock* to control the accesses on x. More concretely, τ_1 acquires the lx lock on x at the beginning and releases it only after executing its **flush** instructions. Similarly, τ_2 acquires the lx lock prior to accessing x. As such, if τ_2 reads 1 for x (i.e. observes $x := 1$ by τ_1) and executes $z := 1$, then it must have acquired lx after it was released by τ_1, i.e. after τ_1 executed its **flush** instructions, as required.

The lx lock is acquired by calling **lock**, implemented as a spin lock in Fig. 5a: the implementation of **lock**(lx, a, v) loops until lx is free (i.e. $lx{=}0$), at which point it acquires it by atomically setting it to the non-zero value v. In order to distinguish which thread currently holds the lock, τ_1 and τ_2 respectively write the distinct values 1 and 2 to lx upon acquiring it. Lastly, the a argument denotes a thread-local register used as the loop flag by **lock**, and is passed on to **lock** by the calling thread.

We present a proof sketch of this program in Fig. 5a, where for brevity we have omitted the RG contexts. As before, our goal is to establish the I invariant, with its first, second and fourth conjuncts describing the chronological order on different versions of x, y and z. The third conjunct states that once τ_1 has finished executing, i.e. it has released the lock ($lx \in \{0, 2\}$) having written 1 to x ($x_v{=}1$), its flush instructions have also executed ($x_s{=}y_s{=}1$). Similarly, the penultimate conjunct states that once τ_2 has written 1 to z ($z_v{=}1$) having read 1 for x, i.e. after acquiring lx once it is released by τ_1 (see above), then the τ_1 **flush** instructions must have already executed ($x_s{=}y_s{=}1$).

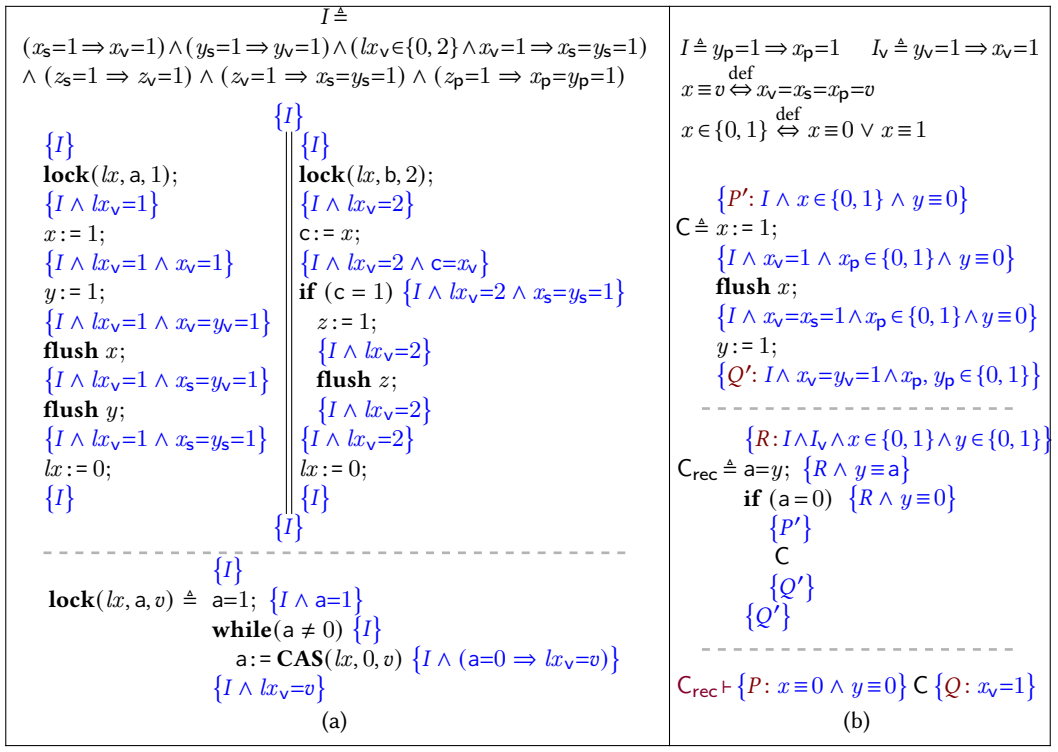

Fig. 5. Proof sketches of Example 3 (a) and Example 4 (b)

The last conjunct describes the desired persist ordering: if the write on z has persisted ($z_p{=}1$), then so must have both writes on x and y ($x_p{=}y_p{=}1$), as required.

Example 4 (Recovery)**.** In Fig. 5b we show how to use the (REC) rule to reason about recovery. The original program C is similar to that in Fig. 1b: first 1 is written to x and persisted by calling **flush**, and then 1 is written to y. As in Fig. 1b, the intervening **flush** ensures that $x := 1$ persists before $y := 1$ and thus $y_p{=}1 \Rightarrow x_p{=}1$ as described by I, assuming initially $x \equiv 0$ (i.e. $x_v{=}x_s{=}x_p{=}0$) and $y \equiv 0$.

As $I \triangleq y_p{=}1 \Rightarrow x_p{=}1$ is invariant throughout the execution (I only concerns persistent versions), the recovery C_{rec} treats the write on y as a flag to ascertain if $x := 1$ has persisted. That is, if y holds 1 upon recovery, then x must also hold 1 and thus C_{rec} simply returns; otherwise, C_{rec} reruns C.

We present a proof sketch of C and C_{rec} in Fig. 5b. We first establish $\langle \top; \top \rangle \vdash \{P'\}C\{Q'\}$ and $\langle \top; \top \rangle \vdash \{R\}C_{rec}\{Q'\}$, and then use (REC) to prove $C_{rec} \vdash \{P\}C\{Q\}$. Note that as mandated by the (REC) premise, we must show $P \Rightarrow P'$, $Q' \Rightarrow Q$, and $Q' \Rightarrow R[\overline{x_p/x_s}]\,[\overline{x_p/x_v}]$ which follow immediately.

5 THE Ix86$_{sim}$ OPERATIONAL SEMANTICS

We present the Ix86$_{sim}$ operational semantics discussed in §2.3. Recall that in Ix86$_{sim}$ a memory operation on x manipulates x_v; when $x \in X$, **flush** x copies x'_v to x'_s for $x' \in X$; x_v may be copied to x_s non-deterministically; and $\overline{x_s}$ may be non-deterministically copied to $\overline{x_p}$. To model this, we define a *translation function* that transforms Px86$_{sim}$ programs to access the instrumented memory.

Translation. Our translation function, $[\![.]\!]$, is defined in Fig. 6 and uses the auxiliary function, $(\!|.|\!)$, to translate sequential programs. As discussed, memory accesses on x are translated to access x_v; conditionals, loops and sequential composition are translated inductively; and **flush** x is translated

$(\!|\mathsf{a}:=\mathsf{e}|\!) \triangleq \mathsf{a}:=\mathsf{e}$ $\quad (\!|\mathsf{a}:=x|\!) \triangleq \mathsf{a}:=x_{\mathsf{v}}$ $\quad (\!|x:=\mathsf{e}|\!) \triangleq x_{\mathsf{v}}:=\mathsf{e}$ $\quad (\!|\mathsf{a}:=\mathbf{CAS}(x,\mathsf{e}_1,\mathsf{e}_2)|\!) \triangleq \mathsf{a}:=\mathbf{CAS}(x_{\mathsf{v}},\mathsf{e}_1,\mathsf{e}_2)$

$(\!|\mathsf{a}:=\mathbf{FAA}(x,\mathsf{e})|\!) \triangleq \mathsf{a}:=\mathbf{FAA}(x_{\mathsf{v}},\mathsf{e})$ $\quad (\!|\mathbf{sfence}|\!) \triangleq \mathbf{skip}$ $\quad (\!|\mathbf{mfence}|\!) \triangleq \mathbf{mfence}$ $\quad (\!|\mathbf{flush}\ x|\!) \triangleq \langle \mathbf{persist}\ X \rangle$

$(\!|\mathbf{if}\ (\mathsf{e})\ \mathbf{then}\ \mathsf{c}_1\ \mathbf{else}\ \mathsf{c}_2|\!) \triangleq \mathbf{if}\ (\mathsf{e})\ \mathbf{then}\ (\!|\mathsf{c}_1|\!)\ \mathbf{else}\ (\!|\mathsf{c}_2|\!)$ $\quad (\!|\mathbf{while}(\mathsf{e})\ \mathsf{c}|\!) \triangleq \mathbf{while}(\mathsf{e})\ (\!|\mathsf{c}|\!)$

$(\!|\mathsf{c}_1;\mathsf{c}_2|\!) \triangleq (\!|\mathsf{c}_1|\!);(\!|\mathsf{c}_2|\!)$ $\quad [\![\mathsf{c}_1 \,||\cdots||\, \mathsf{c}_n]\!] \triangleq (\!|\mathsf{c}_1|\!) \,||\cdots||\, (\!|\mathsf{c}_n|\!) \,||\, \mathsf{c}_{\mathsf{s}} \,||\, \mathsf{c}_{\mathsf{p}}$

with $\quad \mathsf{c}_{\mathsf{s}} \triangleq \mathbf{while}(*)\ \langle \mathbf{pick}\ x;\ x_{\mathsf{s}} := x_{\mathsf{v}} \rangle$ $\quad \mathsf{c}_{\mathsf{p}} \triangleq \mathbf{while}(*)\ \langle \overline{x_{\mathsf{p}} := x_{\mathsf{s}}}; \rangle$

Fig. 6. Ix86_{sim} program translation where we assume $x \in X$

to persist X (when $x \in X$) *atomically*, as indicated by $\langle . \rangle$. That is, in one computation step **persist** X reads the value of x_{v} and subsequently copies it to x_{s} for each $x \in X$, as we describe shortly.

The **mfence** instructions are left unchanged by the translation, while **sfence** instructions are translated as **skip**. This is because as discussed in §3, **sfence** behaves as a no-op in the absence of **flush**/$\mathbf{flush}_{\text{opt}}$. As such, since our translation eliminates **flush** and our language excludes $\mathbf{flush}_{\text{opt}}$, **sfence** is simply translated to **skip**. Lastly, the translation of a concurrent program is obtained from the point-wise translation of each thread, run in parallel with c_{s} and c_{p}. Intuitively, c_{s} models the non-deterministic propagation of x_{v} to x_{s} for an arbitrary x, carried out atomically. Analogously, c_{p} models the non-deterministic propagation of $\overline{x_{\mathsf{s}}}$ to $\overline{x_{\mathsf{p}}}$, for all locations. We write τ_{s} and τ_{p} for the threads executing c_{s} and c_{p}, respectively. Given a program C, we write C^{sp} for $\mathsf{C} \,||\, \mathsf{c}_{\mathsf{s}} \,||\, \mathsf{c}_{\mathsf{p}}$.

We next describe the Ix86_{sim} operational semantics by separating the transitions of its *program* and *storage* subsystems. The former describe the steps in program execution, e.g. how a conditional branch is triggered. The latter describe how the storage subsystem (the non-volatile memory and thread-local buffers in Fig. 2a) determine the execution steps. The Ix86_{sim} operational semantics is then defined by combining the transitions of its program and storage subsystems.

Program Transitions. The Ix86_{sim} program transitions are given at the top of Fig. 7 and are defined via the transitions of their constituent threads. Thread transitions are of the form: $\mathsf{c}, \mathsf{S} \xrightarrow{\tau:l} \mathsf{c}', \mathsf{S}'$, where $\mathsf{c}, \mathsf{c}' \in \textsc{SCom}$ denote translated sequential programs, and $\mathsf{S}, \mathsf{S}' \in \textsc{Stack}$ denote *stacks* mapping registers to values. The $\tau{:}l$ marks the transition by recording the executing thread τ, and the transition *label* l. A label may be ϵ for silent transitions of no-ops; $(\mathtt{R}, x, v)$ for reading v from x; $(\mathtt{W}, x, v)$ for writing v to x; $(\mathtt{U}, x, v, v')$ for a successful RMW (update) modifying the value of x to v' when its value matches v; $\mathtt{MF}$ for executing an **mfence**; $(\mathtt{FL}, X)$ for persisting the X cache line; $(\mathtt{S}, x_{\mathsf{s}}, v)$ for the atomic propagation of v to x_{s}; and $(\mathtt{P}, \overline{x_{\mathsf{p}}^i}, \overline{v^i})$ for the atomic propagation of $\overline{v^i}$ to $\overline{x_{\mathsf{p}}^i}$.

Given an expression e, we write $\mathsf{S}(\mathsf{e})$ for the value to which e evaluates under S; this definition is standard and omitted. Most program transitions are standard. The (P-MF) transition describes executing an **mfence**. The (P-CAS0) transition describes the unsuccessful execution of $\mathbf{CAS}(x, \mathsf{e}_1, \mathsf{e}_2)$; i.e. when the value read (v) is different from $\mathsf{S}(\mathsf{e}_1)$. The (P-CAS1) transition dually describes the successful execution of **CAS**. Note that in the failure case no update takes place and the transition is labelled with a read, and not an update as in the success case. The (P-FAA) transition behaves analogously. When executing **persist** X (i.e. a translated **flush** x with $x \in X$), the volatile-to-synchronous propagation of X is modelled by the $(\mathtt{FL}, X)$ transition in (P-AtomFL). The volatile-to-synchronous propagation of c_{s} is modelled by $(\mathtt{S}, x_{\mathsf{s}}, v)$ in (P-AtomS); *mutatis mutandis* for (P-AtomP).

Storage Transitions. The Ix86_{sim} storage transitions are given at the bottom of Fig. 7 and are of the form: $\mathsf{IM}, \mathsf{B} \xrightarrow{\tau:l} \mathsf{IM}', \mathsf{B}'$, where $\mathsf{IM}, \mathsf{IM}'$ denote the instrumented memory, and B, B' denote the *buffer map*, associating each thread with its *buffer*. Each buffer entry may be of the form: (1) $\langle x_{\mathsf{v}}, v \rangle$, denoting a pending write of value v on x_{v}; or (2) $X \subseteq \textsc{Loc}$, denoting a pending **flush** on cache line

Program transitions: $\textsc{Com} \times \textsc{Stack} \xrightarrow{\textsc{Tid}:\textsc{Lab}\cup\{\epsilon\}} \textsc{Com} \times \textsc{Stack}$ $\quad \mathsf{S} \in \textsc{Stack} \triangleq \textsc{Reg} \xrightarrow{\text{fin}} \textsc{Val}$

$\textsc{Lab} \triangleq \left\{ (\mathtt{R}, x_\mathsf{v}, v), (\mathtt{W}, x_\mathsf{v}, v), (\mathtt{U}, x_\mathsf{v}, v, v'), \mathtt{MF}, (\mathtt{FL}, X), (\mathtt{S}, x_\mathsf{s}, v), (\mathtt{P}, \overline{x_\mathsf{p}^i, v^i}) \;\middle|\; x, x^i \in \textsc{Loc} \wedge X \in \textsc{CL} \wedge v, v', v^i \in \textsc{Val} \right\}$

$$\frac{\mathsf{S}'{=}\mathsf{S}[\mathsf{a} \mapsto \mathsf{S}(\mathsf{e})]}{\mathsf{a} := \mathsf{e}, \mathsf{S} \xrightarrow{\tau:\epsilon} \mathbf{skip}, \mathsf{S}'} \text{(P-Assign)} \qquad \frac{\mathsf{S}'{=}\mathsf{S}[\mathsf{a} \mapsto v]}{\mathsf{a} := x_\mathsf{v}, \mathsf{S} \xrightarrow{\tau:(\mathtt{R}, x_\mathsf{v}, v)} \mathbf{skip}, \mathsf{S}'} \text{(P-Read)} \qquad \frac{}{\mathbf{mfence}, \mathsf{S} \xrightarrow{\tau:\mathtt{MF}} \mathbf{skip}, \mathsf{S}} \text{(P-MF)}$$

$$\frac{\mathsf{S}(\mathsf{e}){=}v}{x_\mathsf{v} := \mathsf{e}, \mathsf{S} \xrightarrow{\tau:(\mathtt{W}, x_\mathsf{v}, v)} \mathbf{skip}, \mathsf{S}} \text{(P-Write)} \qquad \frac{\mathsf{S}(\mathsf{e}_1) \neq v \quad \mathsf{S}'{=}\mathsf{S}[\mathsf{a} \mapsto v]}{\mathsf{a} := \mathbf{CAS}(x_\mathsf{v}, \mathsf{e}_1, \mathsf{e}_2), \mathsf{S} \xrightarrow{\tau:(\mathtt{R}, x_\mathsf{v}, v)} \mathbf{skip}, \mathsf{S}'} \text{(P-CAS0)}$$

$$\frac{\mathsf{S}(\mathsf{e}){=}v \quad \mathsf{S}'{=}\mathsf{S}[\mathsf{a} \mapsto v_0]}{\mathsf{a} := \mathbf{FAA}(x_\mathsf{v}, \mathsf{e}), \mathsf{S} \xrightarrow{\tau:(\mathtt{U}, x_\mathsf{v}, v_0, v_0+v)} \mathbf{skip}, \mathsf{S}'} \text{(P-FAA)} \qquad \frac{\mathsf{S}(\mathsf{e}_1){=}v_1 \quad \mathsf{S}(\mathsf{e}_2){=}v_2 \quad \mathsf{S}'{=}\mathsf{S}[\mathsf{a} \mapsto v_1]}{\mathsf{a} := \mathbf{CAS}(x_\mathsf{v}, \mathsf{e}_1, \mathsf{e}_2), \mathsf{S} \xrightarrow{\tau:(\mathtt{U}, x_\mathsf{v}, v_1, v_2)} \mathbf{skip}, \mathsf{S}'} \text{(P-CAS1)}$$

$$\frac{}{\langle \mathbf{persist}\ X \rangle, \mathsf{S} \xrightarrow{\tau:(\mathtt{FL}, X)} \mathbf{skip}, \mathsf{S}} \text{(P-AtomFL)} \qquad \frac{\mathsf{S}(\mathsf{e}){\neq}0 \Rightarrow \mathsf{c}{=}\mathsf{c}_1 \quad \mathsf{S}(\mathsf{e}){=}0 \Rightarrow \mathsf{c}{=}\mathsf{c}_2}{\mathbf{if}\ (\mathsf{e})\ \mathbf{then}\ \mathsf{c}_1\ \mathbf{else}\ \mathsf{c}_2, \mathsf{S} \xrightarrow{\tau:\epsilon} \mathsf{c}, \mathsf{S}} \text{(P-If)}$$

$$\frac{}{\mathbf{while}(\mathsf{e})\ \mathsf{c}, \mathsf{S} \xrightarrow{\tau:\epsilon} \mathbf{if}\ (\mathsf{e})\ \mathbf{then}\ \mathsf{c}; (\mathbf{while}(\mathsf{e})\ \mathsf{c})\ \mathbf{else}\ \mathbf{skip}, \mathsf{S}} \text{(P-While)} \qquad \frac{}{\mathbf{skip}; \mathsf{c}, \mathsf{S} \xrightarrow{\tau:\epsilon} \mathsf{c}, \mathsf{S}} \text{(P-Seq1)}$$

$$\frac{}{\langle \mathbf{pick}\ x; x_\mathsf{s} := x_\mathsf{v} \rangle, \mathsf{S} \xrightarrow{\tau_\mathsf{s}:(\mathtt{S}, x_\mathsf{s}, v)} \mathbf{skip}, \mathsf{S}} \text{(P-AtomS)} \qquad \frac{\mathsf{c}_1, \mathsf{S} \xrightarrow{\tau:l} \mathsf{c}_1', \mathsf{S}'}{\mathsf{c}_1; \mathsf{c}_2, \mathsf{S} \xrightarrow{\tau:l} \mathsf{c}_1'; \mathsf{c}_2, \mathsf{S}'} \text{(P-Seq2)}$$

$$\frac{}{\langle \overline{x_\mathsf{p}^i := x_\mathsf{s}^i} \rangle, \mathsf{S} \xrightarrow{\tau_\mathsf{p}:(\mathtt{P}, \overline{x_\mathsf{p}^i, v^i})} \mathbf{skip}, \mathsf{S}} \text{(P-AtomP)} \qquad \frac{\mathsf{C}(\tau), \mathsf{S} \xrightarrow{\tau:l} \mathsf{c}, \mathsf{S}'}{\mathsf{C}, \mathsf{S} \xrightarrow{\tau:l} \mathsf{C}[\tau \mapsto \mathsf{c}], \mathsf{S}'} \text{(P-Par)}$$

Storage transitions: $\textsc{IMem} \times \textsc{IBMap} \xrightarrow{\textsc{Tid}:\textsc{Lab}\cup\{\epsilon\}} \textsc{IMem} \times \textsc{IBMap}$ $\quad \mathsf{B} \in \textsc{IBMap} \triangleq \textsc{Tid} \xrightarrow{\text{fin}} \textsc{IBuff}$

$\mathsf{b} \in \textsc{IBuff} \triangleq \textsc{Seq}\,\langle (\textsc{ILoc} \times \textsc{Val}) \cup \mathcal{P}\,(\textsc{Loc}) \rangle$

$$\frac{\mathsf{B}(\tau){=}\mathsf{b}}{\mathsf{IM}, \mathsf{B} \xrightarrow{\tau:(\mathtt{W}, x_\mathsf{v}, v)} \mathsf{IM}, \mathsf{B}[\tau \mapsto \mathsf{b}.\langle x_\mathsf{v}, v \rangle]} \text{(M-Write)} \qquad \frac{\mathsf{B}(\tau){=}\epsilon \quad \mathsf{IM}(x_\mathsf{v}){=}v_1}{\mathsf{IM}, \mathsf{B} \xrightarrow{\tau:(\mathtt{U}, x_\mathsf{v}, v_1, v_2)} \mathsf{IM}[x_\mathsf{v} \mapsto v_2], \mathsf{B}} \text{(M-RMW)}$$

$$\frac{\mathsf{rd}(\mathsf{IM}, \mathsf{B}(\tau), x_\mathsf{v}){=}v}{\mathsf{IM}, \mathsf{B} \xrightarrow{\tau:(\mathtt{R}, x_\mathsf{v}, v)} \mathsf{IM}, \mathsf{B}} \text{(M-Read)} \qquad \frac{\mathsf{B}(\tau){=}\epsilon}{\mathsf{IM}, \mathsf{B} \xrightarrow{\tau:\mathtt{MF}} \mathsf{IM}, \mathsf{B}} \text{(M-MF)} \qquad \frac{\mathsf{IM}(x_\mathsf{v}){=}v}{\mathsf{IM}, \mathsf{B} \xrightarrow{\tau_\mathsf{s}:(\mathtt{S}, x_\mathsf{s}, v)} \mathsf{IM}[x_\mathsf{s} \mapsto v], \mathsf{B}} \text{(M-AtomS)}$$

$$\frac{\forall i.\, \mathsf{IM}(x_\mathsf{s}^i){=}v^i}{\mathsf{IM}, \mathsf{B} \xrightarrow{\tau_\mathsf{p}:(\mathtt{P}, \overline{x_\mathsf{p}^i, v^i})} \mathsf{IM}[\overline{x_\mathsf{p}^i \mapsto v^i}], \mathsf{B}} \text{(M-AtomP)} \qquad \frac{\mathsf{B}(\tau){=}\mathsf{b}}{\mathsf{IM}, \mathsf{B} \xrightarrow{\tau:(\mathtt{FL}, X)} \mathsf{IM}, \mathsf{B}[\tau \mapsto \mathsf{b}.X]} \text{(M-AtomFL)}$$

$$\frac{\mathsf{B}(\tau){=}\langle x_\mathsf{v}, v \rangle.\mathsf{b}}{\mathsf{IM}, \mathsf{B} \xrightarrow{\tau:\epsilon} \mathsf{IM}[x_\mathsf{v} \mapsto v], \mathsf{B}[\tau \mapsto \mathsf{b}]} \text{(M-PropW)} \qquad \frac{\mathsf{B}(\tau){=}X.\mathsf{b} \qquad X{=}\overline{x^i} \qquad \forall i.\, \mathsf{IM}(x_\mathsf{v}^i){=}v^i}{\mathsf{IM}, \mathsf{B} \xrightarrow{\tau:\epsilon} \mathsf{IM}[\overline{x_\mathsf{s}^i \mapsto v^i}], \mathsf{B}[\tau \mapsto \mathsf{b}]} \text{(M-PropFL)}$$

with $\quad \mathsf{rd}(\mathsf{IM}, \mathsf{b}, x_\mathsf{v}) \triangleq \begin{cases} v & \text{if } \exists \mathsf{b}_1, \mathsf{b}_2.\ \mathsf{b}{=}\mathsf{b}_1.\langle x_\mathsf{v}, v \rangle.\mathsf{b}_2 \wedge \forall v'.\ \langle x_\mathsf{v}, v' \rangle \notin \mathsf{b}_2 \\ \mathsf{IM}(x_\mathsf{v}) & \text{otherwise} \end{cases}$

Fig. 7. The Ix86_{sim} program transitions (above); the Ix86_{sim} storage transitions (below)

X. When a thread writes v to x_v, this is recorded in its buffer as the $\langle x_\mathsf{v}, v \rangle$ entry, as described by (M-Write). Similarly, when a thread makes a $(\mathtt{FL}, X)$ transition (i.e. executes **persist** X translated from **flush** x with $x \in X$), this is recorded in its buffer as X, as shown in (M-AtomFL). Recall that

$$\textsc{Com} \vdash \textsc{Conf} \xrightarrow{\textsc{Tid}\times\textsc{Lab}\cup\{\epsilon,\lightning\}} \textsc{Conf} \qquad \textsc{Conf} \triangleq \textsc{Com} \times \textsc{Stack} \times \textsc{IMem} \times \textsc{IBMap}$$

$$\frac{\mathsf{C},\mathsf{S} \xrightarrow{\tau:\epsilon} \mathsf{C}',\mathsf{S}'}{\Delta \vdash \mathsf{C},\mathsf{S},\mathsf{IM},\mathsf{B} \xRightarrow{\tau:\epsilon} \mathsf{C}',\mathsf{S}',\mathsf{IM},\mathsf{B}}\ (\textsc{SilentP}) \qquad \frac{\mathsf{C},\mathsf{S} \xrightarrow{\tau:l} \mathsf{C}',\mathsf{S}' \quad \mathsf{IM},\mathsf{B} \xrightarrow{\tau:l} \mathsf{IM}',\mathsf{B}'}{\Delta \vdash \mathsf{C},\mathsf{S},\mathsf{IM},\mathsf{B} \xRightarrow{\tau:l} \mathsf{C}',\mathsf{S}',\mathsf{IM}',\mathsf{B}'}\ (\textsc{Step})$$

$$\frac{\mathsf{IM},\mathsf{B} \xrightarrow{\tau:\epsilon} \mathsf{IM}',\mathsf{B}'}{\Delta \vdash \mathsf{C},\mathsf{S},\mathsf{IM},\mathsf{B} \xRightarrow{\tau:\epsilon} \mathsf{C},\mathsf{S},\mathsf{IM}',\mathsf{B}'}\ (\textsc{SilentS}) \qquad \frac{\mathsf{B}_0 \triangleq \lambda\tau.\epsilon \quad \mathsf{IM}'{=}\mathsf{IM}[\overline{x_\mathsf{s} \mapsto \mathsf{IM}(x_\mathsf{p})}][\overline{x_\mathsf{v} \mapsto \mathsf{IM}(x_\mathsf{p})}]}{\mathsf{C}_\mathsf{rec} \vdash \mathsf{C},\mathsf{S},\mathsf{IM},\mathsf{B} \xRightarrow{\lightning} \mathsf{C}_\mathsf{rec},\mathsf{S}_0,\mathsf{IM}',\mathsf{B}_0}\ (\textsc{Crash})$$

Fig. 8. The Ix86_{sim} operational semantics

when a thread reads from x_v, it first consults its own buffer, followed by the memory (if no write to x_v is found in the buffer). This lookup chain is captured by $\mathsf{rd}(\mathsf{IM},\mathsf{b},x_\mathsf{v})$ in the premise of (M-Read).

The (M-MF) rule ensures that an **mfence** proceeds only when the buffer of the executing thread is empty, as stipulated by the $\mathsf{B}(\tau){=}\epsilon$ premise. In the (M-RMW) rule, when executing an RMW instruction on x_v, a similar lookup chain is followed to determine the value of x_v, as with a read. To ensure their atomicity, RMW instructions may only proceed when the buffer of the executing thread is drained. Moreover, the resulting update is committed directly to the memory, bypassing the thread buffer. This is to ensure that the resulting update is immediately visible to other threads.

As with (P-AtomS), (M-AtomS) describes the non-deterministic copying of x_v to x_s, with the result written directly to memory, provided that the buffer of the executing thread (i.e. τ_s) is empty; (P-AtomP) behaves analogously. Lastly, (M-PropW) describes the debuffering of a pending write in a thread-local buffer and propagating it to memory. Similarly, (M-PropFL) describes the debuffering of a pending flush on X, where the value of x^i_v (in $\mathsf{IM}(x^i_\mathsf{v})$) is propagated to $\mathsf{IM}(x^i_\mathsf{s})$, for each $x^i \in X$.

Combined Transitions. The Ix86_{sim} operational semantics in Fig. 8 is defined by combining the program and storage transitions, under a *recovery program*, C_rec, run after a crash. The (SilentP) rule describes the case when the program subsystem takes a silent step and thus the storage subsystem is left unchanged; similarly for (SilentS). The (Step) rule describes the case when the program and storage subsystems both take the *same* transition (with the same label) and thus the transition effect is that of their combined effects. Lastly, the (Crash) rule describes the case when the program crashes: the registers (in S) and the thread-local buffers are lost (as they are volatile), and are thus reset; the memory is left largely unchanged (as it is non-volatile); and the execution is restarted with the recovery program. Note that upon a crash the persistent versions in the resulting memory (IM') remain unchanged, while the synchronous and volatile versions are lost and are simply overwritten by the persistent versions. This is because if $x_\mathsf{v} \neq x_\mathsf{p}$ (resp. $x_\mathsf{s} \neq x_\mathsf{p}$) upon a crash, then intuitively the write responsible for the current value of x_v (resp. x_s) has not yet reached the persistent memory and is thus lost after recovery.

Ix86_{sim} Subsumes Px86_{sim}. In Thm. 1 below we show that our Ix86_{sim} model is weaker than Px86_{sim} and subsumes all its behaviours. This then allows us to establish the soundness of POG over the simpler Ix86_{sim} model. To prove that Ix86_{sim} subsumes Px86_{sim}, we show that for all non-instrumented memories M produced by a Px86_{sim} trace, there exists an instrumented memory IM' produced by an Ix86_{sim} trace such that M and IM agree on the (persisted) values of all locations.

Theorem 1. *For all memories* M *produced by a* Px86_{sim} *trace, there exists an instrumented memory* IM' *produced by an* Ix86_{sim} *trace such that:* $\forall x.\ \mathsf{M}(x){=}\mathsf{IM}(x_\mathsf{p})$.

6 POG SOUNDNESS

We prove that the POG proof rules in Fig. 3 are sound. We proceed with several auxiliary definitions.

Assertion Semantics. We define the set of *states* as $\textsc{State} \triangleq \textsc{IMem} \times \textsc{Stack}$, and use σ as a metavariable for states. Assertions are interpreted as sets of states as expected: the instrumented memory provides the interpretation of the $x_{\mathsf{v}}, x_{\mathsf{s}}, x_{\mathsf{p}}$ constants, and the stack provides the interpretation of the a constants. In what follows we write $[\![P]\!]$ for $\{\sigma \mid \sigma \models P\}$.

$\mathcal{R}/\mathcal{G}$ Semantics. $\mathcal{R}/\mathcal{G}$ components are interpreted as relations on states. The guarantee $\mathcal{G}$ is interpreted point-wise as the smallest preorder that admits all constituent guarded assignments. That is, if $\langle x_{\mathsf{v}}, \mathsf{e}, P\rangle \in \mathcal{G}$ and $(\mathsf{IM}, \mathsf{S}) \in [\![P]\!]$, then $((\mathsf{IM}, \mathsf{S}), (\mathsf{IM}', \mathsf{S})) \in [\![\mathcal{G}]\!]^{\mathsf{g}}$, when $\mathsf{IM}'{=}\mathsf{IM}[x_{\mathsf{v}} \mapsto \mathsf{S}(\mathsf{e})]$:

$$[\![\emptyset]\!]^{\mathsf{g}} \triangleq \mathsf{id} \qquad [\![\mathcal{G} \cup \{\langle x_{\mathsf{v}}, \mathsf{e}, P\rangle\}]\!]^{\mathsf{g}} \triangleq \left([\![\mathcal{G}]\!]^{\mathsf{g}} \cup \left\{\left((\mathsf{IM}, \mathsf{S}), (\mathsf{IM}[x_{\mathsf{v}} \mapsto \mathsf{S}(\mathsf{e})], \mathsf{S})\right) \mid (\mathsf{IM}, \mathsf{S}) \models P\right\}\right)^*$$

Dually, the rely $\mathcal{R}$ is interpreted point-wise as the largest relation on states that preserves the stability of all constituent assertions. That is, if $P \in \mathcal{R}$, $\sigma \in [\![P]\!]$ and $(\sigma, \sigma') \in [\![\mathcal{R}]\!]^{\mathsf{r}}$, then $\sigma' \in [\![P]\!]$:

$$[\![\emptyset]\!]^{\mathsf{r}} \triangleq \textsc{State} \times \textsc{State} \qquad [\![\mathcal{R} \cup \{P\}]\!]^{\mathsf{r}} \triangleq [\![\mathcal{R}]\!]^{\mathsf{r}} \cap \{(\sigma, \sigma') \mid \sigma \models P \Rightarrow \sigma' \models P\}$$

Configuration Safety. We define the semantics of POG triples via an auxiliary predicate, $\mathsf{safe}_n(\mathsf{C}^{\mathsf{sp}}, \sigma, R, G, Q, I)$, stating that executing the translated program C^{sp} over the σ state is safe with respect to the interpreted R, G relations, post-states Q and invariant I for up to n steps.

Definition 3 (Configuration safety)**.** For all C, IM, S and $R, G \subseteq \textsc{State} \times \textsc{State}$, $Q \subseteq \textsc{State}$: $\mathsf{safe}_0(\mathsf{C}^{\mathsf{sp}}, (\mathsf{IM}, \mathsf{S}), R, G, Q, I)$ always holds; and $\mathsf{safe}_{n+1}(\mathsf{C}^{\mathsf{sp}}, (\mathsf{IM}, \mathsf{S}), R, G, Q, I)$ holds iff:

(1) $(\mathsf{IM}, \mathsf{S}) \in I$ and (2) if $\mathsf{C}{=}\mathsf{C}_{\mathbf{skip}}$, then $(\mathsf{IM}, \mathsf{S}) \in Q$, where $\mathsf{C}_{\mathbf{skip}} \triangleq \lambda\tau.\mathbf{skip}$

(3) for all σ: if $((\mathsf{IM}, \mathsf{S}), \sigma) \in R \cup \mathsf{A}_{\mathsf{sp}}$, then $\mathsf{safe}_n(\mathsf{C}^{\mathsf{sp}}, \sigma, R, G, Q, I)$

(4) for all $\tau, l, \mathsf{IM}_1, \mathsf{IM}_2, \mathsf{B}_1, \mathsf{B}_2, \mathsf{C}', \mathsf{IM}', \mathsf{S}', l{\neq}\not{l}$:

$$\text{if } \mathsf{C}, \mathsf{IM}_1, \mathsf{B}_1, \mathsf{S} \xRightarrow{\tau:l} \mathsf{C}', \mathsf{IM}_2, \mathsf{B}_2, \mathsf{S}' \wedge \mathsf{IM}{=}\Downarrow(\mathsf{IM}_1, \mathsf{B}_1, \tau) \wedge \mathsf{IM}'{=}\Downarrow(\mathsf{IM}_2, \mathsf{B}_2, \tau)$$
$$\text{then } ((\mathsf{IM}, \mathsf{S}), (\mathsf{IM}', \mathsf{S})) \in G \cup \mathsf{A}_{\mathsf{sp}} \wedge \mathsf{safe}_n((\mathsf{C}')^{\mathsf{sp}}, (\mathsf{IM}', \mathsf{S}'), R, G, Q, I)$$

$$\text{where } \Downarrow(\mathsf{IM}, \mathsf{B}, \tau){=}\mathsf{IM}' \overset{\text{def}}{\Leftrightarrow} (\mathsf{IM}, \mathsf{B}) \xrightarrow{\tau:\epsilon}{}^{*} \mathsf{IM}', \mathsf{B}' \wedge \mathsf{B}'(\tau){=}\epsilon$$
$$\text{and } \mathsf{A}_{\mathsf{sp}} \triangleq \left\{\left((\mathsf{IM}, \mathsf{S}), (\mathsf{IM}[\overline{x_{\mathsf{p}} \mapsto \mathsf{IM}(x_{\mathsf{s}})}], \mathsf{S})\right), \left((\mathsf{IM}, \mathsf{S}), (\mathsf{IM}[y_{\mathsf{s}} \mapsto y_{\mathsf{p}}], \mathsf{S})\right) \mid y \in \mathrm{Loc}\right\}^*$$

Recall that a translated program (via $[\![.]\!]$) is of the form $\mathsf{C}^{\mathsf{sp}} \triangleq \mathsf{C} \,||\, \mathsf{c}_{\mathsf{s}} \,||\, \mathsf{c}_{\mathsf{p}}$. A configuration is always safe for zero steps. For $n{+}1$ steps, a configuration is safe if: (1) $(\mathsf{IM}, \mathsf{S})$ is a state in the invariant I; (2) whenever the program has finished execution (i.e. $\mathsf{C}{=}\mathsf{C}_{\mathbf{skip}}$), then $(\mathsf{IM}, \mathsf{S})$ must be a post-state in Q; (3) whenever the environment changes the state according to the rely (in R) or performs a (volatile-to-synchronous or synchronous-to-persistent) propagation (in A_{sp}), then the resulting configuration remains safe for a further n steps; and (4) whenever the thread performs an $\mathrm{Ix86}_{\mathrm{sim}}$ transition, its changes to the state are either those permitted by the guarantee (in G) or those of propagation (in A_{sp}), and the new configuration remains safe for n more steps.

Note that the $\mathrm{Ix86}_{\mathrm{sim}}$ transitions (Fig. 8) are over an instrumented memory and a buffer map. As such, the memory IM in the pre-state of the thread τ describes several storage pairs of the form $(\mathsf{IM}_1, \mathsf{B}_1)$ such that $\Downarrow(\mathsf{IM}_1, \mathsf{B}_1, \tau){=}\mathsf{IM}$. That is, once the pending entries of τ in $\mathsf{B}_1(\tau)$ are propagated to IM_1 (via $\xrightarrow{\tau:\epsilon}$ storage transitions in rule (M-PropW) of Fig. 8), the resulting memory corresponds to IM. Intuitively, IM denotes the *view* of τ of the storage subsystem, in that τ observes its pending writes and flushes in $\mathsf{B}_1(\tau)$, even though they have not yet reached the memory.

We next define *valid POG judgements*, and show that POG triples (Fig. 3) yield valid judgements. Note that the invariant of a triple, Q_{p}, is obtained from the postcondition Q by erasing the values of registers as well as the volatile/synchronous versions, and replacing them with arbitrary values.

Definition 4. A *judgement* $\langle \mathcal{R}; \mathcal{G} \rangle \Vdash \{P\}\ \mathsf{C}\ \{Q\}$ is valid iff for all $n, \sigma \in [\![P]\!]$, $\mathsf{safe}_n([\![\mathsf{C}]\!], \sigma, [\![\mathcal{R}]\!]^{\mathsf{r}}, [\![\mathcal{G}]\!]^{\mathsf{g}}, [\![Q]\!], [\![Q_{\mathsf{p}}]\!])$ holds with $Q_{\mathsf{p}} \triangleq \exists \overline{v_a, v_{\mathsf{v}}, v_{\mathsf{s}}}.\, Q[\overline{v_a/\mathsf{a}}][\overline{v_{\mathsf{v}}/x_{\mathsf{v}}}][\overline{v_{\mathsf{s}}/x_{\mathsf{s}}}]$. A judgement $\mathsf{C}_{\mathsf{rec}} \Vdash \{P\}\mathsf{C}\{Q\}$ is valid iff for all $(\mathsf{IM}, \mathsf{S}) \in [\![P]\!]$, $\mathsf{IM}', \mathsf{S}'$, if $\mathsf{C}_{\mathsf{rec}} \vdash \mathsf{C}, \mathsf{S}, \mathsf{IM}, \mathsf{B}_0 \Rightarrow^* \mathsf{C}_{\mathbf{skip}}, \mathsf{S}', \mathsf{IM}', \mathsf{B}_0$, then $(\mathsf{IM}', \mathsf{S}') \in [\![Q]\!]$.

Theorem 2 (POG soundness). *For all POG triples* $\langle \mathcal{R}; \mathcal{G} \rangle \vdash \{P\}\mathsf{C}\{Q\}$ *and recovery triples* $\mathsf{C}_{\mathsf{rec}} \vdash \{P\}\ \mathsf{C}\{Q\}$ *in Fig. 3, the POG judgements* $\langle \mathcal{R}; \mathcal{G} \rangle \Vdash \{P\}\ \mathsf{C}\ \{Q\}$ *and* $\mathsf{C}_{\mathsf{rec}} \Vdash \{P\}\ \mathsf{C}\ \{Q\}$ *are valid.*

Finally, we establish the following *adequacy* theorem showing that POG is sound with respect to $\mathrm{Px86}_{\mathrm{sim}}$. Note that it is sufficient to consider *closed* triples with RG contexts $\langle \top; \top \rangle$, i.e. whole programs (not run in parallel with another program) under an empty environment.

Theorem 3 (Adequacy). *For all* $\langle \top; \top \rangle \vdash \{P\}\mathsf{C}\{Q\}$, $(\mathsf{IM}, \mathsf{S}) \models P$, *and* $\mathsf{M}, \mathsf{M}', \mathsf{S}'$, *if* IM *and* M *agree (i.e.* $\forall x.\ \mathsf{M}(x){=}\mathsf{IM}(x_{\mathsf{v}}) = \mathsf{IM}(x_{\mathsf{s}}){=}\mathsf{IM}(x_{\mathsf{p}})$*) and starting from* (M, S) *with empty (thread-local and persistent) buffers, program* C *runs to completion under Px86$_{sim}$ and yields* $(\mathsf{M}', \mathsf{S}')$ *and empty buffers, then there exists* IM' *such that* IM' *and* M' *agree and* $(\mathsf{IM}', \mathsf{S}') \models Q$.

7 A TWO-STEP TRANSFORMATION FOR ELIMINATING $\mathbf{flush}_{\mathbf{opt}}$ INSTRUCTIONS

We describe the most common use-case of $\mathbf{flush}_{\mathrm{opt}}$ instructions, *epoch persistency*, where the use of $\mathbf{flush}_{\mathrm{opt}}$ (rather than **flush**) may prove advantageous for performance. We observe that programs using $\mathbf{flush}_{\mathrm{opt}}$ for epoch persistency follow a certain pattern, and describe how we transform such programs to ones that use **flush** instead, without altering their persistency behaviour.

Epoch Persistency using $\mathbf{flush}_{\mathbf{opt}}$. Recall from §2 that $\mathbf{flush}_{\mathrm{opt}}$ instructions provide weaker ordering constraints and can be reordered e.g. with respect to writes on different cache lines. As such, in order to mitigate the weak ordering constraints on $\mathbf{flush}_{\mathrm{opt}}$ and to ensure their execution by a particular program point, they are typically followed by an **sfence**/**mfence**/RMW in program text; see e.g. Fig. 1d. Indeed, Fig. 1d is an example of *epoch persistency*. More concretely, by combining $\mathbf{flush}_{\mathrm{opt}}$ and **sfence**/**mfence**/RMW instructions, one can divide an execution into distinct *epochs*, where the writes in each epoch may persist in an arbitrary order, while the writes of earlier (in program order) epochs persist before those in later epochs. This is illustrated in Fig. 9a, where $x := 1$ and $y := 1$ both persist before $z := 1$, whereas $x := 1$ and $y := 1$ themselves may persist in either order; i.e. the write on x, y persist in the first epoch before the write on z in the second epoch.

Let L and L' denote locations to be persisted in two consecutive epochs; one can then enforce epoch persistency by following the pattern below in three steps:

(1) executing the writes on L and their corresponding $\mathbf{flush}_{\mathrm{opt}}$ for each cache line, provided that each $\mathbf{flush}_{\mathrm{opt}}$ on a cache line X follows the writes in L on X;
(2) executing an *epoch barrier*, namely an **sfence**, **mfence** or RMW; and (Epoch)
(3) executing the writes on L'.

The combination of $\mathbf{flush}_{\mathrm{opt}}$ instructions in (1) and the epoch barrier in (2) ensures that the writes on L in (1) persist before those on L' in (3) (e.g. $L{=}\{x, y\}$ and $L'{=}\{z\}$ in Fig. 9a, assuming that x and y are in distinct cache lines). Note that in step (1) it is sufficient to execute one $\mathbf{flush}_{\mathrm{opt}}$ per *cache line* as $\mathbf{flush}_{\mathrm{opt}}$ persists all (earlier) pending writes on the same cache line. For instance, if $y := 1$ in Fig. 9a is replaced with $x' := 1$ (where x and x' are on the same cache line), step (1) may simply comprise $x := 1; x' := 1; \mathbf{flush}_{\mathrm{opt}}\ x$ (without a separate $\mathbf{flush}_{\mathrm{opt}}\ x'$). This epoch persistency pattern is used by e.g. Raad et al. [2020] to implement several persistent libraries.

Note that replacing each $\mathbf{flush}_{\mathrm{opt}}$ in Fig. 9a with **flush** also achieves epoch persistency, albeit at a finer granularity (one write per epoch). This is because each **flush** is ordered with respect to all writes and thus introduces a new epoch. Using $\mathbf{flush}_{\mathrm{opt}}$ thus admits more than one write per epoch, and may improve performance as it allows the writes in the same epoch to persist in any order.

$\mathsf{c}_1 \triangleq$ $x := 1;$ **flush**$_{\text{opt}}$ $x;$ $y := 1;$ **flush**$_{\text{opt}}$ $y;$ **sfence**; $z := 1;$	$\mathsf{c}_2 \triangleq$ $x := 1;$ $y := 1;$ **flush**$_{\text{opt}}$ $x;$ **flush**$_{\text{opt}}$ $y;$ **sfence**; $z := 1;$	$\mathsf{c}_3 \triangleq$ $x := 1;$ $y := 1;$ **flush** $x;$ **flush** $y;$ **sfence**; $z := 1;$	$\mathsf{c}_i \;\Big\|\; \mathsf{a} := z;$ **if** (a=1) $w := 1;$
(a)	(b)	(c)	(d)
$\lightning : z{=}1 \Rightarrow x{=}y{=}1$	$\lightning : z{=}1 \Rightarrow x{=}y{=}1$	$\lightning : z{=}1 \Rightarrow x{=}y{=}1$	$\lightning : w{=}1 \Rightarrow x{=}y{=}1$

Fig. 9. Epoch persistency using **flush**$_{\text{opt}}$ (a); the program obtained from c_1 by reordering **flush**$_{\text{opt}}$ x (b); the program obtained from c_2 by converting **flush**$_{\text{opt}}$ to **flush** (c); a common concurrent use-case of **flush**$_{\text{opt}}$ (d), where $\mathsf{c}_i \in \{\mathsf{c}_1, \mathsf{c}_2, \mathsf{c}_3\}$. The c_1, c_2, c_3 programs all have the same persistency behaviour (observing the same values upon recovery), and can be used interchangeably in (d) without changing the persistency behaviour.

Two-Step Transformation. Note that as **flush**$_{\text{opt}}$ instructions are not ordered with respect to writes on different cache lines, each **flush**$_{\text{opt}}$ in step (1) can be reordered after the writes on different cache lines *without changing the persistency behaviour*. For instance, **flush**$_{\text{opt}}$ x in c_1 of Fig. 9a can be reordered past the $y := 1$ write to obtain c_2 in Fig. 9b, where c_1 and c_2 have equivalent persistency behaviours. As such, step (1) of the pattern above can further be split as: (1.a) executing the writes on L; (1.b) executing **flush**$_{\text{opt}}$ for each L cache line.

Indeed, this intuition informs the first step of our transformation towards eliminating **flush**$_{\text{opt}}$: given a program of the form $\mathsf{C} \triangleq \mathsf{c}_a; \textbf{flush}_{\text{opt}}\ x; \mathsf{c}_b; \mathsf{c}$ with $x \in X$, if c is the *first* epoch barrier following **flush**$_{\text{opt}}$ x (i.e. c_b contains no **sfence**/**mfence**/RMW), then one can rewrite C as $\mathsf{C}' \triangleq \mathsf{c}_a; \mathsf{c}_b; \textbf{flush}_{\text{opt}}\ x; \mathsf{c}$ without changing its persistency behaviour (i.e. C and C' yield the same values for all locations upon recovery), *provided that* c_b contains no writes on X and no read instructions. Note that $\mathsf{c}_b \triangleq y := 1$ in Fig. 9a simply meets the stipulated provisos. We shortly elaborate on these provisos on c_b in §7.1, and note that it is always possible to achieve epoch persistency in such a way that fulfils these conditions. Note that one can repeatedly apply this transformation to push down all **flush**$_{\text{opt}}$ in step (1) just before the epoch barrier in (2), thus splitting (1) as (1.a) and (1.b).

Next, we observe that once all **flush**$_{\text{opt}}$ have been pushed down before the epoch barrier, one can almost always replace each **flush**$_{\text{opt}}$ with a corresponding **flush** without altering its persistency behaviour. This is thanks to the strong ordering between each **flush**$_{\text{opt}}$ and the subsequent epoch barrier. For instance, c_2 in Fig. 9b can be transformed to c_3 in Fig. 9c, while leaving its persistency behaviour unchanged. This rewriting constitutes the second and last step of our transformation for eliminating **flush**$_{\text{opt}}$ instructions. In §7.2 we elaborate on the only scenarios under which this rewriting may alter the persistency behaviour, and note that such scenarios do not arise realistically.

Finally, note that this two-step transformation can be used to eliminate **flush**$_{\text{opt}}$ instructions in concurrent programs by applying the transformation to each thread containing **flush**$_{\text{opt}}$. This is illustrated in Fig. 9d, where e.g. c_1 in the left thread can be first transformed to c_2 and subsequently to c_3, while preserving the persistency behaviour of the concurrent program at each step.

7.1 Caveats of Reordering flush$_{\text{opt}}$ after Later Instructions (Transformation Step 1)

Recall that in the first step of our transformation we push down a **flush**$_{\text{opt}}$ x instruction with $x \in X$ just before the epoch barrier, provided that there are *no writes on* X and *no reads* between **flush**$_{\text{opt}}$ x and the epoch barrier, as otherwise this transformation alters the persistency behaviour.

In the case of writes on X, consider the example in Fig. 10a where $x, x' \in X$ and the $x' := 1$ write is between **flush**$_{\text{opt}}$ x and **sfence**. Fig. 10b depicts the program obtained from Fig. 10a by reordering **flush**$_{\text{opt}}$ x after $x' := 1$. Recall that executing **flush**$_{\text{opt}}$ x persists *all earlier* writes on X; as such

$x := 1;$ $\mathbf{flush}_{\mathrm{opt}}\ x;$ $x' := 1;$ $\mathbf{sfence};$ $z := 1;$ (a)	$x := 1;$ $x' := 1;$ $\mathbf{flush}_{\mathrm{opt}}\ x;$ $\mathbf{sfence};$ $z := 1;$ (b)	$x := 1;$ $\mathbf{flush}_{\mathrm{opt}}\ x;$ $\mathbf{sfence};$ $x' := 1;$ $z := 1;$ (c)
$\lightning : z{=}1 \Rightarrow x{=}1 \qquad z{=}1 \wedge x'{=}0$ ✓	$\lightning : z{=}1 \Rightarrow x{=}x'{=}1 \qquad z{=}1 \wedge x'{=}0$ ✗	$\lightning : z{=}1 \Rightarrow x{=}1 \qquad z{=}1 \wedge x'{=}0$ ✓

Fig. 10. An epoch persistency example not following the (Epoch) pattern (a); the program obtained from (a) by reordering $\mathbf{flush}_{\mathrm{opt}}\ x$ after $x' := 1$ (where $x, x' \in X$), thus altering its persistency behaviour (b); a program with equivalent persistency behaviour to (a) that follows the (Epoch) pattern (c).

$\mathbf{flush}_{\mathrm{opt}}\ x$ in Fig. 10b is guaranteed to persist both $x := 1$ and $x' := 1$ (i.e. $z{=}1 \Rightarrow x{=}x'{=}1$ upon recovery), while $\mathbf{flush}_{\mathrm{opt}}\ x$ in Fig. 10a is guaranteed to persist only $x := 1$ (i.e. $z{=}1 \Rightarrow x{=}1$ upon recovery). The two programs may therefore have different persistency behaviours: it is possible to observe $z{=}1 \wedge x'{=}0$ after recovery in Fig. 10a but not in Fig. 10b.

Note that Fig. 10a does not adhere to (Epoch): either $x := 1$ and $x' := 1$ are to persist in the same epoch and the program should have been rewritten as in Fig. 10b, or they are to persist in separate epochs in which case the program could have been rewritten as in Fig. 10c. That is, the intended persistency behaviour of Fig. 10a is ambiguous, and it is better practice to rewrite it as either Fig. 10b or Fig. 10c, both of which adhere to (Epoch). We thus argue that it is possible to achieve epoch persistency *without* an intervening write on X between $\mathbf{flush}_{\mathrm{opt}}\ x$ and its epoch barrier. As such, it is possible to apply our first transformation step while preserving the persistency behaviour. Observe that the first transformation step in both Fig. 10b and Fig. 10c is idempotent ($\mathbf{flush}_{\mathrm{opt}}\ x$ is already before **sfence**) and thus trivially preserves the persistency behaviour.

In the case of reads, consider the example in Fig. 11a where the $\mathtt{a} := x$ read is between $\mathbf{flush}_{\mathrm{opt}}\ x$ and **sfence**. Fig. 11b depicts the program obtained from Fig. 11a by reordering $\mathbf{flush}_{\mathrm{opt}}\ x$ after $\mathtt{a} := x$. Note that the right thread executes $y := 1$ only when $\mathtt{a} := x$ reads 2 from x, which in turn implies that $x := 2$ in the left thread is store-ordered after $x := 1$ in the right (otherwise $\mathtt{a} := x$ would read 1 from x). That is, $y{=}1$ implies the following store order in both Fig. 11a and Fig. 11b: $x := 1 \rightarrow x := 2 \rightarrow \mathtt{a} := x$. In Fig. 11b we further have $\mathtt{a} := x \rightarrow \mathbf{flush}_{\mathrm{opt}}\ x$. Put together, this ensures $x := 1 \rightarrow x := 2 \rightarrow \mathbf{flush}_{\mathrm{opt}}\ x$ when $y{=}1$ in Fig. 11b. Consequently, executing $\mathbf{flush}_{\mathrm{opt}}\ x$ first persists $x := 1$ and then persists $x := 2$, as executing $\mathbf{flush}_{\mathrm{opt}}\ x$ persists all pending writes on x in the store order. As such, $z{=}y{=}1 \Rightarrow x{=}2$ upon recovery in Fig. 11b. By contrast, in Fig. 11a the $\mathtt{a} := x \rightarrow \mathbf{flush}_{\mathrm{opt}}\ x$ order does not hold, and thus $\mathbf{flush}_{\mathrm{opt}}\ x$ is only guaranteed to persist $x := 1$, yielding $z{=}y{=}1 \Rightarrow x \in \{1, 2\}$ upon recovery. The two programs may thus have different persistency behaviours: it is possible to observe $z{=}y{=}x{=}1$ after recovery in Fig. 11a but not in Fig. 11b.

Note that once again Fig. 11a does not adhere to (Epoch) and its intended persistency behaviour is ambiguous. More concretely, either: (1) x, y are on different cache lines, in which case the absence of a corresponding $\mathbf{flush}_{\mathrm{opt}}\ y$ implies that $y := 1$ is to persist in the epoch after $x := 1$, and is thus better practice to rewrite the program as in Fig. 11c; or (2) x, y are on the same cache line but $y := 1$ is to persist in the epoch after $x := 1$ and thus as in the previous case it is clearer to rewrite the program as in Fig. 11c; or (3) x, y are on the same cache line and $y := 1$ is to persist in the same epoch as $x := 1$, in which case $\mathbf{flush}_{\mathrm{opt}}\ x$ must follow $y := 1$. That is, in all three cases it is possible to rewrite Fig. 11c such that adheres to (Epoch) and avoids the intervening read between $\mathbf{flush}_{\mathrm{opt}}$ and the epoch barrier. We thus argue that it is possible to achieve epoch persistency *without* an intervening read between a $\mathbf{flush}_{\mathrm{opt}}$ and the epoch barrier, and thus it is often possible to apply our first transformation step while preserving the persistency behaviour. Finally, we prove that the first step of our transformation is sound in that it does not alter the persistency behaviour.

$x := 2;$ || $x := 1;$ **flush**$_{\text{opt}}$ x; a := x; **if** (a=2) $y := 1$; **sfence**; $z := 1$;	$x := 2;$ || $x := 1;$ a := x; **flush**$_{\text{opt}}$ x; **if** (a=2) $y := 1$; **sfence**; $z := 1$;	$x := 2;$ || $x := 1;$ **flush**$_{\text{opt}}$ x; **sfence**; a := x; **if** (a=2) $y := 1$; $z := 1$;
(a)	(b)	(c)
$\lightning: z{=}y{=}1 \Rightarrow x \in \{1,2\}$ $z{=}y{=}x{=}1$✓	$\lightning: z{=}y{=}1 \Rightarrow x{=}2$ $z{=}y{=}x{=}1$✗	$\lightning: z{=}y{=}1 \Rightarrow x \in \{1,2\}$ $z{=}y{=}x{=}1$✓

Fig. 11. An epoch persistency example not following the (Epoch) pattern (a); the program obtained from (a) by reordering **flush**$_{\text{opt}}$ x after a read (a := x), thus altering its persistency behaviour (b); a program with equivalent persistency behaviour to (a) that follows the (Epoch) pattern (c).

Theorem 4 (Step 1 soundness). *Given* C *with* C(τ)=c_a; **flush**$_{opt}$ x; c_b; c *and* $x \in X$, *if* c *is the first epoch barrier after* **flush**$_{opt}$ x *(i.e.* c_b *contains no* **sfence**/**mfence**/*RMW) and* c_b *contains no writes on* X *and no reads, then* C *and* C′ *have equivalent persistency behaviours, where* C′ $\triangleq$ C[$\tau \mapsto$ c_a; c_b; **flush**$_{opt}$ x; c].

7.2 Caveats of Converting flush$_{\text{opt}}$ to flush (Transformation Step 2)

Recall that once all **flush**$_{\text{opt}}$ instruction have been pushed down just before the epoch barrier, our second transformation step replaces each **flush**$_{\text{opt}}$ with a corresponding **flush**. However, in the case of a *blind persist* this transformation may alter the persistency behaviour of the program.

A blind persist denotes a scenario where a persist operation on x is executed without a previous access (read/write) on x. An example of this is illustrated in Fig. 12a, where **flush**$_{\text{opt}}$ x is issued without any prior access on x (assuming x, y are on different cache lines). Fig. 12b depicts the program obtained from Fig. 12a by replacing **flush**$_{\text{opt}}$ with **flush**. Note that the left thread executes $w := 1$ only when a=2, which in turn implies that $y := 1$ in the left thread is store-ordered before $y := 2$ in the right (otherwise a=2 would not be possible). That is, $w{=}1$ implies the following store order in both Fig. 12a and Fig. 12b: $x := 1 \rightarrow y := 1 \rightarrow y := 2$. As **flush** instructions are ordered with respect to *all* writes, in Fig. 12b we further have $y := 2 \rightarrow$ **flush** x. Put together, this ensures $x := 1 \rightarrow$ **flush** x when $w{=}1$ in Fig. 12b, and thus executing **flush** x persists $x := 1$; i.e. $w{=}z{=}1 \Rightarrow x{=}1$ upon recovery in Fig. 12b. By contrast, as **flush**$_{\text{opt}}$ instructions may be reordered with respect to writes on different cache lines, in Fig. 12a the $y := 2 \rightarrow$ **flush**$_{\text{opt}}$ x order does not hold, and thus **flush**$_{\text{opt}}$ x may not persist $x := 1$, yielding $w{=}z{=}1 \Rightarrow x \in \{0,1\}$ upon recovery. The two programs may thus have different persistency behaviours: it is possible to observe $w{=}z{=}1 \wedge x{=}0$ after recovery in Fig. 12a but not in Fig. 12b.

Note that blind persists are uncommon: persist instructions are expensive and are not typically issued without ascertaining that there is a corresponding write pending to be persisted. More concretely, prior to issuing a **flush**$_{\text{opt}}$/**flush** on x, the existence of pending writes on x is usually ascertained by either reading from x, or by having written to x earlier (in the same thread), as shown in Fig. 12c and Fig. 12d, respectively, where **flush**$_{\text{(opt)}}$ denotes either **flush**$_{\text{opt}}$ or **flush**. For instance, recall that in epoch persistency (Epoch) each persist is non-blind as it is preceded by a write on the same cache line. In more realistic scenarios such as (Epoch), Figs. 12c and 12d, **flush**$_{\text{opt}}$ instructions can thus be replaced with corresponding **flush** without altering the persistency behaviour.

Finally, we prove that the second transformation step is sound (see the accompanying technical appendix for the definition of blind persists).

Theorem 5 (Step 2 soundness). *Given* C *with* C(τ)=c_a; **flush**$_{opt}$ x; c; c_b, *if* **flush**$_{opt}$ *is not blind and* c *is an epoch barrier, then* C *and* C′ $\triangleq$ C[$\tau \mapsto$ c′; **flush** x; c] *have equivalent persistency behaviours.*

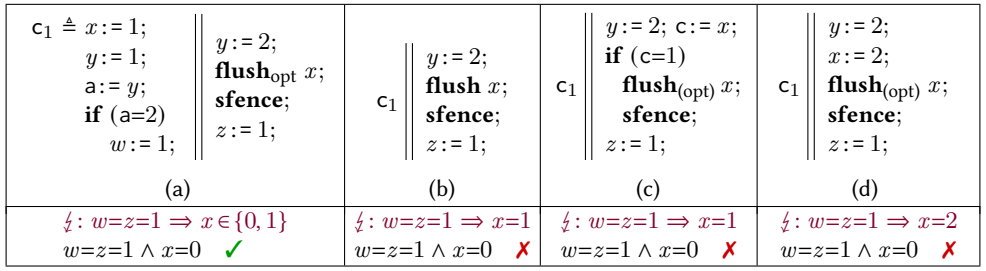

Fig. 12. A blind persist example with $x \in X$, $y \notin X$ (a); the program obtained from (a) by replacing $\mathbf{flush}_{\mathrm{opt}}$ with **flush**, altering its persistency (b); more realistic examples analogous to (a) with non-blind persists, where **flush**/$\mathbf{flush}_{\mathrm{opt}}$ can be used interchangeably (denoted by $\mathbf{flush}_{(\mathrm{opt})}$) without altering the persistency (c, d).

8 CONCLUSIONS, RELATED AND FUTURE WORK

We presented POG, the *first* program logic for reasoning about persistency behaviours under the $\mathrm{Px86}_{\mathrm{sim}}$ fragment that excludes $\mathbf{flush}_{\mathrm{opt}}$ instructions. We used POG to verify several representative examples. To establish the soundness of POG, we developed an intermediate operational model, $\mathrm{Ix86}_{\mathrm{sim}}$, which simplifies $\mathrm{Px86}_{\mathrm{sim}}$ by forgoing its persistent buffer and modelling its persist orderings by tracking three different versions for each location. We demonstrated that $\mathrm{Ix86}_{\mathrm{sim}}$ subsumes $\mathrm{Px86}_{\mathrm{sim}}$ and emulates all its valid behaviours. We then proved that POG is sound with respect to $\mathrm{Ix86}_{\mathrm{sim}}$ and thus also with respect to $\mathrm{Px86}_{\mathrm{sim}}$. As we note below (see future work), $\mathrm{Ix86}_{\mathrm{sim}}$ is a valuable contribution in its own right, as it facilitates automated verification techniques for persistency behaviours. Finally, in order to extend the reasoning principles of POG to the full $\mathrm{Px86}_{\mathrm{sim}}$ that also contains $\mathbf{flush}_{\mathrm{opt}}$ instructions, we presented a two-step transformation mechanism that allows us, in most cases, to rewrite a program using $\mathbf{flush}_{\mathrm{opt}}$ instructions, to an equivalent one that uses **flush** instructions instead without altering its persistency behaviour. As such, to reason about a program C that uses $\mathbf{flush}_{\mathrm{opt}}$, one can first use our transformation to rewrite C to a program C′ with equivalent persistency behaviour, and then use POG to reason about C′.

We based POG on the OGRA program logic [Lahav and Vafeiadis 2015]. As such, since OGRA is proved sound for the release-acquire (RA) consistency model and RA is a weaker model than x86-TSO, POG is also sound for RA. Consequently, POG is incomplete for reasoning about x86-TSO in that it cannot be used to prove the absence of behaviours that are admissible under RA but not x86-TSO. For instance, given the program $\mathsf{C} \triangleq (x := 1; y := 1) \,||\, (y := 2; x := 2)$, we cannot use POG to prove that the final states of C exclude those in which $x{=}1 \wedge y{=}2$ holds.

However, we note that almost all existing program logics in the literature, including those for the strong sequential consistency model, are incomplete (e.g. [Dinsdale-Young et al. 2010; Jones 1983; Jung et al. 2015; Kaiser et al. 2017; Lahav and Vafeiadis 2015; Nanevski et al. 2014; Raad et al. 2015; Svendsen et al. 2018; Turon et al. 2014; Vafeiadis and Narayan 2013]) as their main aim to enable simple high-level reasoning principles that apply to common patterns, albeit at the cost of compromising completeness for certain cases. Nevertheless, as shown in the literature, incompleteness can be remedied to some extent by including ghost state [Jacobs and Piessens 2011]. Specifically, it is possible to prove the example above using ghost state, provided that we also show that introducing such state is sound under $\mathrm{Ix86}_{\mathrm{sim}}$/x86-TSO, which we leave for future work.

Related work. Although existing literature on NVM has grown rapidly in recent years, formally verifying programs and algorithms that operate on NVM has largely remained unexplored. Friedman et al. [2018] developed several persistent queue implementations using the Intel-x86 persist instructions (e.g. **flush**); Zuriel et al. [2019] also developed two persistent set implementations

using Intel-x86 persist instructions. Both [Friedman et al. 2018; Zuriel et al. 2019] argue that their implementations are correct by providing an informal argument at the level of program traces. Derrick et al. [2019] provided a formal correctness proof of the queue implementation by Friedman et al. [2018]; this proof is also at the level of program traces. Moreover, all three of [Derrick et al. 2019; Friedman et al. 2018; Zuriel et al. 2019] assume that the underlying memory model is sequential consistency (SC) [Lamport 1979], rather than Intel x86-TSO. Raad et al. [2019] recently developed a persistent transactional library on top of the ARM architecture; they later adapted this implementation to the Px86_{sim} architecture. In both cases they provide a formal proof of their implementation correctness on top of the corresponding architecture. Nevertheless, these proofs are low-level in that they operate at the level of execution traces, rather than the program syntax.

To our knowledge, no existing work provides a syntactic proof system for verifying the persistency guarantees of concurrent programs, especially in the presence of relaxed (out-of-order) or asynchronous persists. The most closely related work to ours are those of [Chen et al. 2015; Ntzik et al. 2015]. Chen et al. [2015] present crash Hoare logic (CHL) for reasoning about the crashing behaviour of the FSCQ file system. CHL is more restricted than POG in two ways. First, CHL can only be used for *sequential* programs and does not support concurrent reasoning. Second, in contrast to POG's support for explicit **flush** instructions, CHL does not support the Unix explicit persist instruction **fsync**. Ntzik et al. [2015] extend the Views framework [Dinsdale-Young et al. 2013] to support fault conditions. As an extension of Views, this work supports concurrency; however, their support for persistent reasoning is rather limited: (1) it assumes that the underlying memory model is SC and does not account for weak concurrent behaviours; (2) it does not distinguish between stores and persists, and thus assumes that all stores persist synchronously and in the store order; and consequently (3) does not support any explicit persist instructions such as **flush**.

Future work. We plan to build on our work here in several ways. First, we will use POG to verify existing implementations of persistent libraries and data structures such as [Friedman et al. 2018; Intel 2015; Zuriel et al. 2019]. Second, we will build automated techniques such as model checking (MC) for verifying persistency. We plan to do this by extending existing MC algorithms that already support x86-TSO (e.g. [Abdulla et al. 2015; Kokologiannakis et al. 2019a,b]) with the atomic propagation constructs of Ix86_{sim}. This will allow us to leverage cutting-edge MC tools to verify persistency with minimal implementation overhead. Lastly, building on the ideas underpinning POG, we will devise a similar program logic for reasoning about persistency under the ARMv8 architecture [Raad et al. 2019].

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A Ix86$_{\text{sim}}$ SUBSUMES Px86$_{\text{sim}}$

A.1 The Px86$_{\text{man}}$ Event-Annotated Operational Semantics

Types.

Annotated persistent memory

$$M \in \text{AMem} \triangleq \left\{ f \in \text{Loc} \xrightarrow{\text{fin}} W \cup U \;\middle|\; \forall x \in dom(f).\ \texttt{loc}(f(x)) = x \right\}$$

Annotated persistent buffers

$$PB \in \text{APBuff} \triangleq \text{Seq}\,\langle W \cup U \cup FL \rangle$$

Annotated volatile buffers

$$b \in \text{ABuff}_\tau \triangleq \text{Seq}\left\langle \left\{ e \in W \cup FL \cup SF \;\middle|\; \texttt{tid}(e){=}\tau \right\} \right\rangle$$
$$b \in \text{ABuff} \triangleq \bigcup_{\tau \in \text{TId}} \text{ABuff}_\tau$$

Annotated volatile buffer maps

$$B \in \text{ABMap} \triangleq \left\{ B \in \text{TId} \xrightarrow{\text{fin}} \text{ABuff} \;\middle|\; \forall \tau \in dom(B).\ B(\tau) \in \text{ABuff}_\tau \right\}$$

Annotated labels

$$\begin{array}{lll}
\text{ALabels} \ni \lambda ::= & \mathsf{R}\langle r, w \rangle & \text{where } r \in R, w \in W, \texttt{loc}(r){=}\texttt{loc}(w), \texttt{val}_\texttt{r}(r){=}\texttt{val}_\texttt{w}(w) \\
| & \mathsf{U}\langle u, w \rangle & \text{where } u \in U, w \in W, \texttt{loc}(u){=}\texttt{loc}(w), \texttt{val}_\texttt{r}(u){=}\texttt{val}_\texttt{w}(w) \\
| & \mathsf{W}\langle w \rangle & \text{where } w \in W \\
| & \mathsf{MF}\langle mf \rangle & \text{where } mf \in MF \\
| & \mathsf{SF}\langle sf \rangle & \text{where } sf \in SF \\
| & \mathsf{FL}\langle fl \rangle & \text{where } fl \in FL \\
| & \mathsf{B}\langle e \rangle & \text{where } e \in W \cup SF \cup FL \\
| & \mathsf{PB}\langle e \rangle & \text{where } e \in W \cup U \cup FL \\
| & \mathcal{E}\langle \tau \rangle & \text{where } \tau \in \text{TId}
\end{array}$$

$\pi \in \text{Path} \triangleq \text{Seq}\,\langle \text{ALabels} \rangle$ Event paths

$\mathcal{H} \in \text{Hist} \triangleq \text{Seq}\,\langle \text{Path} \rangle$ Histories

Let

$$\begin{array}{r}
\text{AMem} \ni M_0 \triangleq \lambda x.init_x \text{ with } \texttt{lab}(init_x) \triangleq (\texttt{W}, x, 0) \\
\text{APBuff} \ni PB_0 \triangleq \lambda x.\epsilon \\
\text{ABuff} \ni b_0 \triangleq \epsilon \\
\text{ABMap} \ni B_0 \triangleq \lambda \tau.b_0
\end{array}$$

Storage Subsystem.

$$\frac{\texttt{tid}(w){=}\tau \quad B(\tau){=}b}{M, PB, B \xrightarrow{\mathsf{W}\langle w \rangle} M, PB, B[\tau \mapsto b.w]}\ \text{(AM-Write)}$$

$$\frac{\texttt{tid}(r){=}\tau \quad B(\tau){=}b \quad \texttt{loc}(r){=}x \quad \texttt{rd}(M, PB, b, x){=}e}{M, PB, B \xrightarrow{\mathsf{R}\langle r,e \rangle} M, PB, B}\ \text{(AM-Read)}$$

$$\frac{\texttt{tid}(u){=}\tau \quad B(\tau){=}\epsilon \quad \texttt{loc}(u){=}x \quad \texttt{rd}(M, PB, \epsilon, x){=}e}{M, PB, B \xrightarrow{\mathsf{U}\langle u,e \rangle} M, PB.u, B}\ \text{(AM-RMW)}$$

$$\frac{\texttt{tid}(mf){=}\tau \quad B(\tau){=}\epsilon}{M, PB, B \xrightarrow{\mathsf{MF}\langle mf\rangle} M, PB, B} \text{ (AM-MF)}$$

$$\frac{\texttt{tid}(sf){=}\tau \quad B(\tau){=}b}{M, PB, B \xrightarrow{\mathsf{SF}\langle sf\rangle} M, PB, B[\tau \mapsto b.sf]} \text{ (AM-SF)}$$

$$\frac{\texttt{tid}(fl){=}\tau \quad B(\tau){=}b}{M, PB, B \xrightarrow{\mathsf{FL}\langle fl\rangle} M, PB, B[\tau \mapsto b.fl]} \text{ (AM-FL)}$$

$$\frac{B(\tau){=}w.b \qquad w \in W}{M, PB, B \xrightarrow{\mathsf{B}\langle w\rangle} M, PB.w, B[\tau \mapsto b]} \text{ (AM-BPropW)}$$

$$\frac{B(\tau){=}sf.b \qquad sf \in SF}{M, PB, B \xrightarrow{\mathsf{B}\langle sf\rangle} M, PB, B[\tau \mapsto b]} \text{ (AM-BPropSF)}$$

$$\frac{B(\tau){=}fl.b \qquad fl \in FL}{M, PB, B \xrightarrow{\mathsf{B}\langle fl\rangle} M, PB.fl, B[\tau \mapsto b]} \text{ (AM-BPropFL)}$$

$$\frac{\mathsf{PB}{=}\mathsf{PB}_1.w.\mathsf{PB}_2 \qquad w \in W_X \qquad \mathsf{PB}_1 \cap (W_X \cup FL){=}\emptyset}{M, PB, B \xrightarrow{\mathsf{PB}\langle w\rangle} M[x \mapsto w], PB_1.PB_2, B} \text{ (AM-PropW)}$$

$$\frac{\mathsf{PB}{=}\mathsf{PB}_1.e.\mathsf{PB}_2 \qquad e \in FL_X \qquad \mathsf{PB}_1 \cap (W_X \cup FL){=}\emptyset}{M, PB, B \xrightarrow{\mathsf{PB}\langle e\rangle} M, PB_1.PB_2, B} \text{ (AM-PropP)}$$

where

$$\texttt{rd}(M, PB, b, x) \triangleq \begin{cases} e & \text{if } \texttt{rd}_{\mathsf{S}}(b, x) = e \\ e & \text{else if } PB{=}PB_1.e.PB_2 \\ & \text{and } (W_x \cup U_x) \cap PB_2{=}\emptyset \\ & \text{and } e \in W_x \cup U_x \\ M(x) & \text{otherwise} \end{cases} \qquad \texttt{rd}_{\mathsf{S}}(b, x) \triangleq \begin{cases} w & \text{if } \exists b_1, b_2.\ b{=}b_1.w.b_2 \\ & \text{and } \texttt{loc}(w){=}x \\ & \text{and } W_x \cap b_2{=}\emptyset \\ \text{undef} & \text{otherwise} \end{cases}$$

Program Subsystem.

$$\frac{\mathsf{c}_1, \mathsf{S} \xrightarrow{\lambda} \mathsf{c}_1', \mathsf{S}'}{\mathsf{c}_1; \mathsf{c}_2, \mathsf{S} \xrightarrow{\lambda} \mathsf{c}_1'; \mathsf{c}_2, \mathsf{S}'} \text{ (AP-Seq1)} \qquad \frac{}{\textbf{skip}; \mathsf{c}, \mathsf{S} \xrightarrow{\mathcal{E}\langle\tau\rangle} \mathsf{c}, \mathsf{S}} \text{ (AP-Seq2)}$$

$$\frac{\mathsf{S}(\mathsf{e}){\neq}0 \Rightarrow \mathsf{c}{=}\mathsf{c}_1 \quad \mathsf{S}(\mathsf{e}){=}0 \Rightarrow \mathsf{c}{=}\mathsf{c}_2}{\textbf{if } (\mathsf{e}) \textbf{ then } \mathsf{c}_1 \textbf{ else } \mathsf{c}_2, \mathsf{S} \xrightarrow{\mathcal{E}\langle\tau\rangle} \mathsf{c}, \mathsf{S}} \text{ (AP-If)} \qquad \frac{\mathsf{S}(\mathsf{e}){=}v}{\mathsf{a} := \mathsf{e}, \mathsf{S} \xrightarrow{\mathcal{E}\langle\tau\rangle} \textbf{skip}, \mathsf{S}[\mathsf{a} \mapsto v]} \text{ (AP-Assign)}$$

$$\frac{}{\textbf{while}(\mathsf{e})\ \mathsf{c}, \mathsf{S} \xrightarrow{\mathcal{E}\langle\tau\rangle} \textbf{if } (\mathsf{e}) \textbf{ then } \mathsf{c}; (\textbf{while}(\mathsf{e})\ \mathsf{c}) \textbf{ else skip}, \mathsf{S}} \text{ (AP-While)}$$

$$\frac{\mathtt{val_w}(w){=}\mathsf{S}(\mathsf{e}) \quad \mathtt{loc}(w){=}x}{x := \mathsf{e}, \mathsf{S} \xrightarrow{\mathsf{W}\langle w\rangle} \mathbf{skip}, \mathsf{S}} \text{ (AP-WRITE)} \qquad \frac{\mathtt{val_r}(r){=}v \quad \mathtt{loc}(r){=}x}{\mathsf{a} := x, \mathsf{S} \xrightarrow{\mathsf{R}\langle r,w\rangle} \mathbf{skip}, \mathsf{S}[\mathsf{a} \mapsto v]} \text{ (AP-READ)}$$

$$\frac{\mathtt{val_r}(u){=}v \quad \mathtt{val_w}(u){=}v{+}\mathsf{S}(\mathsf{e}) \quad \mathtt{loc}(u){=}x}{\mathsf{a} := \mathbf{FAA}(x, \mathsf{e}), \mathsf{S} \xrightarrow{\mathsf{U}\langle u,w\rangle} \mathbf{skip}, \mathsf{S}[\mathsf{a} \mapsto v]} \text{ (AP-FAA)} \qquad \frac{}{\mathbf{mfence}, \mathsf{S} \xrightarrow{\mathsf{MF}\langle mf\rangle} \mathbf{skip}, \mathsf{S}} \text{ (AP-MFENCE)}$$

$$\frac{\mathtt{val_r}(r){=}v \neq \mathsf{S}(\mathsf{e}_1) \quad \mathtt{loc}(r){=}x}{\mathsf{a} := \mathbf{CAS}(x, \mathsf{e}_1, \mathsf{e}_2), \mathsf{S} \xrightarrow{\mathsf{R}\langle r,w\rangle} \mathbf{skip}, \mathsf{S}[\mathsf{a} \mapsto v]} \text{ (AP-CAS0)} \qquad \frac{\mathtt{val_r}(u){=}v{=}\mathsf{S}(\mathsf{e}_1) \quad \mathtt{val_w}(u){=}\mathsf{S}(\mathsf{e}_2) \quad \mathtt{loc}(u){=}x}{\mathsf{a} := \mathbf{CAS}(x, \mathsf{e}_1, \mathsf{e}_2), \mathsf{S} \xrightarrow{\mathsf{U}\langle u,w\rangle} \mathbf{skip}, \mathsf{S}[\mathsf{a} \mapsto v]}$$

$$\frac{}{\mathbf{sfence}, \mathsf{S} \xrightarrow{\mathsf{SF}\langle sf\rangle} \mathbf{skip}, \mathsf{S}} \text{ (AP-SFENCE)} \qquad \frac{\mathtt{loc}(fl){=}x}{\mathbf{flush}\ x, \mathsf{S} \xrightarrow{\mathsf{FL}\langle fl\rangle} \mathbf{skip}, \mathsf{S}} \text{ (AP-FL)}$$

$$\frac{\mathsf{C}(\tau), \mathsf{S} \xrightarrow{\lambda} \mathsf{c}, \mathsf{S}' \quad \exists e.\ \lambda{=}\mathsf{PB}\langle e\rangle \vee \mathtt{tid}(\lambda){=}\tau}{\mathsf{C}, \mathsf{S} \xrightarrow{\lambda} \mathsf{C}[\tau \mapsto \mathsf{c}], \mathsf{S}'} \text{ (AP-PAR)}$$

where:

$$\mathtt{tid}(\lambda) \triangleq \begin{cases} \tau & \text{if } \lambda{=}\mathcal{E}\langle\tau\rangle \\ \mathtt{tid}(\mathtt{event}(\lambda)) & \text{otherwise} \end{cases}$$

$$\begin{aligned} \mathtt{event}(\mathsf{R}\langle r,w\rangle) &\triangleq r \\ \mathtt{event}(\mathsf{U}\langle u,w\rangle) &\triangleq u \\ \mathtt{event}(\mathsf{W}\langle w\rangle) &\triangleq w \\ \mathtt{event}(\mathsf{MF}\langle mf\rangle) &\triangleq mf \\ \mathtt{event}(\mathsf{SF}\langle sf\rangle) &\triangleq sf \\ \mathtt{event}(\mathsf{FL}\langle fl\rangle) &\triangleq fl \\ \mathtt{event}(\mathsf{B}\langle e\rangle) &\triangleq e \\ \mathtt{event}(\mathsf{PB}\langle e\rangle) &\triangleq e \\ \mathtt{event}(\mathcal{E}\langle\tau\rangle) &\text{ undefined} \end{aligned}$$

The Px86$_{man}$ Event-Annotated Operational Semantics.

$$\frac{\mathsf{c}, \mathsf{S} \xrightarrow{\mathcal{E}\langle\tau\rangle} \mathsf{c}', \mathsf{S}'}{\Delta \vdash \mathsf{c}, \mathsf{S}, M, PB, B, \pi_d, \pi_l \Rightarrow \mathsf{c}', \mathsf{S}' M, PB, B, \pi_d, \pi_l} \text{ (A-SILENTP)}$$

$$\frac{M, PB, B \xrightarrow{\lambda} M', PB', B' \qquad \lambda \in \left\{\mathsf{B}\langle e\rangle, \mathsf{PB}\langle e\rangle\right\} \qquad \mathsf{fresh}(\lambda, \pi_d)}{\Delta \vdash \mathsf{c}, \mathsf{S}, M, PB, B, \pi_d, \lambda.\pi_l \Rightarrow \mathsf{c}, \mathsf{S}, M', PB', B', \pi_d.\lambda, \pi_l} \text{ (A-SILENTS)}$$

$$\frac{\mathsf{c}, \mathsf{S} \xrightarrow{\lambda} \mathsf{c}', \mathsf{S}' \qquad M, PB, B \xrightarrow{\lambda} M', PB', B' \qquad \mathsf{fresh}(\lambda, \pi_d)}{\Delta \vdash \mathsf{c}, \mathsf{S}, M, PB, B, \pi_d, \lambda.\pi_l \Rightarrow \mathsf{c}', \mathsf{S}', M', PB', B', \pi_d.\lambda, \pi_l} \text{ (A-STEP)}$$

$$\frac{\Delta{=}(\mathsf{c}_0, \mathbf{rec})}{\Delta \vdash \mathsf{c}, \mathsf{S}, M, PB, B, \pi_d, \epsilon \Rightarrow \mathbf{rec}(\mathsf{c}_0, M), \mathsf{S}_0, M, PB_0, B_0, \epsilon, \pi} \text{ (A-CRASH)}$$

with

$$\begin{aligned}\mathsf{fresh}(\lambda,\pi) \triangleq\ & \lambda \notin \pi \wedge \forall e, w, w'.\\ & (\lambda{=}\mathsf{R}\langle e, w\rangle \Rightarrow \mathsf{R}\langle e, w'\rangle \notin \pi) \wedge (\lambda{=}\mathsf{U}\langle e, w\rangle \Rightarrow \mathsf{U}\langle e, w'\rangle \notin \pi)\end{aligned}$$

Definition 5.

$$\begin{aligned}\mathsf{wfp}(\pi) \triangleq & \forall \lambda, \pi_1, \pi_2, e, r, e_1, e_2, \lambda_1, \lambda_2, X.\\ & \mathsf{nodups}(\pi)\\ & \pi{=}\pi_1.\mathsf{R}\langle r, e\rangle.\pi_2 \vee \pi{=}\pi_1.\mathsf{U}\langle r, e\rangle.\pi_2 \Rightarrow \mathsf{wfrd}(r, e, \pi_1)\\ & \mathsf{B}\langle e\rangle \in \pi \Rightarrow\\ & \quad \mathsf{W}\langle e\rangle \prec_\pi \mathsf{B}\langle e\rangle \vee \mathsf{SF}\langle e\rangle \prec_\pi \mathsf{B}\langle e\rangle \vee \mathsf{FL}\langle e\rangle \prec_\pi \mathsf{B}\langle e\rangle\\ & \mathsf{PB}\langle e\rangle \in \pi \Rightarrow \mathsf{B}\langle e\rangle \prec_\pi \mathsf{PB}\langle e\rangle \vee \mathsf{U}\langle e, -\rangle \prec_\pi \mathsf{PB}\langle e\rangle\\ & \mathsf{W}\langle e_1\rangle \prec_\pi \mathsf{MF}\langle e_2\rangle \wedge \mathtt{tid}(e_1){=}\mathtt{tid}(e_2) \Rightarrow \mathsf{B}\langle e_1\rangle \prec_\pi \mathsf{MF}\langle e_2\rangle\\ & \mathsf{SF}\langle e_1\rangle \prec_\pi \mathsf{MF}\langle e_2\rangle \wedge \mathtt{tid}(e_1){=}\mathtt{tid}(e_2) \Rightarrow \mathsf{B}\langle e_1\rangle \prec_\pi \mathsf{MF}\langle e_2\rangle\\ & \mathsf{FL}\langle e_1\rangle \prec_\pi \mathsf{MF}\langle e_2\rangle \wedge \mathtt{tid}(e_1){=}\mathtt{tid}(e_2) \Rightarrow \mathsf{B}\langle e_1\rangle \prec_\pi \mathsf{MF}\langle e_2\rangle\\ & \mathsf{W}\langle e_1\rangle \prec_\pi \mathsf{SF}\langle e_2\rangle \wedge \mathtt{tid}(e_1){=}\mathtt{tid}(e_2) \wedge \mathsf{B}\langle e_2\rangle \in \pi \Rightarrow \mathsf{B}\langle e_1\rangle \prec_\pi \mathsf{B}\langle e_2\rangle\\ & \mathsf{SF}\langle e_1\rangle \prec_\pi \mathsf{SF}\langle e_2\rangle \wedge \mathtt{tid}(e_1){=}\mathtt{tid}(e_2) \wedge \mathsf{B}\langle e_2\rangle \in \pi \Rightarrow \mathsf{B}\langle e_1\rangle \prec_\pi \mathsf{B}\langle e_2\rangle\\ & \mathsf{FL}\langle e_1\rangle \prec_\pi \mathsf{SF}\langle e_2\rangle \wedge \mathtt{tid}(e_1){=}\mathtt{tid}(e_2) \wedge \mathsf{B}\langle e_2\rangle \in \pi \Rightarrow \mathsf{B}\langle e_1\rangle \prec_\pi \mathsf{B}\langle e_2\rangle\\ & \mathsf{SF}\langle e_1\rangle \prec_\pi \mathsf{W}\langle e_2\rangle \wedge \mathtt{tid}(e_1){=}\mathtt{tid}(e_2) \wedge \mathsf{B}\langle e_2\rangle \in \pi \Rightarrow \mathsf{B}\langle e_1\rangle \prec_\pi \mathsf{B}\langle e_2\rangle\\ & \mathsf{SF}\langle e_1\rangle \prec_\pi \mathsf{U}\langle e_2, e\rangle \wedge \mathtt{tid}(e_1){=}\mathtt{tid}(e_2) \Rightarrow \mathsf{B}\langle e_1\rangle \prec_\pi \mathsf{U}\langle e_2, e\rangle\\ & \mathsf{SF}\langle e_1\rangle \prec_\pi \mathsf{FL}\langle e_2\rangle \wedge \mathtt{tid}(e_1){=}\mathtt{tid}(e_2) \wedge \mathsf{B}\langle e_2\rangle \in \pi \Rightarrow \mathsf{B}\langle e_1\rangle \prec_\pi \mathsf{B}\langle e_2\rangle\\ & \mathsf{W}\langle e_1\rangle \prec_\pi \mathsf{W}\langle e_2\rangle \wedge \mathtt{tid}(e_1) = \mathtt{tid}(e_2) \wedge \mathsf{B}\langle e_2\rangle \in \pi \Rightarrow \mathsf{B}\langle e_1\rangle \prec_\pi \mathsf{B}\langle e_2\rangle\\ & \mathsf{W}\langle e_1\rangle \prec_\pi \mathsf{U}\langle e_2, e\rangle \wedge \mathtt{tid}(e_1) = \mathtt{tid}(e_2) \Rightarrow \mathsf{B}\langle e_1\rangle \prec_\pi \mathsf{U}\langle e_2, e\rangle\\ & \mathsf{W}\langle e_1\rangle \prec_\pi \mathsf{FL}\langle e_2\rangle \wedge \mathtt{tid}(e_1){=}\mathtt{tid}(e_2) \Rightarrow \mathsf{B}\langle e_1\rangle \prec_\pi \mathsf{B}\langle e_2\rangle\\ & \mathsf{FL}\langle e_1\rangle \prec_\pi \mathsf{W}\langle e_2\rangle \wedge \mathtt{tid}(e_1){=}\mathtt{tid}(e_2) \Rightarrow \mathsf{B}\langle e_1\rangle \prec_\pi \mathsf{B}\langle e_2\rangle\\ & \mathsf{FL}\langle e_1\rangle \prec_\pi \mathsf{U}\langle e_2, e\rangle \wedge \mathtt{tid}(e_1){=}\mathtt{tid}(e_2) \Rightarrow \mathsf{B}\langle e_1\rangle \prec_\pi \mathsf{U}\langle e_2, e\rangle\\ & \mathsf{FL}\langle e_1\rangle \prec_\pi \mathsf{FL}\langle e_2\rangle \wedge \mathtt{tid}(e_1){=}\mathtt{tid}(e_2) \Rightarrow \mathsf{B}\langle e_1\rangle \prec_\pi \mathsf{B}\langle e_2\rangle\\ & e_1{,}e_2 \in W \cup U \wedge \lambda_1 \in \{\mathsf{B}\langle e_1\rangle{,}\mathsf{U}\langle e_1, e\rangle\} \wedge \lambda_2 \in \{\mathsf{B}\langle e_2\rangle{,}\mathsf{U}\langle e_2, e\rangle\} \wedge \lambda_1 \prec_\pi \lambda_2 \wedge \mathtt{loc}(e_1){=}\mathtt{loc}(e_2)\\ & \quad \Rightarrow \mathsf{PB}\langle e_1\rangle \prec_\pi \mathsf{PB}\langle e_2\rangle\\ & e_1 \in W \cup U \wedge e_2 \in FL \wedge \mathtt{loc}(e_1), \mathtt{loc}(e_2) \in X \wedge \lambda_1 \in \{\mathsf{B}\langle e_1\rangle, \mathsf{U}\langle e_1, e\rangle\}\\ & \wedge \lambda_2{=}\mathsf{B}\langle e_2\rangle \wedge \lambda_1 \prec_\pi \lambda_2\\ & \quad \Rightarrow \mathsf{PB}\langle e_1\rangle \prec_\pi \mathsf{PB}\langle e_2\rangle\\ & e_1 \in FL \wedge e_2 \in D \wedge \lambda_1{=}\mathsf{B}\langle e_1\rangle \wedge \lambda_2 \in \{\mathsf{B}\langle e_2\rangle, \mathsf{U}\langle e_2, e\rangle\} \wedge \lambda_1 \prec_\pi \lambda_2\\ & \quad \Rightarrow \mathsf{PB}\langle e_1\rangle \prec_\pi \mathsf{PB}\langle e_2\rangle\end{aligned}$$

where

$$\mathsf{nodups}(\pi) \triangleq \forall \pi_1, \pi_2, \lambda.\ \pi = \pi_1.\lambda.\pi_2 \Rightarrow \mathsf{fresh}(\lambda, \pi_1.\pi_2)$$

$$\begin{aligned}\mathsf{wfrd}(r,e,\pi) \triangleq\ & \exists \pi_1, \lambda.\ \pi = -.\lambda.\pi_1 \\ & \wedge (\lambda{=}\mathsf{B}\langle e\rangle \vee \lambda{=}\mathsf{U}\langle e,-\rangle \vee (\lambda{=}\mathsf{W}\langle e\rangle \wedge \mathtt{tid}(e) = \mathtt{tid}(r))) \\ & \wedge \begin{pmatrix} (\lambda{=}\mathsf{B}\langle e\rangle \vee \lambda{=}\mathsf{U}\langle e,-\rangle) \Rightarrow \\ \left\{\mathsf{B}\langle e'\rangle, \mathsf{U}\langle e',-\rangle \in \pi_1 \;\middle|\; \mathtt{loc}(e'){=}\mathtt{loc}(r)\right\} = \emptyset \\ \wedge \left\{e' \;\middle|\; \begin{matrix}\mathsf{W}\langle e'\rangle \in \pi_1 \wedge \mathsf{B}\langle e'\rangle \notin \pi_1 \\ \wedge\, \mathtt{loc}(e'){=}\mathtt{loc}(r) \wedge \mathtt{tid}(e'){=}\mathtt{tid}(r)\end{matrix}\right\} = \emptyset \end{pmatrix} \\ & \wedge \begin{pmatrix} \lambda{=}\mathsf{W}\langle e\rangle \Rightarrow \\ \mathsf{B}\langle e\rangle \notin \pi_1 \wedge \left\{\mathsf{W}\langle e'\rangle \in \pi_1 \;\middle|\; \begin{matrix}\mathtt{loc}(e'){=}\mathtt{loc}(r) \wedge \\ \mathtt{tid}(e'){=}\mathtt{tid}(r)\end{matrix}\right\} = \emptyset \end{pmatrix}\end{aligned}$$

Definition 6.

$$\mathsf{wf}(M, PB, B, \pi) \overset{\text{def}}{\Leftrightarrow} \mathsf{pbuff}(PB_0, \pi) = PB \wedge \mathsf{bmap}(B_0, \pi) = B \wedge \mathsf{wfp}(\pi)$$

where

$$\mathsf{pbuff}(PB, \epsilon) \triangleq PB$$

$$\mathsf{pbuff}(PB, \lambda.\pi) \triangleq \begin{cases} \mathsf{pbuff}(PB.e, \pi) & \text{if } \exists e.\ \lambda \in \{\mathsf{B}\langle e\rangle, \mathsf{U}\langle e,-\rangle\} \wedge \mathsf{PB}\langle e\rangle \notin \pi \\ \mathsf{pbuff}(PB, \pi) & \text{otherwise} \end{cases}$$

$$\mathsf{bmap}(B, \epsilon) \triangleq B$$

$$\mathsf{bmap}(B, \pi.\lambda) \triangleq \begin{cases} \mathsf{bmap}(B[\tau \mapsto B(\tau).e], \pi) & \text{if } \exists e, \tau.\ \lambda{=}\mathsf{W}\langle e\rangle \wedge \mathtt{tid}(e){=}\tau \wedge \mathsf{B}\langle e\rangle \notin \pi \\ \mathsf{bmap}(B[\tau \mapsto B(\tau).\langle \mathsf{fl}, e\rangle], \pi) & \text{if } \exists e, \tau.\ \lambda{=}\mathsf{FL}\langle e\rangle \wedge \mathtt{tid}(e){=}\tau \wedge \mathsf{B}\langle e\rangle \notin \pi \\ \mathsf{bmap}(B[\tau \mapsto B(\tau).\langle \mathsf{sf}, e\rangle], \pi) & \text{if } \exists e, \tau.\ \lambda{=}\mathsf{SF}\langle e\rangle \wedge \mathtt{tid}(e){=}\tau \wedge \mathsf{B}\langle e\rangle \notin \pi \\ \mathsf{bmap}(B, \pi) & \text{otherwise} \end{cases}$$

Lemma 1. *For all* **rec**, $\mathsf{C}, \mathsf{C}', \mathsf{S}, \mathsf{S}', M, M', PB, PB', B, B', \pi_d, \pi_d', \pi_l, \pi_l'$*:*

(1) $\mathsf{wf}(M_0, PB_0, B_0, \epsilon)$

(2) if $\mathbf{rec} \vdash \mathsf{C}, \mathsf{S}, M, PB, B, \pi_d, \pi_l \Rightarrow \mathsf{C}', \mathsf{S}', M', PB', B', \pi_d', \pi_l'$ *and* $\mathsf{wf}(M, PB, B, \pi_d)$*, and* $\mathsf{wfp}(\pi_d.\pi_l)$*, then* $\mathsf{wf}(M', PB', B', \pi_d')$

(3) if $\mathbf{rec} \vdash \mathsf{C}, \mathsf{S}, M_0, PB_0, B_0, \epsilon, \pi_l \Rightarrow^* \mathsf{C}_{\mathbf{skip}}, \mathsf{S}', M, PB, B, \pi_d, \pi_l'$*, then* $\mathsf{wf}(M, PB, B, \pi_d)$ *and* $\mathsf{wfp}(\pi_d'.\pi_l')$.

Proof. The first part simply follows from the definitions of M_0, PB_0, B_0. The second part follows straightforwardly by induction on the structure of $\Rightarrow$. The last part follows from the previous two parts and induction on the length of $\Rightarrow^*$. □

Lemma 2 (Repeated from [Raad et al. 2020])**.** *For all* $\mathsf{C}, \mathsf{C}', \mathsf{S}, \mathsf{S}', \mathsf{M}, \mathsf{M}'$*, if* $\mathsf{C}, \mathsf{S}, \mathsf{M}, \mathsf{PB}_0, \mathsf{B}_0 \rightarrow^* \mathsf{C}', \mathsf{S}', \mathsf{M}', -, -$*, then there exists* M, M', π *such that:* $\mathsf{C}, \mathsf{S}, M, PB_0, B_0, \epsilon, \pi \rightarrow^* \mathsf{C}', \mathsf{S}', M', -, -, \pi, \epsilon$ *and for all* x*:* $\mathsf{M}(x){=}\mathtt{val}_{\mathtt{w}}(M(x)) \wedge \mathsf{M}'(x){=}\mathtt{val}_{\mathtt{w}}(M'(x))$.

Proof. See [Raad et al. 2020]. □

A.2 Ix86$_{\mathsf{sim}}$ Event-Annotated Operational Semantics

Types.

$$IM \in \textsc{AIMem} \triangleq \left\{ f \in \textsc{ILoc} \xrightarrow{\text{fin}} W \cup U \;\middle|\; \begin{array}{l} \forall x \in dom(f). \\ \mathtt{loc}(f(x))=x \end{array} \right\} \qquad \text{Instrumented Persistent Memory}$$

$$ib \in \textsc{AIBuff}_\tau \triangleq \textsc{Seq}\left\langle \left\{ e, X \;\middle|\; e \in W \wedge \mathtt{tid}(e)=\tau \wedge X \subseteq \textsc{Loc} \right\} \right\rangle \qquad \text{Instrumented Buffers}$$

$$ib \in \textsc{AIBuff} \triangleq \bigcup_{\tau \in \textsc{Tid}} \textsc{AIBuff}_\tau$$

$$IB \in \textsc{AIBMap} \triangleq \left\{ f \in \textsc{Tid} \xrightarrow{\text{fin}} \textsc{AIBuff} \;\middle|\; \begin{array}{l} \forall \tau \in dom(f). \\ f(\tau) \in \textsc{AIBuff}_\tau \end{array} \right\} \qquad \text{Instrumented Buffer Maps}$$

Program Subsystem.

$$\frac{\mathsf{c}_1, \mathsf{S} \xrightarrow{\lambda} \mathsf{c}_1', \mathsf{S}'}{\mathsf{c}_1; \mathsf{c}_2, \mathsf{S} \xrightarrow{\lambda} \mathsf{c}_1'; \mathsf{c}_2, \mathsf{S}'} \text{ (AIP-Seq1)} \qquad \frac{}{\textbf{skip}; \mathsf{c}, \mathsf{S} \xrightarrow{\mathcal{E}\langle\tau\rangle} \mathsf{c}, \mathsf{S}} \text{ (AIP-Seq2)}$$

$$\frac{\mathsf{S}(\mathsf{e})\neq 0 \Rightarrow \mathsf{c}=\mathsf{c}_1 \quad \mathsf{S}(\mathsf{e})=0 \Rightarrow \mathsf{c}=\mathsf{c}_2}{\textbf{if } (\mathsf{e}) \textbf{ then } \mathsf{c}_1 \textbf{ else } \mathsf{c}_2, \mathsf{S} \xrightarrow{\mathcal{E}\langle\tau\rangle} \mathsf{c}, \mathsf{S}} \text{ (AIP-If)} \qquad \frac{\mathsf{S}(\mathsf{e})=v}{\mathsf{a} := \mathsf{e}, \mathsf{S} \xrightarrow{\mathcal{E}\langle\tau\rangle} \textbf{skip}, \mathsf{S}[\mathsf{a} \mapsto v]} \text{ (AIP-Assign)}$$

$$\frac{}{\textbf{while}(\mathsf{e})\ \mathsf{c}, \mathsf{S} \xrightarrow{\mathcal{E}\langle\tau\rangle} \textbf{if } (\mathsf{e}) \textbf{ then } \mathsf{c}; (\textbf{while}(\mathsf{e})\ \mathsf{c}) \textbf{ else skip}, \mathsf{S}} \text{ (AIP-While)}$$

$$\frac{\mathtt{val}_{\mathtt{w}}(w)=\mathsf{S}(\mathsf{e}) \quad \mathtt{loc}(w)=x_{\mathsf{v}}}{x_{\mathsf{v}} := \mathsf{e}, \mathsf{S} \xrightarrow{\mathsf{W}\langle w\rangle} \textbf{skip}, \mathsf{S}} \text{ (AIP-Write)} \qquad \frac{\mathtt{val}_{\mathtt{r}}(r)=v \quad \mathtt{loc}(r)=x_{\mathsf{v}}}{\mathsf{a} := x_{\mathsf{v}}, \mathsf{S} \xrightarrow{\mathsf{R}\langle r,w\rangle} \textbf{skip}, \mathsf{S}[\mathsf{a} \mapsto v]} \text{ (AIP-Read)}$$

$$\frac{\mathtt{val}_{\mathtt{r}}(u)=v \quad \mathtt{val}_{\mathtt{w}}(u)=v+\mathsf{S}(\mathsf{e}) \quad \mathtt{loc}(u)=x_{\mathsf{v}}}{\mathsf{a} := \textbf{FAA}(x_{\mathsf{v}}, \mathsf{e}), \mathsf{S} \xrightarrow{\mathsf{U}\langle u,w\rangle} \textbf{skip}, \mathsf{S}[\mathsf{a} \mapsto v]} \text{ (AIP-FAA)} \qquad \frac{}{\textbf{mfence}, \mathsf{S} \xrightarrow{\mathsf{MF}\langle mf\rangle} \textbf{skip}, \mathsf{S}} \text{ (AIP-MFence)}$$

$$\frac{\mathtt{val}_{\mathtt{r}}(r)=v \neq \mathsf{S}(\mathsf{e}_1) \quad \mathtt{loc}(r)=x_{\mathsf{v}}}{\mathsf{a} := \textbf{CAS}(x_{\mathsf{v}}, \mathsf{e}_1, \mathsf{e}_2), \mathsf{S} \xrightarrow{\mathsf{R}\langle r,w\rangle} \textbf{skip}, \mathsf{S}[\mathsf{a} \mapsto v]} \text{ (AIP-CAS0)}$$

$$\frac{\mathtt{val}_{\mathtt{r}}(u)=v=\mathsf{S}(\mathsf{e}_1) \quad \mathtt{val}_{\mathtt{w}}(u)=\mathsf{S}(\mathsf{e}_2) \quad \mathtt{loc}(u)=x_{\mathsf{v}}}{\mathsf{a} := \textbf{CAS}(x_{\mathsf{v}}, \mathsf{e}_1, \mathsf{e}_2), \mathsf{S} \xrightarrow{\mathsf{U}\langle u,w\rangle} \textbf{skip}, \mathsf{S}[\mathsf{a} \mapsto v]} \text{ (AIP-CAS1)}$$

$$\frac{\mathtt{tid}(w_s)=\tau_{\mathsf{s}} \qquad \mathtt{loc}(w_s)=x_{\mathsf{s}} \qquad \mathtt{loc}(w_v)=x_{\mathsf{v}} \qquad \mathtt{val}_{\mathtt{w}}(w_s)=\mathtt{val}_{\mathtt{w}}(w_v)}{\langle \textbf{pick } x; x_{\mathsf{s}} := x_{\mathsf{v}} \rangle, \mathsf{S} \xrightarrow{\mathsf{S}\langle w_s, w_v\rangle} \textbf{skip}, \mathsf{S}} \text{ (AIP-AtomS)}$$

$$\frac{}{\langle \textbf{persist } X \rangle, \mathsf{S} \xrightarrow{\mathsf{SFL}\langle \tau, X\rangle} \textbf{skip}, \mathsf{S}} \text{ (AIP-AtomFL)}$$

$$\frac{\forall i.\ \mathtt{tid}(w_p^i){=}\tau_\mathsf{p} \wedge \mathtt{loc}(w_p^i){=}x_\mathsf{p}^i \wedge \mathtt{loc}(w_s^i){=}x_\mathsf{s}^i \wedge \mathtt{val}_\mathtt{w}(w_p^i){=}\mathtt{val}_\mathtt{w}(w_s^i)}{\langle\overline{x_\mathsf{p}^i := x_\mathsf{s}^i}\rangle, \mathsf{S} \xrightarrow{\mathsf{P}\langle\overline{w_p^i, w_s^i}\rangle} \mathbf{skip}, \mathsf{S}} \text{ (AIP-AtomP)}$$

$$\frac{\mathsf{C}(\tau), \mathsf{S} \xrightarrow{\lambda} \mathsf{c}, \mathsf{S}' \quad \mathtt{tid}(\lambda){=}\tau}{\mathsf{C}, \mathsf{S} \xrightarrow{\lambda} \mathsf{C}[\tau \mapsto \mathsf{c}], \mathsf{S}'} \text{ (AIP-Par)}$$

where:

$$\mathtt{tid}(\lambda) \triangleq \begin{cases} \tau & \text{if } \lambda{=}\mathcal{E}\langle\tau\rangle \\ \tau_\mathsf{s} & \text{if } \lambda{=}\mathsf{S}\langle -, -\rangle \\ \tau_\mathsf{p} & \text{if } \lambda{=}\mathsf{P}\langle\overline{-,-}\rangle \\ \tau & \text{if } \lambda{=}\mathsf{SFL}\langle\tau, X\rangle \\ \mathtt{tid}(e) & \text{if } \lambda \in \{\mathsf{R}\langle e, -\rangle, \mathsf{U}\langle e, -\rangle, \mathsf{W}\langle e\rangle, \mathsf{MF}\langle e\rangle\} \\ \text{undefined} & \text{otherwise} \end{cases}$$

Storage Subsystem. $\text{AIMem} \times \text{AIBMap} \xrightarrow{\text{ALabels}} \text{AIMem} \times \text{AIBMap}$

$$\frac{\mathtt{tid}(w){=}\tau \quad IB(\tau){=}ib}{IM, IB \xrightarrow{\mathsf{W}\langle w\rangle} IM, IB[\tau \mapsto ib.w]} \text{ (AIM-Write)}$$

$$\frac{\mathtt{tid}(r){=}\tau \quad IB(\tau){=}ib \quad \mathtt{rd}(IM, ib, x_\mathsf{v}){=}e}{IM, IB \xrightarrow{\mathsf{R}\langle r,e\rangle} IM, IB} \text{ (AIM-Read)}$$

$$\frac{\mathtt{tid}(mf){=}\tau \quad IB(\tau){=}\epsilon}{IM, IB \xrightarrow{\mathsf{MF}\langle mf\rangle} IM, IB} \text{ (AIM-MF)}$$

$$\frac{\mathtt{tid}(u){=}\tau \quad IB(\tau){=}\epsilon \quad \mathtt{rd}(IM, \epsilon, x_\mathsf{v}){=}e}{IM, IB \xrightarrow{\mathsf{U}\langle u,e\rangle} IM[x_\mathsf{v} \mapsto u], IB} \text{ (AIM-RMW)}$$

$$\frac{IB(\tau){=}ib}{IM, IB \xrightarrow{\mathsf{SFL}\langle\tau,X\rangle} IM, IB[\tau \mapsto ib.X]} \text{ (AIM-AtomFL)}$$

$$\frac{\mathtt{tid}(w_s){=}\tau_\mathsf{s} \quad IB(\tau_\mathsf{s}){=}\epsilon \quad \mathtt{loc}(w_s){=}x_\mathsf{s} \quad \mathtt{rd}(IM, \epsilon, x_\mathsf{v}){=}w_v}{IM, IB \xrightarrow{\mathsf{S}\langle w_s, w_v\rangle} IM[x_\mathsf{s} \mapsto w_s], IB} \text{ (AIM-AtomS)}$$

$$\frac{\forall i.\ \mathtt{tid}(w_p^i){=}\tau_\mathsf{p} \quad IB(\tau_\mathsf{p}){=}\epsilon \quad \forall i.\ \mathtt{loc}(w_p^i){=}x_\mathsf{p}^i \quad \forall i.\ \mathtt{rd}(IM, \epsilon, x_\mathsf{s}^i){=}w_s^i}{IM, IB \xrightarrow{\mathsf{P}\langle\overline{w_p^i, w_s^i}\rangle} IM[\overline{x_\mathsf{p} \mapsto w_p^i}], IB} \text{ (AIM-AtomP)}$$

$$\frac{IB(\tau){=}w.ib \quad \mathtt{loc}(w){=}x_\mathsf{o} \quad x_\mathsf{o} \in \{x_\mathsf{v}, x_\mathsf{s}, x_\mathsf{p}\}}{IM, IB \xrightarrow{\mathsf{B}\langle w\rangle} IM[x_\mathsf{o} \mapsto w], IB[\tau \mapsto ib]} \text{ (AIM-BPropW)}$$

$$\frac{IB(\tau){=}X.ib \quad X{=}\overline{x^i} \subseteq \text{Loc} \quad \forall i.\ \mathtt{loc}(w^i){=}x_\mathsf{s}^i \wedge \mathtt{val}_\mathtt{w}(w^i){=}IM(x_\mathsf{v}^i) \wedge \mathtt{tid}(w^i){=}\tau}{IM, IB \xrightarrow{\mathsf{B}\langle E\rangle} IM[\overline{x_\mathsf{s}^i \mapsto w^i}], IB[\tau \mapsto ib]} \text{ (AIM-BPropFL)}$$

where given $x_\mathsf{o} \in \{x_\mathsf{v}, x_\mathsf{s}, x_\mathsf{p}\}$:

$$\mathtt{rd}(IM, ib, x_o) \triangleq \begin{cases} w & \text{if } \exists ib_1, ib_2.\ ib{=}ib_1.w.ib_2 \wedge\ w \in W_{x_\mathsf{o}} \wedge W_{x_\mathsf{o}} \cap ib_2{=}\emptyset \\ IM(x_\mathsf{o}) & \text{otherwise} \end{cases}$$

Ix86$_{\mathsf{sim}}$ Event-Annotated Operational Semantics

$$\frac{\mathsf{C},\mathsf{S} \xrightarrow{\mathcal{E}\langle\tau\rangle} \mathsf{C}',\mathsf{S}'}{\Delta \vdash \mathsf{C},\mathsf{S},IM,IB,\pi \Rightarrow \mathsf{C}',\mathsf{S}',IM,IB,\pi} \text{ (AI-SilentP)}$$

$$\frac{IM,IB \xrightarrow{\lambda} IM',IB' \quad \mathsf{fresh}(\lambda,\pi) \quad \lambda \in \{\mathsf{B}\langle e\rangle, \mathsf{B}\langle E\rangle \mid e \in W \wedge E \subseteq W\}}{\Delta \vdash \mathsf{C},\mathsf{S},IM,IB,\pi \Rightarrow \mathsf{C},\mathsf{S},IM',IB',\pi.\mathsf{B}\langle e\rangle} \text{ (AI-SilentS)}$$

$$\frac{\mathsf{C},\mathsf{S} \xrightarrow{\lambda} \mathsf{C}',\mathsf{S}' \quad IM,IB \xrightarrow{\lambda} IM',IB' \quad \mathsf{fresh}(\lambda,\pi)}{\Delta \vdash \mathsf{C},\mathsf{S},IM,IB,\pi \Rightarrow \mathsf{C}',\mathsf{S}',IM',IB',\pi.\lambda} \text{ (AI-Step)}$$

$$\frac{IM'{=}IM\,[\overline{x_{\mathsf{s}} \mapsto x_{\mathsf{p}}}]\,[\overline{x_{\mathsf{v}} \mapsto x_{\mathsf{p}}}] \quad \Delta{=}(\mathsf{C}_0,\mathbf{rec}) \quad IB_0 \triangleq \lambda\tau.\epsilon \quad M \in \mathtt{M}(IM')}{\Delta \vdash \mathsf{C},\mathsf{S},IM,IB,\pi \Rightarrow [\![\mathbf{rec}(\mathsf{C}_0,M)]\!],\mathsf{S}_0,IM',IB_0,\epsilon} \text{ (AI-Crash)}$$

where

$$\mathtt{M}(IM) \triangleq \begin{cases} \left\{ M \;\middle|\; \begin{array}{l} \forall x,e,e'.\ IM(x_{\mathsf{p}}){=}e \wedge M(x){=}e' \Rightarrow \\ \quad \mathtt{val}_{\mathtt{w}}(e){=}\mathtt{val}_{\mathtt{w}}(e') \end{array} \right\} & \text{if } \forall x.\ IM(x_{\mathsf{v}}){=}IM(x_{\mathsf{s}}){=}IM(x_{\mathsf{p}}) \\ \emptyset & \text{otherwise} \end{cases}$$

and $\mathsf{fresh}(\mathsf{B}\langle E\rangle,\pi) \overset{\text{def}}{\Leftrightarrow} \mathsf{B}\langle E\rangle \notin \pi \wedge \forall e \in E.\ \mathsf{B}\langle e\rangle \notin \pi \wedge \forall \mathsf{B}\langle E'\rangle \in \pi.\ e \notin E'$.

Definition 7.

$$\mathsf{wf}(IM,IB,\pi) \overset{\text{def}}{\Leftrightarrow} \mathtt{bmap}(IB_0,\pi) = IB \wedge \mathsf{wfp}(\pi)$$

Lemma 3. *For all* $\mathsf{C},\mathsf{C}',\mathsf{S},\mathsf{S}',IM,IM',\pi$, *if* $\mathsf{C},\mathsf{S},IM,IB_0,\epsilon \Rightarrow^* \mathsf{C}',\mathsf{S}',IM',-,-$, *then there exists* IM,IM' *such that:* $\mathsf{C},\mathsf{S},\mathsf{IM},\mathsf{B}_0 \Rightarrow^* \mathsf{C}',\mathsf{S}',\mathsf{IM}',-$ *and for all* x *and* $x_{\mathsf{o}} \in \{x_{\mathsf{v}},x_{\mathsf{s}},x_{\mathsf{p}}\}$: $\mathsf{IM}(x_{\mathsf{o}}){=}\mathtt{val}_{\mathtt{w}}(IM(x_{\mathsf{o}})) \wedge \mathsf{IM}'(x_{\mathsf{o}}){=}\mathtt{val}_{\mathtt{w}}(IM'(x_{\mathsf{o}}))$.

Proof. Follows by straightforward induction on the structure of $\Rightarrow^*$. □

Lemma 4. *For all* **rec**, $\mathsf{C},\mathsf{C}',\mathsf{S},\mathsf{S}',IM,IM',IB,IB',\pi,\pi'$:

(1) $\mathsf{wf}(IM_0,IB_0,\epsilon)$
(2) if $\mathbf{rec} \vdash \mathsf{C},\mathsf{S},IM,IB,\pi \Rightarrow \mathsf{C}',\mathsf{S}',IM',IB',\pi'$ *and* $\mathsf{wf}(IM,IB,\pi)$, *then* $\mathsf{wf}(IM',IB',\pi')$
(3) if $\mathbf{rec} \vdash \mathsf{C},\mathsf{S}_0,IM_0,IB_0,\epsilon \Rightarrow^* \mathsf{C}_{\mathbf{skip}},\mathsf{S},IM,IB,\pi$, *then* $\mathsf{wf}(IM,IB,\pi)$

Proof. The first part simply follows from the definitions of IM_0, IB_0. The second part follows straightforwardly by induction on the structure of $\Rightarrow$. The last part follows from the previous two parts and induction on the length of $\Rightarrow^*$. □

Notation. In what follows we write WU for $W \cup U$.

A.3 Definitions

Definition 8.

$$(M, PB, B, \pi_d, \pi_l) \approx (IM, IB, \pi) \overset{\text{def}}{\Leftrightarrow} (M, PB, \pi_d, \pi_l) \approx IM \wedge B \approx IB \wedge (\pi_d, \pi_l) \approx \pi$$

$$\begin{array}{l}(M, PB, \pi_d, \pi_l) \approx IM \overset{\text{def}}{\Leftrightarrow} \\ \quad \forall x, e'.\ IM(x_{\mathsf{v}})\!=\!e' \wedge \mathsf{leqMaxELoc}(\pi_d, \pi_l, x) \Rightarrow \\ \qquad \exists e \in WU_x.\ \mathtt{val}_{\mathtt{w}}(e)\!=\!\mathtt{val}_{\mathtt{w}}(e') \wedge \mathsf{maxPW}(M, PB, \pi_l, x, e) \\ \quad \wedge \forall x, e'.\ IM(x_{\mathsf{s}})\!=\!e' \wedge \mathsf{leqMaxE}(\pi_d, \pi_l) \wedge \mathsf{hasMaxELoc}(\pi_d, \pi_l, x) \Rightarrow \\ \qquad \exists e \in WU_x.\ \mathtt{val}_{\mathtt{w}}(e)\!=\!\mathtt{val}_{\mathtt{w}}(e') \wedge \mathsf{maxPW}(M, PB, \pi_l, x, e) \\ \quad \wedge \forall x, e'.\ IM(x_{\mathsf{p}})\!=\!e' \Rightarrow \\ \qquad \exists e \in WU_x.\ \mathtt{val}_{\mathtt{w}}(e)\!=\!\mathtt{val}_{\mathtt{w}}(e') \wedge \mathsf{maxPW}(M, PB, \pi_l, x, e)\end{array}$$

$$\begin{array}{l}\mathsf{leqMaxELoc}(\pi_d, \pi_l, x) \overset{\text{def}}{\Leftrightarrow} \exists m.\ \mathtt{loc}(m)\!=\!x \wedge \mathsf{maxELoc}(m, \pi_d.\pi_l) \\ \qquad\qquad \wedge \forall \lambda \in \{\mathsf{B}\langle m\rangle, \mathsf{U}\langle m, -\rangle\}.\ \lambda \in \pi_d \Rightarrow \pi_d\!=\!-.\lambda\end{array}$$

$$\begin{array}{l}\mathsf{leqMaxE}(\pi_d, \pi_l) \overset{\text{def}}{\Leftrightarrow} \exists m.\ \mathsf{maxE}(m, \pi_d.\pi_l) \\ \qquad\qquad \wedge \forall \lambda \in \{\mathsf{B}\langle m\rangle, \mathsf{U}\langle m, -\rangle\}.\ \lambda \in \pi_d \Rightarrow \pi_d\!=\!-.\lambda\end{array}$$

$$\begin{array}{l}\mathsf{hasMaxELoc}(\pi_d, \pi_l, x) \overset{\text{def}}{\Leftrightarrow} \exists m.\ \mathtt{loc}(m)\!=\!x \wedge \mathsf{maxELoc}(m, \pi_d.\pi_l) \\ \qquad\qquad \wedge \exists \lambda \in \{\mathsf{B}\langle m\rangle, \mathsf{U}\langle m, -\rangle\}.\ \lambda \in \pi_d\end{array}$$

$$\begin{array}{l}\mathsf{maxPW}(M, PB, \pi_l, x, e) \overset{\text{def}}{\Leftrightarrow} \\ \quad (M(x)\!=\!e \wedge \forall e' \in PB \cap WU_x.\ \mathsf{PB}\langle e'\rangle \notin \pi_l) \\ \quad \vee (\exists PB'.\ PB\!=\!-.e.PB' \wedge \mathsf{PB}\langle e\rangle \in \pi_l \wedge \forall e' \in PB \cap WU_x.\ \mathsf{PB}\langle e'\rangle \notin \pi_l)\end{array}$$

$$\begin{array}{l}\mathsf{maxELoc}(e, \pi) \overset{\text{def}}{\Leftrightarrow} \mathsf{PB}\langle e\rangle \in \pi \\ \qquad\qquad \wedge \nexists e' \in WU.\ \mathsf{PB}\langle e\rangle \prec_\pi \mathsf{PB}\langle e'\rangle \wedge \mathtt{loc}(e)\!=\!\mathtt{loc}(e')\end{array}$$

$$\begin{array}{l}\mathsf{maxE}(e, \pi) \overset{\text{def}}{\Leftrightarrow} \mathsf{maxELoc}(e, \pi) \\ \qquad\qquad \wedge\ \forall e' \neq e.\ \forall \lambda \in \{\mathsf{B}\langle e\rangle, \mathsf{U}\langle e, -\rangle\}, \lambda' \in \{\mathsf{B}\langle e'\rangle, \mathsf{U}\langle e', -\rangle\}. \\ \qquad\qquad\quad \mathsf{maxELoc}(e', \pi) \Rightarrow \lambda \not\leftarrow_\lambda \lambda'\end{array}$$

$$\begin{array}{l}(\pi_d, \pi_l) \approx \pi \overset{\text{def}}{\Leftrightarrow} \\ \quad \pi_d\!=\!\pi\!=\!\epsilon \\ \quad \vee\ \exists w_1, w_2 \in W, \pi'_d, \pi'.\ \pi_d\!=\!\mathsf{W}\langle w_1\rangle.\pi'_d \wedge \pi\!=\!\mathsf{W}\langle w_2\rangle.\pi' \wedge w_1 \approx w_2 \wedge \pi'_d \approx \pi' \\ \quad \vee\ \exists r_1, r_2 \in R, \pi'_d, \pi'.\ \pi_d\!=\!\mathsf{R}\langle r_1, -\rangle.\pi'_d \wedge \pi\!=\!\mathsf{R}\langle r_2, -\rangle.\pi' \wedge r_1 \approx_r r_2 \wedge \pi'_d \approx \pi' \\ \quad \vee\ \exists u_1, u_2 \in U, \pi'_d, \pi', \pi''.\ \pi_d\!=\!\mathsf{U}\langle u_1, -\rangle.\pi'_d \wedge \pi\!=\!\mathsf{U}\langle u_2, -\rangle.\pi''.\pi' \wedge u_1 \approx_u u_2 \\ \qquad\qquad \wedge\ \pi'_d \approx \pi' \wedge \mathsf{path}_{\mathsf{sa}}(u_1, \pi_d.\pi_l, \pi'') \\ \quad \vee\ \exists mf \in MF, \pi'_d, \pi'.\ \pi_d\!=\!\mathsf{MF}\langle mf\rangle.\pi'_d \wedge \pi\!=\!\mathsf{MF}\langle mf\rangle.\pi' \wedge \pi'_d \approx \pi' \\ \quad \vee\ \exists sf \in SF, \pi'_d.\ \pi_d\!=\!\mathsf{SF}\langle sf\rangle.\pi'_d \wedge \pi'_d \approx \pi \\ \quad \vee\ \exists X, \pi'_d, \pi'.\ \exists fl \in FL_X.\ \pi_d\!=\!\mathsf{FL}\langle fl\rangle.\pi'_d \wedge \pi\!=\!\mathsf{SFL}\langle \mathtt{tid}(fl), X\rangle.\pi' \wedge \pi'_d \approx \pi' \\ \quad \vee\ \exists \pi'_d, \pi', \pi''.\ \exists w_1, w_2 \in W.\ \pi_d\!=\!\mathsf{B}\langle w_1\rangle.\pi'_d \wedge \pi\!=\!\mathsf{B}\langle w_2\rangle.\pi''.\pi' \wedge w_1 \approx_w w_2 \\ \qquad\qquad \wedge\ \pi'_d \approx \pi' \wedge \mathsf{path}_{\mathsf{sa}}(w_1, \pi_d.\pi_l, \pi'') \\ \quad \vee\ \exists sf \in SF, \pi'_d.\ \pi_d\!=\!\mathsf{B}\langle sf\rangle.\pi'_d \wedge \underline{\pi'_d} \approx \pi \\ \quad \vee\ \exists X, \overline{x^i}, \pi'_d, \pi', \pi''.\ \exists fl \in FL_X, \overline{w^i_s}.\ X\!=\!\overline{x^i} \wedge \pi_d\!=\!\mathsf{B}\langle fl\rangle.\pi'_d \wedge \pi\!=\!\mathsf{B}\langle \overline{w^i_s}\rangle.\pi''.\pi' \\ \qquad\qquad \wedge\ \forall i.\ \mathtt{tid}(w^i_s)\!=\!\mathtt{tid}(fl) \wedge \mathtt{loc}(w^i_s)\!=\!x^i_{\mathsf{s}} \\ \qquad\qquad \wedge\ \pi'_d \approx \pi' \wedge \mathsf{path}_{\mathsf{sa}}(fl, \pi_d.\pi_l, \pi'') \\ \quad \vee\ \exists \pi'_d, e.\ \pi_d\!=\!\mathsf{PB}\langle e\rangle.\pi'_d \wedge \pi'_d \approx \pi\end{array}$$

$$\begin{aligned}
\mathsf{path}_{\mathsf{sa}}(e,\pi,\pi') \overset{\text{def}}{\Leftrightarrow}\ & \neg\mathsf{maxELoc}(e,\pi) \wedge \pi'{=}\epsilon \\
& \vee\ \mathsf{maxELoc}(e,\pi) \wedge \neg\mathsf{maxE}(e,\pi) \wedge e \notin FL \wedge \exists x, w_s, w_v.\\
& \qquad \mathtt{loc}(e){=}x \wedge \pi'{=}\mathsf{S}\langle w_s, w_v\rangle \\
& \qquad \wedge\ \mathtt{loc}(w_v){=}x_{\mathsf{v}} \wedge \mathtt{loc}(w_s){=}x_{\mathsf{s}} \wedge \mathtt{val}_{\mathtt{w}}(w_s){=}\mathtt{val}_{\mathtt{w}}(w_v) \wedge \mathtt{tid}(w_s){=}\tau_{\mathsf{s}} \\
& \vee\ \mathsf{maxELoc}(e,\pi) \wedge \neg\mathsf{maxE}(e,\pi) \wedge e \in \underline{FL} \wedge \pi'{=}\epsilon \\
& \vee\ \mathsf{maxE}(e,\pi) \wedge e \notin FL \wedge \exists x, \overline{y^j}, w_s, w_v, \overline{w_p^j}, \overline{w_s^j}.\\
& \qquad \mathtt{loc}(e){=}x \wedge \pi'{=}\mathsf{S}\langle w_s, w_v\rangle.\mathsf{P}\langle \overline{w_p^j}, \overline{w_s^j}\rangle \\
& \qquad \wedge\ \mathtt{loc}(w_v){=}x_{\mathsf{v}} \wedge \mathtt{loc}(w_s){=}x_{\mathsf{s}} \wedge \mathtt{val}_{\mathtt{w}}(w_s){=}\mathtt{val}_{\mathtt{w}}(w_v) \wedge \mathtt{tid}(w_s){=}\tau_{\mathsf{s}} \\
& \qquad \wedge\ \mathsf{Loc}{=}\overline{y^j} \wedge \forall i.\ \mathtt{loc}(w_s^j){=}y_{\mathsf{s}}^j \wedge \mathtt{loc}(w_p^j){=}y_{\mathsf{p}}^j \wedge \mathtt{val}_{\mathtt{w}}(w_p^j){=}\mathtt{val}_{\mathtt{w}}(w_s^j) \wedge \mathtt{tid}(w_p^j){=}\tau_{\mathsf{p}} \\
& \vee\ \mathsf{maxE}(e,\pi) \wedge e \in FL \wedge \exists \overline{x^i}, \overline{w_p^i}, \overline{w_s^i}.\\
& \qquad \mathsf{Loc}{=}\overline{x^i} \wedge \pi'{=}\mathsf{P}\langle \overline{w_p^i}, \overline{w_s^i}\rangle \wedge\ \forall i.\mathtt{loc}(w_s^i){=}x_{\mathsf{s}}^i \wedge \mathtt{loc}(w_p^i){=}x_{\mathsf{p}}^i \wedge \mathtt{val}_{\mathtt{w}}(w_p^i){=}\mathtt{val}_{\mathtt{w}}(w_s^i) \wedge \mathtt{t}
\end{aligned}$$

$$w_1 \approx_w w_2 \overset{\text{def}}{\Leftrightarrow} \exists x.\ w_1 \in W_x \wedge w_2 \in W_{x_{\mathtt{v}}} \wedge \mathtt{tid}(w_1){=}\mathtt{tid}(w_2) \wedge \mathtt{val}_{\mathtt{w}}(w_1){=}\mathtt{val}_{\mathtt{w}}(w_2)$$

$$r_1 \approx_r r_2 \overset{\text{def}}{\Leftrightarrow} \exists x.\ r_1 \in R_x \wedge r_2 \in R_{x_{\mathtt{v}}} \wedge \mathtt{tid}(r_1){=}\mathtt{tid}(r_2) \wedge \mathtt{val}_{\mathtt{r}}(r_1){=}\mathtt{val}_{\mathtt{r}}(r_2)$$

$$\begin{aligned}
u_1 \approx_u u_2 \overset{\text{def}}{\Leftrightarrow}\ & \exists x.\ u_1 \in U_x \wedge u_2 \in U_{x_{\mathtt{v}}} \wedge \mathtt{tid}(u_1){=}\mathtt{tid}(u_2) \\
& \qquad \wedge\ \mathtt{val}_{\mathtt{r}}(u_1){=}\mathtt{val}_{\mathtt{r}}(u_2) \wedge \mathtt{val}_{\mathtt{w}}(u_1){=}\mathtt{val}_{\mathtt{w}}(u_2)
\end{aligned}$$

$$B \approx IB \overset{\text{def}}{\Leftrightarrow} \forall \tau.\ B(\tau) \approx IB(\tau)$$

$$\begin{aligned}
b \approx ib \overset{\text{def}}{\Leftrightarrow}\ & b{=}ib{=}\epsilon \\
& \vee\ \exists sf \in SF, b'.\ b{=}sf.b' \wedge b' \approx ib \\
& \vee\ \exists x, b', ib'.\ \exists w_1, w_2 \in W.\ b{=}w_1.b' \wedge ib{=}w_2.ib' \wedge w_1 \approx_w w_2 \wedge b' \approx ib' \\
& \vee\ \exists X, b', ib'.\ \exists fl \in FL_X.\ b{=}fl.b' \wedge ib{=}X.ib' \wedge b' \approx ib'
\end{aligned}$$

A.4 Proof

Lemma 5. *For all* $\Delta, \mathsf{C}, \mathsf{S}, M, PB, B, \pi_d, \pi_l, \mathsf{C}', \mathsf{S}', M', PB', B', \pi_d', \pi_l', IM, IB, \pi$*:*
if:

- $(M, PB, B, \pi_d, \pi_l) \approx (IM, IB, \pi)$
- $\mathsf{wf}(M, PB, B, \pi_d) \wedge \mathsf{wfp}(\pi_d.\pi_l) \wedge \mathsf{wf}(IM, IB, \pi) \wedge IB(\tau_{\mathsf{s}}){=}IB(\tau_{\mathsf{p}}){=}\epsilon$
- $\Delta \vdash \mathsf{C}, \mathsf{S}, M, PB, B, \pi_d, \pi_l \Rightarrow \mathsf{C}', \mathsf{S}', M', PB', B', \pi_d', \pi_l'$

then there exists IM', IB', π' *such that:*

- $(M', PB', B', \pi_d', \pi_l') \approx (IM', IB', \pi')$
- $\mathsf{wf}(M', PB', B', \pi_d') \wedge \mathsf{wfp}(\pi_d'.\pi_l') \wedge \mathsf{wf}(IM', IB', \pi') \wedge IB'(\tau_{\mathsf{s}}){=}IB'(\tau_{\mathsf{p}}){=}\epsilon$
- $\Delta \vdash [\![\mathsf{C}]\!], \mathsf{S}, IM, IB, \pi \Rightarrow^* [\![\mathsf{C}']\!], \mathsf{S}', IM', IB', \pi'$

Proof. Pick arbitrary $\mathsf{C}, \mathsf{S}, M, PB, B, \pi_d, \pi_l, \mathsf{C}', \mathsf{S}', M', PB', B', \pi_d', \pi_l', IM, IB, \pi$ such that:

$$(M, PB, B, \pi_d, \pi_l) \approx (IM, IB, \pi) \tag{1}$$

$$\mathsf{wf}(M, PB, B, \pi_d) \text{ and } \mathsf{wf}(IM, IB, \pi) \text{ and } \mathsf{wfp}(\pi_d.\pi_l) \tag{2}$$

$$IB(\tau_{\mathsf{s}}){=}IB(\tau_{\mathsf{p}}){=}\epsilon \tag{3}$$

$$\Delta \vdash \mathsf{C}, \mathsf{S}, M, PB, B, \pi_d, \pi_l \Rightarrow \mathsf{C}', \mathsf{S}', M', PB', B', \pi_d', \pi_l' \tag{4}$$

From Lemma 1 and (4) we have $\mathsf{wf}(M', PB', B', \pi_d') \wedge \mathsf{wfp}(\pi_d'.\pi_l')$. In what follows we demonstrate that there exist IM', IB', π' such that $\Delta \vdash [\![\mathsf{C}]\!], \mathsf{S}, IM, IB, \pi \Rightarrow^* [\![\mathsf{C}']\!], \mathsf{S}', IM', IB', \pi'$, and thus from

Lemma 4 we have $\mathsf{wf}(IM', IB', \pi')$. To establish the remaining conjuncts, we proceed by induction on the structure of $\Rightarrow$.

1. Case (A-SilentP)
From the premise of (A-SilentP) we then know $\mathsf{C} \xrightarrow{\mathcal{E}\langle\tau\rangle} \mathsf{C}'$ for some $\tau \notin \{\tau_\mathsf{s}, \tau_\mathsf{p}\}$ and that $M'{=}M$, $B'{=}B$, $PB'{=}PB$, $\pi'_d{=}\pi_d$ and $\pi'_l{=}\pi_l$. It is then straightforward to demonstrate that $[\![\mathsf{C}]\!], \mathsf{S} \xrightarrow{\mathcal{E}\langle\tau\rangle} [\![\mathsf{C}']\!], \mathsf{S}'$. As such, from (AI-SilentP) we have $\Delta \vdash [\![\mathsf{C}]\!], \mathsf{S}, IM, IB, \pi \Rightarrow [\![\mathsf{C}']\!], \mathsf{S}', IM, IB, \pi$. That is, there exists IM', IB', π' such that $IM'{=}IM$, $IB'{=}IB$, $\pi'{=}\pi$ and $\Delta \vdash [\![\mathsf{C}]\!], IM, IB, \pi \Rightarrow [\![\mathsf{C}']\!], IM', IB', \pi'$. Moreover, from (1) and the definition of IB' we have $IB'(\tau_\mathsf{s}){=}IB'(\tau_\mathsf{p}){=}\epsilon$, as required. Finally, from (1) we have $(M', PB', B', \pi'_d, \pi'_l) \approx (IM', IB', \pi')$, as required.

2. Case (A-Step), $\lambda{=}\mathsf{W}\langle w\rangle$ for some $w \in W$
From (A-Step) and (AM-Write) we then know that there exists τ, b such that $\mathtt{tid}(w){=}\tau \notin \{\tau_\mathsf{s}, \tau_\mathsf{p}\}$ and $B(\tau){=}b$, and that $M'{=}M$, $PB'{=}PB$, $B'{=}B[\tau \mapsto b.w]$, $\mathsf{S}'{=}\mathsf{S}$, $\pi'_d{=}\pi_d.\lambda$, $\pi_l{=}\lambda.\pi'_l$ and $\mathsf{C}, \mathsf{S} \xrightarrow{\mathsf{W}\langle w\rangle} \mathsf{C}', \mathsf{S}$. Let $\mathtt{loc}(w){=}x$ and $\mathtt{val}_\mathtt{w}(w){=}v$. Pick a fresh event $w' \in W$ such that $\mathtt{loc}(w'){=}x_\mathsf{v}$, $\mathtt{tid}(w'){=}\tau$ and $\mathtt{val}_\mathtt{w}(w'){=}v$, and let $\lambda'{=}\mathsf{W}\langle w'\rangle$. It is then straightforward to demonstrate that $[\![\mathsf{C}]\!], \mathsf{S} \xrightarrow{\lambda'} [\![\mathsf{C}']\!], \mathsf{S}'$. Let $IB(\tau){=}ib$ and let $IM'{=}IM$, $IB'{=}IB[\tau \mapsto ib.w']$, $\pi'{=}\pi.\lambda'$. From (AIM-Write) we then know $IM, IB \xrightarrow{\lambda'} IM', IB'$. As such, from (AI-Step) we have $\Delta \vdash [\![\mathsf{C}]\!], \mathsf{S}, IM, IB, \pi \Rightarrow [\![\mathsf{C}']\!], \mathsf{S}', IM', IB', \pi'$. Moreover, from (1) and the definition of IB' we have $IB'(\tau_\mathsf{s}){=}IB'(\tau_\mathsf{p}){=}\epsilon$, as required. Finally, from the definition of $\approx$ and (1) we have $(M', PB', B', \pi'_d, \pi'_l) \approx (IM', IB', \pi')$, as required.

3. Case (A-Step), $\lambda{=}\mathsf{R}\langle r, e\rangle$ for some $r \in R, e \in WU$
Let $\mathtt{loc}(r){=}x$ and $\mathtt{val}_\mathtt{r}(r){=}v$. From (A-Step) and (AM-Read) we know there exists τ such that $\mathtt{tid}(r){=}\tau \notin \{\tau_\mathsf{s}, \tau_\mathsf{p}\}$, $M'{=}M$, $PB'{=}PB$, $B'{=}B'$, $\mathsf{S}'{=}\mathsf{S}[\mathsf{a} \mapsto v]$, $\pi'_d{=}\pi_d.\lambda$, $\pi_l{=}\lambda.\pi'_l$, $\mathtt{rd}(M, PB, B(\tau), x){=}e$ and $\mathsf{C} \xrightarrow{\mathsf{R}\langle r,e\rangle} \mathsf{C}'$. On the other hand, from the first two conjuncts of $(M, PB, B, \pi_d, \pi_l) \approx (IM, IB, \pi)$ in (1) we know there exists e' such that $\mathtt{rd}(IM, IB(\tau), x_\mathsf{v}){=}e'$ and that $\mathtt{val}_\mathtt{w}(e) \approx \mathtt{val}_\mathtt{w}(e'){=}v$. Pick a fresh $r' \in R$ such that $\mathtt{loc}(r'){=}x_\mathsf{v}$, $\mathtt{tid}(r'){=}\tau$ and $\mathtt{val}_\mathtt{r}(r'){=}v$, and let $\lambda'{=}\mathsf{R}\langle r', e'\rangle$. It is then straightforward to demonstrate that $[\![\mathsf{C}]\!], \mathsf{S} \xrightarrow{\lambda'} [\![\mathsf{C}']\!], \mathsf{S}'$. Let $IM'{=}IM$, $IB'{=}IB$, $\pi'{=}\pi.\lambda'$. From (AIM-Read) we then know $IM, IB \xrightarrow{\lambda'} IM', IB'$. As such, from (AI-Step) we have $\Delta \vdash [\![\mathsf{C}]\!], \mathsf{S}, IM, IB, \pi \Rightarrow [\![\mathsf{C}']\!], \mathsf{S}', IM', IB', \pi'$. Moreover, from (1) and the definition of IB' we have $IB'(\tau_\mathsf{s}){=}IB'(\tau_\mathsf{p}){=}\epsilon$, as required. Finally, from the definition of $\approx$ and (1) we have $(M', PB', B', \pi'_d, \pi'_l) \approx (IM', IB', \pi')$, as required.

4. Case (A-Step), $\lambda{=}\mathsf{U}\langle u, e\rangle$ for some $u \in U, e \in W \cup U$
The proof of this case is analogous to that of case 10 below and is thus omitted.

5. Case (A-Step), $\lambda{=}\mathsf{MF}\langle mf\rangle$ for some $mf \in MF$
From (A-Step) and (AM-MF) we then know that there exists τ such that $\mathtt{tid}(mf){=}\tau \notin \{\tau_\mathsf{s}, \tau_\mathsf{p}\}$ and $B(\tau){=}\epsilon$, and that $M'{=}M$, $PB'{=}PB$, $B'{=}B$, $\mathsf{S}'{=}\mathsf{S}$, $\pi'_d{=}\pi_d.\lambda$, $\pi_l{=}\lambda.\pi'_l$ and $\mathsf{C}, \mathsf{S} \xrightarrow{\mathsf{MF}\langle mf\rangle} \mathsf{C}'$. Pick a fresh event $mf' \in MF$ such that $\mathtt{tid}(mf'){=}\tau$ and let $\lambda'{=}\mathsf{MF}\langle mf'\rangle$. It is then straightforward to demonstrate that $[\![\mathsf{C}]\!], \mathsf{S} \xrightarrow{\lambda'} [\![\mathsf{C}']\!], \mathsf{S}'$. From (1) we then know $IB(\tau){=}\epsilon$. Let $IM'{=}IM$,

$IB'{=}IB$, $\pi'{=}\pi.\lambda'$. From (AIM-MF) we then know $IM, IB \xrightarrow{\lambda'} IM', IB'$. As such, from (AI-Step) we have $\Delta \vdash [\![\mathsf{C}]\!], \mathsf{S}, IM, IB, \pi \Rightarrow [\![\mathsf{C}']\!], \mathsf{S}', IM', IB', \pi'$. Moreover, from (1) and the definition of IB' we have $IB'(\tau_{\mathsf{s}}){=}IB'(\tau_{\mathsf{p}}){=}\epsilon$, as required. Finally, from the definition of $\approx$ and (1) we have $(M', PB', B', \pi'_d, \pi'_l) \approx (IM', IB', \pi')$, as required.

6. Case (A-Step), $\lambda{=}\mathsf{SF}\langle sf\rangle$ for some $sf \in SF$
From (A-Step) and (AM-SF) we then know that there exists τ, b such that $\mathtt{tid}(sf){=}\tau \notin \{\tau_{\mathsf{s}}, \tau_{\mathsf{p}}\}$ and $B(\tau){=}b$, and that $M'{=}M$, $PB'{=}PB$, $B'{=}B[\tau \mapsto b.sf]$, $\mathsf{S}'{=}\mathsf{S}$, $\pi'_d{=}\pi_d.\lambda$, $\pi_l{=}\lambda.\pi'_l$ and $\mathsf{C}, \mathsf{S} \xrightarrow{\mathsf{SF}\langle sf\rangle} \mathsf{C}', \mathsf{S}'$. Let $\lambda'{=}\mathcal{E}\langle\tau\rangle$. It is then straightforward to demonstrate that $[\![\mathsf{C}]\!], \mathsf{S} \xrightarrow{\lambda'} [\![\mathsf{C}']\!], \mathsf{S}'$. Let $IM'{=}IM$, $IB'{=}IB$, $\pi'{=}\pi$. As such, from (AI-SilentP) we have $\Delta \vdash [\![\mathsf{C}]\!], \mathsf{S}, IM, IB, \mathsf{S}', \pi \Rightarrow [\![\mathsf{C}']\!], IM', IB', \pi'$. Moreover, from (1) and the definition of IB' we have $IB'(\tau_{\mathsf{s}}){=}IB'(\tau_{\mathsf{p}}){=}\epsilon$, as required. Finally, from the definition of $\approx$ and (1) we have $(M', PB', B', \pi'_d, \pi'_l) \approx (IM', IB', \pi')$, as required.

7. Case (A-Step), $\lambda{=}\mathsf{FL}\langle fl\rangle$ for some $fl \in FL$
From (A-Step) and (AM-FL) we then know that there exists τ, b such that $\mathtt{tid}(fl){=}\tau \notin \{\tau_{\mathsf{s}}, \tau_{\mathsf{p}}\}$, $B(\tau){=}b$, $M'{=}M$, $PB'{=}PB$, $B'{=}B[\tau \mapsto b.fl]$, $\mathsf{S}'{=}\mathsf{S}$, $\pi'_d{=}\pi_d.\lambda$, $\pi_l{=}\lambda.\pi'_l$ and $\mathsf{C}, \mathsf{S} \xrightarrow{\mathsf{FL}\langle fl\rangle} \mathsf{C}', \mathsf{S}'$. Let $\mathtt{loc}(fl){=}x \in X$.

Let $\lambda'{=}\mathsf{SFL}\langle\mathtt{tid}(fl), X\rangle$. It is then straightforward to show that $[\![\mathsf{C}]\!] \xrightarrow{\lambda'} [\![\mathsf{C}']\!]$. Let $IB(\tau){=}ib$ and let $IM'{=}IM$, $IB'{=}IB[\tau \mapsto ib.X]$, $\pi'{=}\pi.\lambda'$. From (AIM-AtomFL) we then know $IM, IB \xrightarrow{\lambda'} IM', IB'$. As such, from (AI-Step) we have $\Delta \vdash [\![\mathsf{C}]\!], \mathsf{S}, IM, IB, \pi \Rightarrow [\![\mathsf{C}']\!], \mathsf{S}', IM', IB', \pi'$. Moreover, from (1) and the definition of IB' we have $IB'(\tau_{\mathsf{s}}){=}IB'(\tau_{\mathsf{p}}){=}\epsilon$, as required. Finally, from the definition of $\approx$ and (1) we have $(M', PB', B', \pi'_d, \pi'_l) \approx (IM', IB', \pi')$, as required.

8. Case (A-SilentS), $\lambda{=}\mathsf{B}\langle sf\rangle$ for some $sf \in SF$
From (A-SilentS) and (AM-BPropSF) we then know that there exists τ, b such that $\mathtt{tid}(sf){=}\tau \notin \{\tau_{\mathsf{s}}, \tau_{\mathsf{p}}\}$, $B(\tau){=}sf.b$, $M'{=}M$, $PB'{=}PB$, $B'{=}B[\tau \mapsto b]$, $\mathsf{S}'{=}\mathsf{S}$, $\pi'_d{=}\pi_d.\lambda$, $\pi_l{=}\lambda.\pi'_l$ and $\mathsf{C}'{=}\mathsf{C}$. From (1) and the definition of B we then know $b \approx IB(\tau)$. Let $IM'{=}IM$, $IB'{=}IB$, $\pi'{=}\pi$. We then trivially have $\Delta \vdash [\![\mathsf{C}]\!], \mathsf{S}, IM, IB, \pi \Rightarrow [\![\mathsf{C}']\!], \mathsf{S}', IM', IB', \pi'$ Moreover, from (1) and the definition of IB' we have $IB'(\tau_{\mathsf{s}}){=}IB'(\tau_{\mathsf{p}}){=}\epsilon$, as required. Finally, from the definition of $\approx$ and (1) we have $(M', PB', B', \pi'_d, \pi'_l) \approx (IM', IB', \pi')$, as required.

9. Case (A-SilentS), $\lambda{=}\mathsf{B}\langle fl\rangle$ for some $fl \in FL$
From (A-SilentS) and (AM-BPropFL) we know there exist τ, b such that $\mathtt{tid}(fl){=}\tau \notin \{\tau_{\mathsf{s}}, \tau_{\mathsf{p}}\}$, $B(\tau){=}fl.b$, $M'{=}M$, $PB'{=}PB.fl$, $B'{=}B[\tau \mapsto b]$, $\pi'_d{=}\pi_d.\lambda$, $\pi_l{=}\lambda.\pi'_l$, $\mathsf{C}'{=}\mathsf{C}$ and $\mathsf{S}'{=}\mathsf{S}$. Let $\mathtt{loc}(fl){=}x \in X{=}\overline{x^i}$. From (1) we know $B \approx IB$ and consequently that there exists ib such that $IB(\tau){=}X.ib$ and $b \approx ib$. Pick fresh $\overline{w^i_s} \in W$ such that for all i: $\mathtt{loc}(w^i_s){=}x^i_{\mathsf{s}}$, $\mathtt{tid}(w^i_s){=}\mathtt{tid}(fl)$ and $\mathtt{val}_{\mathtt{w}}(w^i_s){=}\mathsf{IM}(x^i_{\mathsf{v}})$. Let $\lambda'{=}\mathsf{B}\langle\overline{w^i_s}\rangle$. Note that $\pi'_d.\pi'_l{=}\pi_d.\pi_l$. There are three cases to consider: 1) $\neg\mathsf{maxELoc}(\pi'_d.\pi'_l, fl)$; or 2) $\mathsf{maxELoc}(\pi'_d.\pi'_l, fl) \wedge \neg\mathsf{maxE}(\pi'_d.\pi'_l, fl)$; or 3) $\mathsf{maxE}(\pi'_d.\pi'_l, fl)$.

In case (1) let $IM'{=}IM[\overline{x^i_{\mathsf{s}} \mapsto w^i_s}]$, $IB'{=}IB[\tau \mapsto ib]$, $\pi'{=}\pi.\lambda'$. From (AIM-BPropFL) we then know $IM, IB \xrightarrow{\lambda'} IM', IB'$. As such, from (AI-SilentS) we have $\Delta \vdash [\![\mathsf{C}]\!], \mathsf{S}, IM, IB, \pi \Rightarrow [\![\mathsf{C}']\!], \mathsf{S}', IM', IB', \pi'$. Note that since $b \approx ib$, from the definition of $\approx$ and (1) we have $B' \approx IB'$. Similarly, from

the definitions of π'_d, π'_l, π' and (1) we also have $(\pi'_d, \pi'_l) \approx \pi'$. Moreover, from (1) and the definition of IB' we have $IB'(\tau_\mathsf{s})=IB'(\tau_\mathsf{p})=\epsilon$, as required. Finally, in what follows we show that $(M', PB', \pi'_d, \pi'_l) \approx IM'$, thus demonstrating $(M', PB', B', \pi'_d, \pi'_l) \approx (IM', IB', \pi')$, as required.

Note that from (1) and the definitions of M', PB', π'_d, π'_l we immediately know that the first and third conjuncts of $(M', PB', \pi'_d, \pi'_l) \approx IM'$ hold. To show that the second conjunct holds, let us assume $\mathsf{leqMaxE}(\pi'_d, \pi'_l)$; i.e. pick mm such that $\mathsf{maxE}(\pi'_d.\pi'_l, mm)$ and for all λ_{mm}, if $\lambda_{mm} \in \{\mathsf{B}\langle mm\rangle, \mathsf{U}\langle mm, -\rangle\}$ and $\lambda_{mm} \in \pi'_d$ then $\pi'_d = - .\lambda_{mm}$. That is, if $\lambda_{mm} \in \pi'_d$ then $\lambda_{mm}=\lambda'$ and thus $mm=fl$; i.e. $\mathsf{maxE}(\pi'_d.\pi'_l, fl)$, contradicting the assumption of case (1). We thus conclude that $\lambda_{mm} \notin \pi'_d$. Also let us pick y such that $\mathsf{hasMaxELoc}(\pi'_d, \pi'_l, y)$ holds; i.e. pick m, e, λ_m such that $\mathsf{maxELoc}(\pi'_d.\pi'_l, m), \mathtt{loc}(m)=y, \lambda_m \in \{\mathsf{B}\langle m\rangle, \mathsf{U}\langle m, -\rangle\}, \lambda_m \in \pi'_d$ and $IM'(y_\mathsf{s})=e$. There are now two cases to consider: a) $y_\mathsf{s} \notin \overline{x^i_\mathsf{s}}$, i.e. $y \notin X$; or b) $y_\mathsf{s} \in \overline{x^i_\mathsf{s}}$, i.e. $y \in X$. In case (a) since $IM'(y_\mathsf{s})=IM(y_\mathsf{s})$, the desired result follows from (1) and the definitions of π'_d and π'_l. In case (b), we proceed as follows. Note that since $\mathsf{maxELoc}(\pi'_d.\pi'_l, m)$ and $\neg\mathsf{maxELoc}(\pi'_d.\pi'_l, fl)$ we know that $fl \neq m$. As such, since $\lambda_m \in \pi'_d$, we know that $\lambda_m \prec_{\pi'_d} \lambda'$, i.e. $\lambda_m \prec_{\pi'_d.\pi'_l} \lambda'$. Consequently, since $\mathtt{loc}(fl), \mathtt{loc}(m) \in X$, $\mathsf{PB}\langle m\rangle \in \pi'_d.\pi'_l$ (from $\mathsf{maxELoc}(\pi'_d.\pi'_l, m)$), from (2) we know that $\mathsf{PB}\langle m\rangle \prec_{\pi'_d.\pi'_l} \mathsf{PB}\langle fl\rangle$ or $\mathsf{PB}\langle fl\rangle \notin \pi'_d.\pi'_l$. The former however contradicts the definition of $\mathsf{maxELoc}(\pi'_d.\pi'_l, m)$. As such, we know that $\mathsf{PB}\langle fl\rangle \notin \pi'_d.\pi'_l$.

On the other hand, from (2), the definitions of π'_d, π'_l, and since $\mathsf{PB}\langle mm\rangle \in \pi'_d.\pi'_l$ (from the definition of $\mathsf{maxE}(\pi'_d.\pi'_l, m)$) and $\lambda_{mm} \notin \pi'_d$, we know that $\lambda_{mm} \in \pi'_l$. That is, $\lambda_m \prec_{\pi'_d.\pi'_l} \lambda' \prec_{\pi'_d.\pi'_l} \lambda_{mm}$. As such, since $\lambda' \prec_{\pi'_d.\pi'_l} \lambda_{mm}$, from the definition of $\mathsf{wfp}(.)$ and (2) we know that $\mathsf{PB}\langle fl\rangle \prec_{\pi'_d.\pi'_l} \mathsf{PB}\langle mm\rangle$. This however leads to a contradiction as earlier we established that $\mathsf{PB}\langle fl\rangle \notin \pi'_d.\pi'_l$.

In case (2) let $IM'=IM[\overline{x^i_\mathsf{s} \mapsto w^i_s}]$, $IB'=IB[\tau \mapsto ib]$, $\pi'=\pi.\lambda'$. From (AIM-BPropFL) we then know $IM, IB \xrightarrow{\lambda'} IM', IB'$. As such, from (AI-SilentS) we have $\Delta \vdash [\![\mathsf{C}]\!], \mathsf{S}, IM, IB, \pi \Rightarrow [\![\mathsf{C}']\!], \mathsf{S}', IM', IB', \pi'$. Note that since $IB(\tau_\mathsf{s})=\epsilon$ (from (3)), we also have $IB'(\tau_\mathsf{s})=\epsilon$. Moreover, $IB'(\tau_\mathsf{p})=IB(\tau_\mathsf{p})=\epsilon$. Finally, note that since $b \approx ib$, from the definition of $\approx$ and (1) we have $B' \approx IB$. Similarly, from the definitions of π'_d, π'_l, π' and (1) we also have $(\pi'_d, \pi'_l) \approx \pi'$. In what follows we show that $(M', PB', \pi'_d, \pi'_l) \approx IM'$, thus demonstrating $(M', PB', B', \pi'_d, \pi'_l) \approx (IM', IB', \pi')$, as required.

Note that from (1) and the definitions of M', PB', π'_d, π'_l we immediately know that the first and third conjuncts of $(M', PB', \pi'_d, \pi'_l) \approx IM'$ hold. To show that the second conjunct holds, let us assume $\mathsf{leqMaxE}(\pi'_d, \pi'_l)$; i.e. pick mm such that $\mathsf{maxE}(\pi'_d.\pi'_l, mm)$ and for all λ_{mm}, if $\lambda_{mm} \in \{\mathsf{B}\langle mm\rangle, \mathsf{U}\langle mm, -\rangle\}$ and $\lambda_{mm} \in \pi'_d$ then $\pi'_d = - .\lambda_{mm}$. That is, if $\lambda_{mm} \in \pi'_d$ then $\lambda_{mm}=\lambda'$ and thus $mm=fl$; i.e. $\mathsf{maxE}(\pi'_d.\pi'_l, fl)$, contradicting the assumption of case (2). We thus conclude that $\lambda_{mm} \notin \pi'_d$. Also let us pick y such that $\mathsf{hasMaxELoc}(\pi'_d, \pi'_l, y)$ holds; i.e. pick m, e, λ_m such that $\mathsf{maxELoc}(\pi'_d.\pi'_l, m), \mathtt{loc}(m)=y, \lambda_m \in \{\mathsf{B}\langle m\rangle, \mathsf{U}\langle m, -\rangle\}, \lambda_m \in \pi'_d$ and $IM'(y_\mathsf{s})=e$. There are now two cases to consider: a) $y_\mathsf{s} \notin \overline{x^i_\mathsf{s}}$, i.e. $y \notin X$; or b) $y_\mathsf{s} \in \overline{x^i_\mathsf{s}}$, i.e. $y \in X$ and there exists k such that $y_\mathsf{s}=x^k_\mathsf{s}$. In case (a) since $IM'(y_\mathsf{s})=IM(y_\mathsf{s})$, the desired result follows from (1) and the definitions of π'_d and π'_l. In case (b), since $\mathsf{maxELoc}(\pi'_d.\pi'_l, fl)$ holds, we thus know that $m=fl$ and $\lambda_m=\lambda$. Since $IM'(x^k_\mathsf{s})=w^k_s$, it thus suffices to show that there exists $e' \in WU_{x^k}$ such that $\mathtt{val}_\mathtt{w}(e')=\mathtt{val}_\mathtt{w}(w^k_s)$ and $(M'(x)=e' \wedge \forall a \in PB' \cap WU_{x^k}.\ \mathsf{PB}\langle a\rangle \notin \pi'_l)$ or $(\exists PB''.\ PB'=-.e'.PB'' \wedge \mathsf{PB}\langle e'\rangle \in \pi'_l \wedge \forall a \in PB'' \cap WU_{x^k}.\ \mathsf{PB}\langle a\rangle \notin \pi'_l)$. We then proceed as follows. Recall that $\mathtt{val}_\mathtt{w}(w^k_s)=\mathtt{val}_\mathtt{w}(w^k_v)$ and that $IM(x^k_v)=w^k_v$. As such, from the first conjunct of $(M, PB, \pi_d, \pi_l) \approx IM$ in (1) we know there exists $e' \in WU_{x^k}$ such that $\mathtt{val}_\mathtt{w}(e')=\mathtt{val}_\mathtt{w}(w^k_v)$ and $(M(x)=e' \wedge \forall a \in PB \cap WU_{x^k}.\ \mathsf{PB}\langle a\rangle \notin \pi_l)$ or $(\exists PB''.\ PB=-.e'.PB'' \wedge \mathsf{PB}\langle e'\rangle \in \pi_l \wedge \forall a \in PB'' \cap WU_{x^k}.\ \mathsf{PB}\langle a\rangle \notin \pi_l)$. Consequently, from the definitions of M', PB', π'_l and since $\mathtt{val}_\mathtt{w}(w^k_s)=\mathtt{val}_\mathtt{w}(w^k_v)$, we know there exists

$e' \in WU_{x^k}$ such that $\mathtt{val_w}(e')=\mathtt{val_w}(w_s^k)$ and $(M'(x)=e' \wedge \forall a \in PB' \cap WU_{x^k}.\ \mathsf{PB}\langle a\rangle \notin \pi_l') \text{ or } (\exists PB''.\ PB'=-.e'.PB'' \wedge \mathsf{PB}\langle e'\rangle \in \pi_l' \wedge \forall a \in PB'' \cap WU_{x^k}.\ \mathsf{PB}\langle a\rangle \notin \pi_l')$, as required.

In case (3) let $IM''=IM[\overline{x_\mathsf{s}^i \mapsto w_s^i}]$ and $IB'=IB[\tau \mapsto ib]$. When Loc=$\overline{y}$, For each $y \in \overline{y}$=Loc: let $IM''(y_\mathsf{s})=w_s^y$; pick $w_p^y \in W_{y_\mathsf{p}}$ such that $\mathtt{val_w}(w_p^y)=\mathtt{val_w}(w_s^y)$, $\mathtt{tid}(w_p^y)=\tau_\mathsf{p}$; and let $\lambda_p=\mathsf{P}\langle\overline{w_p^y, w_s^y}\rangle$.

Let $\pi'=\pi.\lambda'.\lambda_p$ and $IM'=IM''[\overline{y_\mathsf{p} \mapsto w_p^y}]$ Since $IB(\tau_\mathsf{s})=IB(\tau_\mathsf{p})=\epsilon$, from (AIM-BPropFL) and (AIM-AtomA) we then know $IM, IB \xrightarrow{\lambda'} \xrightarrow{\lambda_p} IM', IB'$. Similarly, it is straightforward to show that $[\![\mathsf{C}]\!], \mathsf{S} \xrightarrow{\lambda_p} [\![\mathsf{C}']\!], \mathsf{S}'$. As such, from (AI-SilentS) and (AI-Step) we have $\Delta \vdash [\![\mathsf{C}]\!], \mathsf{S}, IM, IB, \pi \Rightarrow^* [\![\mathsf{C}']\!], \mathsf{S}', IM', IB', \pi'$. Note that since $IB(\tau_\mathsf{s})=\epsilon$ (from (3)), we also have $IB'(\tau_\mathsf{s})=IB'(\tau_\mathsf{p})=\epsilon$. Finally, note that since $b \approx ib$, from the definition of $\approx$ and (1) we have $B' \approx IB$. Similarly, from the definitions of π_d', π_l', π' and (1) we also have $(\pi_d', \pi_l') \approx \pi'$. In what follows we show that $(M', PB', \pi_d', \pi_l') \approx IM'$, thus demonstrating $(M', PB', B', \pi_d', \pi_l') \approx (IM', IB', \pi')$, as required.

Note that from (1) and the definitions of M', PB', π_d', π_l' we immediately know that the first conjunct of $(M', PB', \pi_d', \pi_l') \approx IM'$ holds. Establishing the second conjunct of $(M', PB', \pi_d', \pi_l') \approx IM'$ is analogous to that in case (2) and is omitted here. To show that the third conjunct holds, pick an arbitrary location $y \in \overline{y}$ and a such that $IM'(y_\mathsf{p})=a$. From the definition of IM' we then know that $a=w_p^y$. Recall that $\mathtt{val_w}(w_p^y)=\mathtt{val_w}(w_s^y)$ and that $IM'(y_\mathsf{s})=IM''(y_\mathsf{s})=w_s^y$. As such, from the second conjunct of $(M', PB', \pi_d', \pi_l') \approx IM'$ established above we know there exists $e' \in WU_y$ such that $\mathtt{val_w}(e')=\mathtt{val_w}(w_s^y)$ and $(M'(y)=e' \wedge \forall a \in PB' \cap WU_y.\ \mathsf{PB}\langle a\rangle \notin \pi_l') \text{ or } (\exists PB''.\ PB'=-.e'.PB'' \wedge \mathsf{PB}\langle e'\rangle \in \pi_l' \wedge \forall a \in PB'' \cap WU_y.\ \mathsf{PB}\langle a\rangle \notin \pi_l')$. Consequently, since $\mathtt{val_w}(w_p^y)=\mathtt{val_w}(w_s^y)$, we know there exists $e' \in WU_y$ such that $\mathtt{val_w}(e')=\mathtt{val_w}(w_p^y)$ and $(M'(y)=e' \wedge \forall a \in PB' \cap WU_y.\ \mathsf{PB}\langle a\rangle \notin \pi_l') \text{ or } (\exists PB''.\ PB'=-.e'.PB'' \wedge \mathsf{PB}\langle e'\rangle \in \pi_l' \wedge \forall a \in PB'' \cap WU_y.\ \mathsf{PB}\langle a\rangle \notin \pi_l')$, as required.

10. Case (A-SilentS), $\lambda=\mathsf{B}\langle a\rangle$ for some $a \in W$

From (A-SilentS) and (AM-BPropW) we then know that there exists τ, b such that $\mathtt{tid}(a)=\tau \notin \{\tau_\mathsf{s}, \tau_\mathsf{p}\}$, $B(\tau)=a.b$, $M'=M$, $PB'=PB.a$, $B'=B[\tau \mapsto b]$, $\pi_d'=\pi_d.\lambda$, $\pi_l=\lambda.\pi_l'$, $\mathsf{C}'=\mathsf{C}$ and $\mathsf{S}'=\mathsf{S}$. Let $\mathtt{loc}(a)=x$. From (1) we know that $B \approx IB$ and consequently that there exists ib and a' such that $a \approx_w a'$, $IB(\tau)=a'.ib$, $\mathtt{tid}(a')=\mathtt{tid}(a)$, and $b \approx ib$. Note that $\pi_d'.\pi_l'=\pi_d.\pi_l$. There are now three cases to consider: 1) $\neg\mathsf{maxELoc}(\pi_d'.\pi_l', a)$; or 2) $\mathsf{maxELoc}(\pi_d'.\pi_l', a) \wedge \neg\mathsf{maxE}(\pi_d'.\pi_l', a)$; or 3) $\mathsf{maxE}(\pi_d'.\pi_l', a)$.

In case (1), Let $\lambda'=\mathsf{B}\langle a'\rangle$ and let $IM'=IM[x_\mathsf{v} \mapsto a']$, $IB'=IB[\tau \mapsto ib]$, $\pi'=\pi.\lambda'$. From (AIM-BPropW) we then know $IM, IB \xrightarrow{\lambda'} IM', IB'$. As such, from (AI-SilentS) we have $\Delta \vdash [\![\mathsf{C}]\!], IM, IB, \pi \Rightarrow [\![\mathsf{C}']\!], IM', IB', \pi'$. Note that since $b \approx ib$, from the definition of $\approx$ and (1) we have $B' \approx IB$. Similarly, from the definitions of π_d', π_l', π' and (1) we also have $(\pi_d', \pi_l') \approx \pi'$. Moreover, from (1) and the definition of IB' we have $IB'(\tau_\mathsf{s})=IB'(\tau_\mathsf{p})=\epsilon$, as required. Finally, in what follows we show that $(M', PB', \pi_d', \pi_l') \approx IM'$, thus demonstrating $(M', PB', B', \pi_d', \pi_l') \approx (IM', IB', \pi')$, as required.

Note that from (1) and the definitions of M', PB', π_d', π_l' we immediately know that the second and third conjuncts of $(M', PB', \pi_d', \pi_l') \approx IM'$ hold. To show that the first conjunct holds, pick an arbitrary y such that $\mathsf{leqMaxELoc}(\pi_d', \pi_l', y)$ holds; i.e. pick m, e such that $\mathtt{loc}(m)=y$, $IM'(y_\mathsf{v})=e$, $\mathsf{maxELoc}(\pi_d'.\pi_l', m)$ and for all λ_m, if $\lambda_m \in \{\mathsf{B}\langle m\rangle, \mathsf{U}\langle m, -\rangle\}$ and $\lambda_m \in \pi_d'$ then $\pi_d'=-.\lambda_m$. That is, if $\lambda_m \in \pi_d'$ then $\lambda_m=\lambda'$ and thus $m=a$; i.e. $\mathsf{maxELoc}(\pi_d'.\pi_l', a)$, contradicting the assumption of case (1). We thus conclude that $\lambda_m \notin \pi_d'$. As such, since $\mathsf{maxELoc}(\pi_d'.\pi_l', m)$ and $\mathsf{wfp}(\pi_d.\pi_l)$ (from (2)), from the definitions of π_d', π_l' we know $\lambda_m, \mathsf{PB}\langle m\rangle \in \pi_l'$. That is, $\lambda=\mathsf{B}\langle a\rangle \prec_{\pi_d'.\pi_l'} \lambda_m$. Moreover, since for

each location the tso and nvo orders agree, from $\mathsf{wfp}(\pi_d.\pi_l)$ in (2) and the definitions of π'_d, π'_l we know $\mathsf{PB}\langle a\rangle \prec_{\pi'_d.\pi'_l} \mathsf{PB}\langle m\rangle$, i.e. $\mathsf{PB}\langle a\rangle \in \pi'_l$.

There are now two cases to consider: a) $y_{\mathtt{v}}{\neq}x_{\mathtt{v}}$, i.e. $y \neq x$; or b) $y_{\mathtt{v}}{=}x_{\mathtt{v}}$, i.e. $y{=}x$. In case (a) since $IM'(y_{\mathtt{v}}){=}IM(y_{\mathtt{v}})$, the desired result follows from (1) and the definitions of π'_d and π'_l. In case (b), we know that $IM'(y_{\mathtt{v}}){=}IM'(x_{\mathtt{v}}){=}a'$. As established above, we then know that $a \approx_w a'$, that $PB'{=}-.a$, and that $\mathsf{PB}\langle a\rangle \in \pi'_l$. That is, there exists a and $PB''{=}\epsilon$ such that $\mathtt{val_w}(a){=}\mathtt{val_w}(a')$, $PB'{=}-.a.PB''$, $\mathsf{PB}\langle a\rangle \in \pi'_l$, and $\forall c \in PB'' \cap WU_y.\ \mathsf{PB}\langle c\rangle \notin \pi'_l$, as required.

In case (2) let $\lambda'{=}\mathsf{B}\langle a'\rangle$. Recall that $IM(x_{\mathtt{v}}){=}a'$; pick $w_s \in W_{x_{\mathtt{s}}}$ such that $\mathtt{val_w}(w_s){=}\mathtt{val_w}(a')$, $\mathtt{tid}(w_s){=}\tau_{\mathtt{s}}$, Let $\lambda_s{=}\mathsf{S}\langle w_s, a'\rangle$. Let $IM'{=}IM[x_{\mathtt{s}} \mapsto w_s]$, $IB' = IB[\tau \mapsto ib]$, $\pi'{=}\pi.\lambda'.\lambda_s$. Since $IB(\tau_{\mathtt{s}}){=}\epsilon$, from (AIM-BPropW) and (AIM-AtomS) we then know $IM, IB \xrightarrow{\lambda'}\xrightarrow{\lambda_s} IM', IB'$. Similarly, it is straightforward to show that $[\![\mathsf{C}]\!], \mathsf{S} \xrightarrow{\lambda_s} [\![\mathsf{C}']\!], \mathsf{S}'$. As such, from (AI-SilentS) and (AI-Step) we have $\Delta \vdash [\![\mathsf{C}]\!], \mathsf{S}, IM, IB, \pi \Rightarrow^* [\![\mathsf{C}']\!], \mathsf{S}', IM', IB', \pi'$. Note that since $IB(\tau_{\mathtt{s}}){=}\epsilon$ (from (3)), we also have $IB'(\tau_{\mathtt{s}}){=}\epsilon$. Moreover, $IB'(\tau_{\mathtt{p}}){=}IB(\tau_{\mathtt{p}}){=}\epsilon$. Finally, note that since $b \approx ib$, from the definition of $\approx$ and (1) we have $B' \approx IB$. Similarly, from the definitions of π'_d, π'_l, π' and (1) we also have $(\pi'_d, \pi'_l) \approx \pi'$. In what follows we show that $(M', PB', \pi'_d, \pi'_l) \approx IM'$, thus demonstrating $(M', PB', B', \pi'_d, \pi'_l) \approx (IM', IB', \pi')$, as required.

Note that from (1) and the definitions of M', PB', π'_d, π'_l we immediately know that the third conjunct of $(M', PB', \pi'_d, \pi'_l) \approx IM'$ holds. To show that the first conjunct holds, pick an arbitrary y such that $\mathsf{leqMaxELoc}(\pi'_d, \pi'_l, y)$ holds; i.e. pick m, e such that $\mathtt{loc}(m){=}y$, $IM'(y_{\mathtt{v}}){=}e$, $\mathsf{maxELoc}(\pi'_d.\pi'_l, m)$ and for all λ_m, if $\lambda_m \in \{\mathsf{B}\langle m\rangle, \mathsf{U}\langle m, -\rangle\}$ and $\lambda_m \in \pi'_d$ then $\pi'_d{=}-.\lambda_m$. There are now two cases to consider: a) $y_{\mathtt{v}}{\neq}x_{\mathtt{v}}$, i.e. $y \neq x$; or b) $y_{\mathtt{v}}{=}x_{\mathtt{v}}$, i.e. $y{=}x$. In case (a) since $IM'(y_{\mathtt{v}}){=}IM(y_{\mathtt{v}})$, the desired result follows from (1) and the definitions of π'_d and π'_l. In case (b), we know that $IM'(y_{\mathtt{v}}){=}IM'(x_{\mathtt{v}}){=}a'$. Moreover, we know that if $\lambda_m \in \{\mathsf{B}\langle m\rangle, \mathsf{U}\langle m, -\rangle\}$ and $\lambda_m \in \pi'_d$ then $\pi'_d{=}-.\lambda_m$. That is, if $\lambda_m \in \pi'_d$ then $\lambda_m{=}\lambda'$ and thus $m{=}a$. As such, since $\mathsf{maxELoc}(\pi'_d.\pi'_l, m)$ holds, we know that $\mathsf{PB}\langle a\rangle \in \pi'_l$. As established above, we also know that $a \approx_w a'$, that $PB'{=}-.a$, and that $\mathsf{PB}\langle a\rangle \in \pi'_l$. That is, there exists a and $PB''{=}\epsilon$ such that $\mathtt{val_w}(a){=}\mathtt{val_w}(a')$, $PB'{=}{-}.a.PB''$, $\mathsf{PB}\langle a\rangle \in \pi'_l$, and $\forall c \in PB'' \cap WU_y.\ \mathsf{PB}\langle c\rangle \notin \pi'_l$, as required.

To show the second conjunct holds, let us assume $\mathsf{leqMaxE}(\pi'_d, \pi'_l)$; i.e. pick mm such that $\mathsf{maxE}(\pi'_d.\pi'_l, mm)$ and for all λ_{mm}, if $\lambda_{mm} \in \{\mathsf{B}\langle mm\rangle, \mathsf{U}\langle mm, -\rangle\}$ and $\lambda_{mm} \in \pi'_d$ then $\pi'_d{=}-.\lambda_{mm}$. That is, if $\lambda_{mm} \in \pi'_d$ then $\lambda_{mm}{=}\lambda'$ and thus $mm{=}fl$; i.e. $\mathsf{maxE}(\pi'_d.\pi'_l, fl)$, contradicting the assumption of case (2). We thus conclude that $\lambda_{mm} \notin \pi'_d$. Also let us pick y such that $\mathsf{hasMaxELoc}(\pi'_d, \pi'_l, y)$ holds; i.e. pick m, e, λ_m such that $\mathsf{maxELoc}(\pi'_d.\pi'_l, m)$, $\mathtt{loc}(m){=}y$, $\lambda_m \in \{\mathsf{B}\langle m\rangle, \mathsf{U}\langle m, -\rangle\}$, $\lambda_m \in \pi'_d$ and $IM'(y_{\mathtt{s}}){=}e$. There are now two cases to consider: a) $y_{\mathtt{s}}{\neq}x_{\mathtt{s}}$, i.e. $y \neq x$; or b) $y_{\mathtt{s}}{=}x_{\mathtt{s}}$, i.e. $y{=}x$. In case (a) since $IM'(y_{\mathtt{s}}){=}IM(y_{\mathtt{s}})$, the desired result follows from (1) and the definitions of π'_d and π'_l. In case (b), since $\mathsf{maxELoc}(\pi'_d.\pi'_l, a)$ holds, we thus know that $m{=}a$ and $\lambda_m{=}\lambda$. Since $IM'(x_{\mathtt{s}}){=}w_s$, it thus suffices to show that there exists $e \in WU_x$ such that $\mathtt{val_w}(e){=}\mathtt{val_w}(w)_s$ and $(M'(x){=}e \wedge \forall c \in PB' \cap WU_x.\ \mathsf{PB}\langle c\rangle \notin \pi'_l)$ or $(\exists PB''.\ PB'{=}-.e.PB'' \wedge \mathsf{PB}\langle e\rangle \in \pi'_l \wedge \forall c \in PB'' \cap WU_x.\ \mathsf{PB}\langle c\rangle \notin \pi'_l)$. We then proceed as follows. Recall that $\mathtt{val_w}(w_s){=}\mathtt{val_w}(a')$ and that $IM'(x_{\mathtt{v}}){=}a'$. As such, from the first conjunct of $(M', PB', \pi'_d, \pi'_l) \approx IM'$ established above we know there exists $e \in WU_x$ such that $\mathtt{val_w}(e){=}\mathtt{val_w}(a')$ and $(M(x){=}e \wedge \forall c \in PB' \cap WU_x.\ \mathsf{PB}\langle c\rangle \notin \pi'_l)$ or $(\exists PB''.\ PB'{=}-.e.PB'' \wedge \mathsf{PB}\langle e\rangle \in \pi_l \wedge \forall c \in PB'' \cap WU_x.\ \mathsf{PB}\langle c\rangle \notin \pi'_l)$. Consequently, since $\mathtt{val_w}(w_s){=}\mathtt{val_w}(a')$, we know there exists $e \in WU_x$ such that $\mathtt{val_w}(e){=}\mathtt{val_w}(w_s)$ and $(M'(x){=}e \wedge \forall c \in PB' \cap WU_x.\ \mathsf{PB}\langle c\rangle \notin \pi'_l)$ or $(\exists PB''.\ PB'{=}-.e.PB'' \wedge \mathsf{PB}\langle e\rangle \in \pi'_l \wedge \forall c \in PB'' \cap WU_x.\ \mathsf{PB}\langle c\rangle \notin \pi'_l)$, as required.

In case (3) let $\lambda'{=}\mathsf{B}\langle a'\rangle$. Recall that $IM(x_{\mathtt{v}}){=}a'$; pick $w_s \in W_{x_{\mathtt{s}}}$ such that $\mathtt{val}_{\mathtt{w}}(w_s){=}\mathtt{val}_{\mathtt{w}}(a')$, $\mathtt{tid}(w_s){=}\tau_{\mathtt{s}}$. Let $\lambda_s{=}\mathsf{S}\langle w_s, a'\rangle$, $IM''{=}IM[x_{\mathtt{s}} \mapsto w_s]$. When $\mathrm{Loc}{=}\overline{y}$, For each $y \in \overline{y}{=}\mathrm{Loc}$: let $IM''(y_{\mathtt{s}}){=}w_s^y$; pick $w_p^y \in W_{y_{\mathtt{p}}}$ such that $\mathtt{val}_{\mathtt{w}}(w_p^y){=}\mathtt{val}_{\mathtt{w}}(w_s^y)$, $\mathtt{tid}(w_p^y) = \tau_{\mathtt{p}}$; and let $\lambda_p{=}\mathsf{P}\langle\overline{w_p^y, w_s^y}\rangle$. Let $\pi'{=}\pi.\lambda'.\lambda_s.\pi''$ with $\pi''{=}\lambda_p$; $IM'{=}IM''[\overline{y_{\mathtt{p}} \mapsto w_p^y}]$; $IB' = IB[\tau \mapsto ib]$. Since $IB(\tau_{\mathtt{s}}){=}IB(\tau_{\mathtt{p}}){=}\epsilon$, from (AIM-BPropW), (AIM-AtomS) and (AIM-AtomA) we then know $IM, IB \xrightarrow{\lambda'}\xrightarrow{\lambda_s}\xrightarrow{\lambda_p} IM', IB'$. Similarly, it is straightforward to show that $[\![\mathsf{C}]\!], \mathsf{S} \xrightarrow{\lambda_s}\xrightarrow{\lambda_p} [\![\mathsf{C}']\!], \mathsf{S}'$. As such, from (AI-SilentS) and (AI-Step) we have $\Delta \vdash [\![\mathsf{C}]\!], \mathsf{S}, IM, IB, \pi \Rightarrow^* [\![\mathsf{C}']\!], \mathsf{S}', IM', IB', \pi'$.

Establishing the first and second conjuncts of $(M', PB', \pi'_d, \pi'_l) \approx IM'$ is analogous to that in case (2) and is omitted here. To show that the third conjunct holds, pick an arbitrary location $y \in \overline{y}$ and a such that $IM'(y_{\mathtt{p}}){=}a$. From the definition of IM' we then know that $a{=}w_p^y$. Recall that $\mathtt{val}_{\mathtt{w}}(w_p^y){=}\mathtt{val}_{\mathtt{w}}(w_s^y)$ and that $IM'(y_{\mathtt{s}}){=}IM''(y_{\mathtt{s}}){=}w_s^y$. As such, from the second conjunct of $(M', PB', \pi'_d, \pi'_l) \approx IM'$ established above we know there exists $e \in WU_y$ such that $\mathtt{val}_{\mathtt{w}}(e){=}\mathtt{val}_{\mathtt{w}}(w_s^y)$ and $(M'(y){=}e \wedge \forall c \in PB' \cap WU_y.\ \mathsf{PB}\langle c\rangle \notin \pi'_l)$ or $(\exists PB''.\ PB'{=}-.e.PB'' \wedge \mathsf{PB}\langle e\rangle \in \pi'_l \wedge \forall c \in PB'' \cap WU_y.\ \mathsf{PB}\langle c\rangle \notin \pi'_l)$. Consequently, since $\mathtt{val}_{\mathtt{w}}(w_p^y){=}\mathtt{val}_{\mathtt{w}}(w_s^y)$, we know there exists $e \in WU_y$ such that $\mathtt{val}_{\mathtt{w}}(e){=}\mathtt{val}_{\mathtt{w}}(w_p^y)$ and $(M'(y){=}e \wedge \forall c \in PB' \cap WU_y.\ \mathsf{PB}\langle c\rangle \notin \pi'_l)$ or $(\exists PB''.\ PB'{=}-.e.PB'' \wedge \mathsf{PB}\langle e\rangle \in \pi'_l \wedge \forall c \in PB'' \cap WU_y.\ \mathsf{PB}\langle c\rangle \notin \pi'_l)$, as required.

11. Case (A-SilentS), $\lambda{=}\mathsf{PB}\langle w\rangle$ for some $w \in WU$

From (A-SilentS) and (AM-PropW) we then know that there exists τ, x such that $\mathtt{loc}(w){=}x$, $\mathtt{tid}(w){=}\tau \notin \{\tau_{\mathtt{s}}, \tau_{\mathtt{p}}\}$, $PB{=}w.PB'$, $M'{=}M[x \mapsto w]$, $B'{=}B$, $\pi'_d{=}\pi_d.\lambda$, $\pi_l{=}\lambda.\pi'_l$, $\mathsf{C}'{=}\mathsf{C}$ and $\mathsf{S}'{=}\mathsf{S}$ Let $IM'{=}IM$, $IB'{=}IB$, $\pi'{=}\pi$. We thus have $\Delta \vdash [\![\mathsf{C}]\!], \mathsf{S}, IM, IB, \pi \Rightarrow^* [\![\mathsf{C}']\!], IM', \mathsf{S}', IB', \pi'$. From (1) we then simply have $B' \approx IB$. Similarly, from the definitions of π'_d, π'_l, π' and (1) we also have $(\pi'_d, \pi'_l) \approx \pi'$. Moreover, from (1) and the definition of IB' we have $IB'(\tau_{\mathtt{s}})IB'(\tau_{\mathtt{p}}){=}\epsilon$, as required. Finally, in what follows we show that $(M', PB', \pi'_d, \pi'_l) \approx IM'$, thus demonstrating $(M', PB', B', \pi'_d, \pi'_l) \approx (IM', IB', \pi')$, as required.

To show the first conjunct, pick y such that $\mathsf{leqMaxELoc}(\pi'_d, \pi'_l, y)$ holds; i.e. pick m, e' such that $\mathtt{loc}(m){=}y$, $IM'(y_{\mathtt{v}}){=}e'$, $\mathsf{maxELoc}(\pi'_d.\pi'_l, m)$ and for all $\lambda_m \in \{\mathsf{B}\langle m\rangle, \mathsf{U}\langle m, -\rangle\}$, if $\lambda_m \in \pi'_d$ then $\pi'_d{=}-.\lambda_m$. As $IM'{=}IM$, from (1) we then know there exists e such that $\mathtt{val}_{\mathtt{w}}(e){=}\mathtt{val}_{\mathtt{w}}(e')$ and $M(y){=}e \wedge \forall a \in PB \cap WU_y.\ \mathsf{PB}\langle a\rangle \notin \pi_l$ or $\exists PB''.\ PB{=}-.e.PB'' \wedge \mathsf{PB}\langle e\rangle \in \pi_l \wedge \forall a \in PB'' \cap WU_y.\ \mathsf{PB}\langle a\rangle \notin \pi_l)$. There are now two cases to consider: a) $y_{\mathtt{v}}{\neq}x_{\mathtt{v}}$, i.e. $y \neq x$; or b) $y_{\mathtt{v}}{=}x_{\mathtt{v}}$, i.e. $y{=}x$.

In case (a) since $M'(y){=}M(y)$, the desired result follows immediately from the definition of PB'. In case (b), since $w \in WU_x$, $w \in PB$ and $\mathsf{PB}\langle e\rangle \in \pi_l$, we know there exists e, PB'' such that $\mathtt{val}_{\mathtt{w}}(e){=}\mathtt{val}_{\mathtt{w}}(e')$, $PB{=}-.e.PB'' \wedge \mathsf{PB}\langle e\rangle \in \pi_l \wedge \forall a \in PB'' \cap WU_x.\ \mathsf{PB}\langle a\rangle \notin \pi_l$. That is, there exists e, PB_1, PB_2 such that $\mathtt{val}_{\mathtt{w}}(e){=}\mathtt{val}_{\mathtt{w}}(e')$, $PB{=}PB_1.e.PB_2 \wedge \mathsf{PB}\langle e\rangle \in \pi_l \wedge \forall a \in PB_2 \cap WU_x.\ \mathsf{PB}\langle a\rangle \notin \pi_l$. Now either i) $PB_1{\neq}\epsilon$; or ii) $PB_1{=}\epsilon$.

In case (i), we know $e{\neq}w$ and thus $\mathsf{PB}\langle e\rangle \neq \lambda$. As such, from the definitions of PB', π'_l we know there exists e, PB'', such that $\mathtt{val}_{\mathtt{w}}(e){=}\mathtt{val}_{\mathtt{w}}(e')$ and $PB'{=}-.e.PB'' \wedge \mathsf{PB}\langle e\rangle \in \pi'_l \wedge \forall a \in PB'' \cap WU_x.\ \mathsf{PB}\langle a\rangle \notin \pi'_l$, as required. In case (ii), from the definition of PB we know that $e{=}w$ and $PB'{=}PB_2$, and thus $M'(x){=}e$. That is, there exists e such that $\mathtt{val}_{\mathtt{w}}(e){=}\mathtt{val}_{\mathtt{w}}(e')$ and $M'(x){=}e \wedge \forall a \in PB' \cap WU_x.\ \mathsf{PB}\langle a\rangle \notin \pi'_l$, as required.

The proofs of the second and third conjuncts are analogous and omitted here.

12. Case (A-SilentS), $\lambda{=}\mathsf{PB}\langle fl\rangle$ for some $fl \in FL$

From (A-SilentS) and (AM-PropP) we then know there exists τ such that $\mathtt{tid}(fl){=}\tau \notin \{\tau_{\mathtt{s}}, \tau_{\mathtt{p}}\}$,

$PB{=}fl.PB'$, $M'{=}M$, $B'{=}B$, $\pi'_d{=}\pi_d.\lambda$, $\pi_l{=}\lambda.\pi'_l$, $\mathsf{C}'{=}\mathsf{C}$ and $\mathsf{S}'{=}\mathsf{S}$. Let $IM'{=}IM$, $IB'{=}IB$, $\pi'{=}\pi$. We thus have $\Delta \vdash [\![\mathsf{C}]\!], \mathsf{S}, IM, IB, \pi \Rightarrow^* [\![\mathsf{C}']\!], \mathsf{S}', IM', IB', \pi'$. From (1) we then simply have $B' \approx IB$. Similarly, from the definitions of π'_d, π'_l, π' and (1) we also have $(\pi'_d, \pi'_l) \approx \pi'$. Moreover, from (1) and the definition of IB' we have $IB'(\tau_\mathsf{s})IB'(\tau_\mathsf{p}){=}\epsilon$, as required. Finally, from (1) and the definitions of $M', PB', \pi'_d, \pi'l$ and λ we have $(M', PB', \pi'_d, \pi'_l) \approx IM'$, thus demonstrating $(M', PB', B', \pi'_d, \pi'_l) \approx (IM', IB', \pi')$, as required.

13. Case (A-Crash)
From (A-Crash) we then know that $\pi_l{=}\epsilon$, $PB'{=}\epsilon$, $M'{=}M$, $B'{=}\lambda\tau.\epsilon$, $\pi'_d{=}\epsilon$ and $\mathsf{C}'{=}\mathbf{rec}(\mathsf{C}_0, M)$, where $\Delta{=}(\mathsf{C}_0, \mathbf{rec})$ and $\mathsf{S}'{=}\mathsf{S}_0$. Let $IM'{=}IM[\overline{x_\mathsf{s} \mapsto x_\mathsf{p}}][\overline{x_\mathsf{v} \mapsto x_\mathsf{p}}]$, $IB'{=}\lambda\tau.\epsilon$, $\pi'{=}\epsilon$. From the definitions of IB', B', π'_d, π' we then simply have $B \approx IB$ and $(\pi'_d, \pi'_l) \approx \pi'$. Moreover, from the definition of IB' we have $IB'(\tau_\mathsf{s}){=}\epsilon$ and $IB'(\tau_\mathsf{p}){=}\epsilon$, as required. We next show that $(M', PB', \pi'_d, \pi'_l) \approx IM'$, thus demonstrating $(M', PB', B', \pi'_d, \pi'_l) \approx (IM', IB', \pi')$, as required.

Pick arbitrary x, w such that $IM'(x_\mathsf{p}){=}w$. From the definition of IM' we then know $IM(x_\mathsf{p}){=}w$. As such, from the third conjunct of (1) and since $\pi_l{=}\epsilon$, we know there exists $e \in WU_x$ such that $\mathtt{val}_\mathtt{w}(w){=}\mathtt{val}_\mathtt{w}(e)$ and $M(x){=}e$. That is, since $M'{=}M$ and $PB'{=}\epsilon$, there exists $e \in WU_x$ such that $\mathtt{val}_\mathtt{w}(w){=}\mathtt{val}_\mathtt{w}(e)$, $M'(x){=}e$ and $PB' \cap WU_x{=}\emptyset$, thus establishing the third conjunct of $(M', PB', \pi'_d, \pi'_l) \approx IM'$. Moreover, as $IM'(x_\mathsf{s}){=}IM'(x_\mathsf{v}){=}IM'(x_\mathsf{p})$, in doing so we have also established the first and second conjuncts of $(M', PB', \pi'_d, \pi'_l) \approx IM'$.

Finally, note that since for all x: $IM'(x_\mathsf{s}){=}IM'(x_\mathsf{v}){=}IM'(x_\mathsf{p})$ and $\mathtt{val}_\mathtt{w}(IM'(x_\mathsf{p})){=}\mathtt{val}_\mathtt{w}(M(x))$, we know $M \in \mathtt{M}(IM')$. As such, from (AI-Crash) and the definitions of IM', IB', C' we have $[\![\mathsf{C}]\!], \mathsf{S}, IM, IB, \pi \Rightarrow^* [\![\mathsf{C}']\!], \mathsf{S}', IM', IB', \pi'$, as required.

□

Lemma 6. *For all* $\Delta, \mathsf{C}, \mathsf{S}, \mathsf{S}', M, PB, B, \pi_d, \pi_l, \mathsf{C}', M', PB', B', \pi'_d, \pi'_l, IM, IB, \pi$, *if:*

- $(M, PB, B, \pi_d, \pi_l) \approx (IM, IB, \pi)$
- $\mathsf{wf}(M, PB, B, \pi_d) \wedge \mathsf{wfp}(\pi_d.\pi_l) \wedge \mathsf{wf}(IM, IB, \pi) \wedge IB(\tau_\mathsf{s}){=}IB(\tau_\mathsf{p}){=}\epsilon$
- $\Delta \vdash \mathsf{C}, \mathsf{S}, M, PB, B, \pi_d, \pi_l \Rightarrow^* \mathsf{C}', \mathsf{S}', M', PB', B', \pi'_d, \pi'_l$ *and* $\pi'_l{=}\epsilon$

then there exists IM', IB', π' *such that:*

- $\Delta \vdash [\![\mathsf{C}]\!], \mathsf{S}, IM, IB, \pi \Rightarrow^* [\![\mathsf{C}']\!], \mathsf{S}', IM', IB', \pi'$
- $\forall x.\ M'(x){=}IM'(x_\mathsf{p})$

Proof. As $\pi'_l{=}\epsilon$, the desired result follows as a corollary of Lemma 5 by induction on the length of $\Rightarrow^*$. □

Lemma 7. *For all* $\Delta, \mathsf{C}, \mathsf{S}, \mathsf{S}', M, \pi_l, M', IM$, *if*

- $\forall x.\ IM(x_\mathsf{v}){=}IM(x_\mathsf{s}){=}IM(x_\mathsf{p}) \wedge M(x) \approx_v IM(x_\mathsf{p})$; *and*
- $\Delta \vdash \mathsf{C}, \mathsf{S}, M, PB_0, B_0, \epsilon, \pi_l \Rightarrow^* \mathsf{C}', \mathsf{S}', M', PB', B', \pi_d, \epsilon$

then here exists IM', IB', π' *such that:*

- $\Delta \vdash [\![\mathsf{C}]\!], \mathsf{S}, IM, IB_0, \pi \Rightarrow^* [\![\mathsf{C}']\!], \mathsf{S}', IM', IB', \pi'$
- $\forall x.\ M'(x) \approx_v IM'(x_\mathsf{p})$

where $e_1 \approx_v e_2 \overset{def}{\Leftrightarrow} \mathtt{val}_\mathtt{w}(w){=}\mathtt{val}_\mathtt{w}(e')$

Proof. Follows immediately as a corollary of Lemma 6. □

Theorem 6. *For all* $\Delta, \mathsf{C}, \mathsf{M}, \mathsf{M}', \mathsf{IM}, \mathsf{S}, \mathsf{S}'$, *if*

- $\forall x.\ \mathsf{IM}(x_{\mathsf{v}})=\mathsf{IM}(x_{\mathsf{s}})=\mathsf{IM}(x_{\mathsf{p}}) \wedge \mathsf{M}(x)=\mathsf{IM}(x_{\mathsf{p}})$; *and*
- $\Delta \vdash \mathsf{C}, \mathsf{S}, \mathsf{M}, \mathsf{PB}_0, \mathsf{B}_0 \rightarrow^* \mathsf{C}_{\mathbf{skip}}, \mathsf{S}', \mathsf{M}', -, -$

then here exists IM' *such that:*

- $\Delta \vdash [\![\mathsf{C}]\!], \mathsf{S}, \mathsf{IM}, \mathsf{B}_0 \Rightarrow^* [\![\mathsf{C}_{\mathbf{skip}}]\!], \mathsf{S}', \mathsf{IM}', -$
- $\forall x.\ \mathsf{M}'(x)=\mathsf{IM}'(x_{\mathsf{p}})$

Proof. Pick an arbitrary $\Delta, \mathsf{C}, \mathsf{M}, \mathsf{M}', \mathsf{IM}$ such that $\forall x.\ \mathsf{IM}(x_{\mathsf{v}})=\mathsf{IM}(x_{\mathsf{s}})=\mathsf{IM}(x_{\mathsf{p}}) \wedge \mathsf{M}(x)=\mathsf{IM}(x_{\mathsf{p}})$; and $\Delta \vdash \mathsf{C}, \mathsf{M}, \mathsf{PB}_0, \mathsf{B}_0 \rightarrow^* \mathsf{C}_{\mathbf{skip}}, \mathsf{M}', \mathsf{S}', -$. From Lemma 2 we then know that there exists M, M' such that: $\mathsf{C}, \mathsf{S}, M, PB_0, B_0, \epsilon, \pi \rightarrow^* \mathsf{C}_{\mathbf{skip}}, \mathsf{S}', M', -, -, -, \epsilon$ and for all x: $\mathsf{M}(x)=\mathtt{val}_{\mathtt{w}}(M(x)) \wedge \mathsf{M}'(x)=\mathtt{val}_{\mathtt{w}}(M'(x))$. Pick IM and for all x pick e such that $IM(x_{\mathsf{v}})=IM(x_{\mathsf{s}})=IM(x_{\mathsf{p}})=e$ and $\mathtt{val}_{\mathtt{w}}(e)=\mathsf{IM}(x_{\mathsf{p}})=\mathsf{M}(x)=\mathtt{val}_{\mathtt{w}}(M(x))$. As such, we have $\forall x.\ M(x) \approx_v IM(x_{\mathsf{p}})$. Consequently, from Lemma 7 we know there exists IM' such that $\Delta \vdash [\![\mathsf{C}]\!], \mathsf{S}, IM, IB_0, \pi \Rightarrow^* [\![\mathsf{C}_{\mathbf{skip}}]\!], \mathsf{S}', IM', -, -$ and $\forall x.\ M'(x) \approx_v IM'(x_{\mathsf{p}})$. From Lemma 3 we then know there exists $\mathsf{IM}, \mathsf{IM}'$ such that: $\mathsf{C}, \mathsf{S}, \mathsf{IM}, \mathsf{B}_0 \Rightarrow^* [\![\mathsf{C}_{\mathbf{skip}}]\!], \mathsf{S}', \mathsf{IM}', -$ and for all x and $x_{\mathsf{o}} \in \{x_{\mathsf{v}}, x_{\mathsf{s}}, x_{\mathsf{p}}\}$: $\mathsf{IM}(x_{\mathsf{o}})=\mathtt{val}_{\mathtt{w}}(IM(x_{\mathsf{o}})) \wedge \mathsf{IM}'(x_{\mathsf{o}})=\mathtt{val}_{\mathtt{w}}(IM'(x_{\mathsf{o}}))$. Consequently, for all x since we have $\mathsf{IM}'(x_{\mathsf{p}})=\mathtt{val}_{\mathtt{w}}(IM'(x_{\mathsf{p}}))$, $M'(x) \approx_v IM'(x_{\mathsf{p}})$ and $\mathsf{M}'(x)=\mathtt{val}_{\mathtt{w}}(M'(x))$, we have for all x: $\mathsf{IM}'(x_{\mathsf{p}})=\mathsf{M}'(x)$, as required. □

Definition 9. A triple $\langle \mathcal{R}; \mathcal{G} \rangle \vdash \{P\}\ \mathsf{C}\ \{Q\}$ is *closed* iff $\langle \mathcal{R}; \mathcal{G} \rangle = \langle \top; \top \rangle$.

B POG SOUNDNESS

Definition 10. Given a set of registers $A \subseteq \textsc{Reg}$, two stacks S, S' are *equivalent on* A, written $\mathsf{S} \sim_A \mathsf{S}'$, iff: $\forall \mathsf{a} \in A.\ \mathsf{S}(\mathsf{a})=\mathsf{S}'(\mathsf{a})$. A set of states P is *stack-stable* under a set of registers A, written s-stable(P, A), iff: $\forall \mathsf{IM}, \mathsf{S}, \mathsf{S}'.\ (\mathsf{IM}, \mathsf{S}) \in P \wedge \mathsf{S} \sim_A \mathsf{S}' \Rightarrow (\mathsf{IM}, \mathsf{S}') \in P$.

A relation R on states is *stack-stable* under a set of registers A, written s-stable(R, A), iff:

$$\forall \mathsf{IM}, \mathsf{IM}', \mathsf{S}, \mathsf{S}', \mathsf{S}''.\ ((\mathsf{IM}, \mathsf{S}), (\mathsf{IM}', \mathsf{S}')) \in R \wedge \mathsf{S} \sim_A \mathsf{S}'' \Rightarrow ((\mathsf{IM}, \mathsf{S}), (\mathsf{IM}', \mathsf{S}'')) \in R$$

In what follows we write $\overline{A}$ for $\textsc{Reg} \setminus A$.

Proposition 1. *For all* $\mathsf{C}, \mathsf{C}', \mathsf{IM}, \mathsf{IM}', \mathsf{B}, \mathsf{B}', \mathsf{S}, \mathsf{S}', Q, n, \mathcal{R}, \mathcal{G}, A$*:*

- *if:* $\mathsf{C}, \mathsf{S}, \mathsf{IM}, \mathsf{B} \overset{-}{\Longrightarrow} \mathsf{C}', \mathsf{S}', \mathsf{IM}', \mathsf{B}'$
  then: $\mathtt{fr}(\mathsf{C}') \subseteq \mathtt{fr}(\mathsf{C}) \wedge \mathtt{wr}(\mathsf{C}') \subseteq \mathtt{wr}(\mathsf{C}) \wedge \mathsf{S} \sim_{\overline{\mathtt{wr}(\mathsf{C})}} \mathsf{S}'$
- *if:* $\mathsf{S} \sim_{\mathtt{fr}(\mathsf{e})} \mathsf{S}'$
  then: $\mathsf{S}(\mathsf{e})=\mathsf{S}'(\mathsf{e})$
- *if:* $\mathsf{S} \sim_{\mathtt{fr}(Q)} \mathsf{S}'$
  then: $\forall \mathsf{IM}.\ \mathsf{IM}, \mathsf{S} \models Q \Leftrightarrow \mathsf{IM}, \mathsf{S}' \models Q$
- *if:* $\mathsf{safe}_n(\mathsf{C}, \mathsf{IM}, \mathsf{S}, \mathcal{R}, \mathcal{G}, Q, I)$, $\mathtt{fr}(\mathsf{C}) \subseteq A$, s-stable$(Q, A)$, s-stable$(\mathcal{R}, A)$ *and* $\mathsf{S} \sim_A \mathsf{S}'$
  then: $\mathsf{safe}_n(\mathsf{C}, \mathsf{IM}, \mathsf{S}', \mathcal{R}, \mathcal{G}, Q, I)$

Proposition 2. *For all* $P, Q, R, \mathsf{IM}, \mathsf{S}, \mathsf{a}, x, \mathsf{e}, \mathsf{C}, \mathcal{R}, \mathcal{G}$*:*

- *if* $(\mathsf{IM}, \mathsf{S}) \in [\![Q[\mathsf{e}/\mathsf{a}]]\!] \cup [\![Q[\mathsf{e}/x_\mathsf{v}]]\!] \cup [\![Q[\mathsf{e}/x_\mathsf{s}]]\!]$*, then* $(\mathsf{IM}, \mathsf{S}) \in [\![Q_\mathsf{p}]\!]$
- *if* $P \Rightarrow Q_\mathsf{p}$*, then* $P_\mathsf{p} \Rightarrow Q_\mathsf{p}$
- *if* $\langle \mathcal{R}; \mathcal{G} \rangle \Vdash \{P\}\ \mathsf{C}\ \{Q\}$ *is a valid POG judgement, then* $P \subseteq Q_\mathsf{p}$
- *if* $\langle \mathcal{R}; \mathcal{G} \rangle \Vdash \{P\}\ \mathsf{C}\ \{Q\}$ *is a valid POG judgement, then* $P_\mathsf{p} \subseteq Q_\mathsf{p}$

Proof. The first two parts follow from the definitions of $[\![.]\!]$ and Q_p. The third part follows from the definitions of valid judgements and safe. The last part follows from the second and third parts. □

Lemma 8. *For all* $\mathsf{IM}, \mathsf{B}, \mathsf{IM}', \mathsf{B}', \tau$*, if* $\mathsf{IM}, \mathsf{B} \xrightarrow{\tau:\epsilon} \mathsf{IM}', \mathsf{B}'$*, then* $\Downarrow(\mathsf{IM}', \mathsf{B}', \tau)=\Downarrow(\mathsf{IM}, \mathsf{B}, \tau)$.

Proof. Follows from the definition of $\Downarrow(.)$ and the shape of $\xrightarrow{\tau:\epsilon}$ transitions.

Lemma 9. *For all* $n, \mathsf{C}, \mathsf{IM}, \mathsf{S}, \mathcal{R}, \mathcal{G}, Q$*: if* $\mathsf{safe}_{n+1}(\mathsf{C}, \mathsf{IM}, \mathsf{S}, \mathcal{R}, \mathcal{G}, Q, I)$ *holds, then* $\mathsf{safe}_n(\mathsf{C}, \mathsf{IM}, \mathsf{S}, \mathcal{R}, \mathcal{G}, Q, I)$ *holds.*

Proof. By straightforward induction on n.

We lift the definition of syntactic stability (on assertions) to semantic stability (on sets of instrumented states).

Definition 11 (Semantic stability)**.** A set of instrumented memories P is *stable*, written stable(P), iff:

$$\forall (\mathsf{IM}, \mathsf{S}) \in P.\ (\mathsf{IM}[\overline{x_\mathsf{p} \mapsto \mathsf{IM}(x_\mathsf{s})}], \mathsf{S}) \in P \wedge \forall x.\ (\mathsf{IM}[x_\mathsf{s} \mapsto x_\mathsf{v}], \mathsf{S}) \in P$$

Similarly, we lift the definition of syntactic non-interference to semantic non-interference.

Definition 12 (Semantic non-interference)**.** Given the tuples $\langle \mathcal{R}_1, \mathcal{G}_1 \rangle$ and $\langle \mathcal{R}_2, \mathcal{G}_2 \rangle$ where $\mathcal{R}_1, \mathcal{G}_1, \mathcal{R}_2, \mathcal{G}_2$ are relations on states, $\langle \mathcal{R}_1, \mathcal{G}_1 \rangle$ and $\langle \mathcal{R}_2, \mathcal{G}_2 \rangle$ are *non-interfering* iff for all σ_1, σ_2:

- $(\sigma_1, \sigma_2) \in \mathcal{G}_1 \Rightarrow (\sigma_1, \sigma_2) \in \mathcal{R}_2$; and
- $(\sigma_1, \sigma_2) \in \mathcal{G}_2 \Rightarrow (\sigma_1, \sigma_2) \in \mathcal{R}_1$.

It is straightforward to show that for all S, if $\langle \mathcal{R}_1, \mathcal{G}_1\rangle$ and $\langle \mathcal{R}_2, \mathcal{G}_2\rangle$ are syntactically non-interfering, then $\langle [\![\mathcal{R}_1]\!]^{\mathsf{r}}, [\![\mathcal{G}_1]\!]^{\mathsf{r}}\rangle$ and $\langle [\![\mathcal{R}_2]\!]^{\mathsf{r}}, [\![\mathcal{G}_2]\!]^{\mathsf{g}}\rangle$ are semantically non-interfering.

Lemma 10. *For all $\mathcal{R}_1, \mathcal{G}_1, \mathcal{R}_2, \mathcal{G}_2, \mathsf{S}$, if $\langle \mathcal{R}_1, \mathcal{G}_1\rangle$ and $\langle \mathcal{R}_2, \mathcal{G}_2\rangle$ are syntactically non-interfering, then $\langle [\![\mathcal{R}_1]\!]^{\mathsf{r}}, [\![\mathcal{G}_1]\!]^{\mathsf{r}}\rangle$ and $\langle [\![\mathcal{R}_2]\!]^{\mathsf{r}}, [\![\mathcal{G}_2]\!]^{\mathsf{g}}\rangle$ are semantically non-interfering.*

Proof. Pick arbitrary $\mathcal{R}_1, \mathcal{G}_1, \mathcal{R}_2, \mathcal{G}_2, \mathsf{S}$ such that $\langle \mathcal{R}_1, \mathcal{G}_1\rangle$ and $\langle \mathcal{R}_2, \mathcal{G}_2\rangle$ are syntactically non-interfering. We then need to show $\langle [\![\mathcal{R}_1]\!]^{\mathsf{r}}, [\![\mathcal{G}_1]\!]^{\mathsf{r}}\rangle$ and $\langle [\![\mathcal{R}_2]\!]^{\mathsf{r}}, [\![\mathcal{G}_2]\!]^{\mathsf{g}}\rangle$ are semantically non-interfering.

To show the first part. Pick an arbitrary σ_1, σ_2 such that $(\sigma_1, \sigma_2) \in [\![\mathcal{G}_1]\!]^{\mathsf{g}}$. From the definition of $[\![\mathcal{G}_1]\!]^{\mathsf{g}}$ we then know there exists $x_{\mathsf{v}}, \mathsf{e}, P, \mathsf{IM}_1, \mathsf{IM}_2, \mathsf{S}$ such that $\langle x_{\mathsf{v}}, \mathsf{e}, P\rangle \in \mathcal{G}_1$, $\sigma_1{=}(\mathsf{IM}, \mathsf{S}) \in [\![P]\!]$ and $\sigma_2{=}(\mathsf{IM}_2, \mathsf{S})$ with $\mathsf{IM}_2{=}\mathsf{IM}_1[x_{\mathsf{v}} \mapsto \mathsf{S}(\mathsf{e})]$. We next demonstrate that for an arbitrary $R \in \mathcal{R}_2$ we have $(\sigma_1, \sigma_2) \in [\![R]\!]^{\mathsf{r}}$, thus from the definition of $[\![.]\!]^{\mathsf{r}}$ establishing that $(\sigma_1, \sigma_2) \in [\![\mathcal{R}]\!]^{\mathsf{r}}$, as required.

Pick an arbitrary $R \in \mathcal{R}_2$. There are two cases to consider: 1) $\sigma_1 \notin [\![R]\!]$; or 2) $\sigma_1 \in [\![R]\!]$. In case (1), from the definition of $[\![.]\!]^{\mathsf{r}}$ we trivially have $(\sigma_1, \sigma_2) \in [\![R]\!]^{\mathsf{r}}$, as required. In case (2), since $\sigma_1 \in [\![P]\!]$ with $\langle x_{\mathsf{v}}, \mathsf{e}, P\rangle \in \mathcal{G}_1$, and $\sigma_1 \in [\![R]\!]$ with $R \in \mathcal{R}_2$, from the definition of syntactic non-interference we have $\sigma_1 \in [\![R[\mathsf{e}/x_{\mathsf{v}}]]\!]$. As such, given the definition of σ_2 we also have $\sigma_2 \in [\![R]\!]$. Consequently, since $\sigma_1 \in [\![R]\!]$ and $\sigma_2 \in [\![R]\!]$, from the definition of $[\![.]\!]^{\mathsf{r}}$ we have $(\sigma_1, \sigma_2) \in [\![R]\!]^{\mathsf{r}}$, as required. □

Lemma 11 (Skip safety)**.** *For all n, $\mathsf{IM}, \mathsf{S}, \mathcal{R}, \mathcal{G}, Q, I$, if $(\mathsf{IM}, \mathsf{S}) \in Q$ and $\mathsf{wf}(\mathcal{R}, \mathcal{G}, Q, I)$, then $\mathsf{safe}_n(\mathbf{skip}^{\mathsf{sp}}, \mathsf{IM}, \mathsf{S}, \mathcal{R}, \mathcal{G}, Q, I)$ holds, where*

$$\mathsf{wf}(\mathcal{R}, \mathcal{G}, Q, I) \overset{def}{\Leftrightarrow} \mathsf{stable}(Q) \wedge \mathsf{closed}(Q, \mathcal{R}) \wedge \mathcal{G}^*{=}\mathcal{G} \wedge Q \subseteq I$$
$$\mathsf{closed}(Q, \mathcal{R}) \overset{def}{\Leftrightarrow} \forall \sigma, \sigma'.\ \sigma \in Q \wedge (\sigma, \sigma') \in \mathcal{R} \Rightarrow \sigma' \in Q$$

Proof. We proceed by induction on n.

Base case $n{=}0$
Pick arbitrary $\mathsf{IM}, \mathsf{S}, \mathcal{R}, \mathcal{G}, Q, I$. It then suffices to show: $\mathsf{safe}_0(\mathbf{skip}^{\mathsf{sp}}, \mathsf{IM}, \mathsf{S}, \mathcal{R}, \mathcal{G}, Q, I)$, which follows trivially from the definition of safe_0.

Inductive case $n{=}m{+}1$

$$\forall \mathsf{IM}, \mathsf{S}, \mathcal{R}, \mathcal{G}, Q, I.\ \mathsf{IM} \in Q \wedge \mathsf{wf}(\mathcal{R}, \mathcal{G}, Q, I) \Rightarrow \mathsf{safe}_m(\mathbf{skip}^{\mathsf{sp}}, \mathsf{IM}, \mathsf{S}, \mathcal{R}, \mathcal{G}, Q, I) \qquad \text{(I.H.)}$$

Pick arbitrary $\mathsf{IM}, \mathsf{S}, \mathcal{R}, \mathcal{G}, Q, I$ such that $(\mathsf{IM}, \mathsf{S}) \in Q$ and $\mathsf{wf}(\mathcal{R}, \mathcal{G}, Q, I)$. Property (1) follow immediately since $(\mathsf{IM}, \mathsf{S}) \in Q$ and $Q \subseteq I$.

To show property (4), pick arbitrary $\tau, \mathsf{IM}_1, \mathsf{IM}_2, \mathsf{B}_1, \mathsf{B}_2, \mathsf{C}', \mathsf{IM}', \mathsf{S}', l{\neq}\not{\iota}$ such that $\mathsf{IM}{=}\Downarrow(\mathsf{IM}_1, \mathsf{B}_1, \tau)$, $\mathsf{IM}'{=}\Downarrow(\mathsf{IM}_2, \mathsf{B}_2, \tau)$, $\mathsf{C}, \mathsf{SIM}_1, \mathsf{B}_1 \overset{\tau:l}{\Longrightarrow} \mathsf{C}', \mathsf{S}', \mathsf{IM}_2, \mathsf{B}_2$. As $\mathsf{C}{=}\mathbf{skip}$, from the definition of $\overset{\tau:l}{\Longrightarrow}$ we then know $\mathsf{C}'{=}\mathbf{skip}$, $\mathsf{S}{=}\mathsf{S}'$ and $\mathsf{IM}_1, \mathsf{B}_1 \xrightarrow{\tau:\epsilon} \mathsf{IM}_2, \mathsf{B}_2$. As such, from Lemma 8 we know $\mathsf{IM}{=}\Downarrow(\mathsf{IM}_1, \mathsf{B}_1, \tau) = \Downarrow(\mathsf{IM}_2, \mathsf{B}_2, \tau){=}\mathsf{IM}'$. Consequently, as $\mathcal{G}$ is reflexive (from $\mathsf{wf}(\mathcal{R}, \mathcal{G}, Q)$), we know $((\mathsf{IM}, \mathsf{S}), (\mathsf{IM}', \mathsf{S})) \in \mathcal{G}$, as required. Moreover, since $\mathsf{IM}'{=}\mathsf{IM}$ and $\mathsf{S}'{=}\mathsf{S}$, and $(\mathsf{IM}, \mathsf{S}) \in Q$, from (I.H.) we have $\mathsf{safe}_m(\mathbf{skip}^{\mathsf{sp}}, \mathsf{IM}', \mathsf{S}', \mathcal{R}, \mathcal{G}, Q, I)$, as required.

To show property (3), pick arbitrary $\mathsf{IM}', \mathsf{S}'$ such that $((\mathsf{IM}, \mathsf{S}), (\mathsf{IM}', \mathsf{S}')) \in \mathcal{R}$. Since $\mathsf{closed}(Q, \mathcal{R})$ holds (from $\mathsf{wf}(\mathcal{R}, \mathcal{G}, Q)$), we then know that $(\mathsf{IM}', \mathsf{S}') \in Q$. As such, from (I.H.) we have $\mathsf{safe}_m(\mathbf{skip}^{\mathsf{sp}}, \mathsf{IM}', \mathsf{S}', \mathcal{R}, \mathcal{G}, Q, I)$, as required.

Similarly, let $\mathsf{IM}_a{=}\mathsf{IM}[\overline{x_{\mathsf{p}} \mapsto \mathsf{IM}(x_{\mathsf{s}})}]$, $\mathsf{IM}_s{=}\mathsf{IM}[x_{\mathsf{s}} \mapsto \mathsf{IM}(x_v)]$ for an arbitrary x. As $\mathsf{stable}(Q)$ and $(\mathsf{IM}, \mathsf{S}) \in Q$, from the definition of stability we then know $(\mathsf{IM}_s, \mathsf{S}), (\mathsf{IM}_a, \mathsf{S}) \in Q$, and thus from (I.H.) we have $\mathsf{safe}_m(\mathbf{skip}^{\mathsf{sp}}, \mathsf{IM}_s, \mathsf{S}, \mathcal{R}, \mathcal{G}, Q, I)$ and $\mathsf{safe}_m(\mathbf{skip}^{\mathsf{sp}}, \mathsf{IM}_a, \mathsf{S}, \mathcal{R}, \mathcal{G}, Q, I)$, as required. □

Lemma 12 (Rely contraction). *For all* $n, \mathsf{C}, \mathsf{IM}, \mathsf{S}, \mathcal{R}_1, \mathcal{R}_2, \mathcal{G}, Q, I$*: if* $\mathsf{safe}_n(\mathsf{C}^{\mathsf{sp}}, \mathsf{IM}, \mathsf{S}, \mathcal{R}_1, \mathcal{G}, Q, I)$*, then* $\mathsf{safe}_n(\mathsf{C}^{\mathsf{sp}}, \mathsf{IM}, \mathsf{S}, \mathcal{R}_1 \cap \mathcal{R}_2, \mathcal{G}, Q, I)$.

Proof. By induction on n.

Base case n=0
Pick arbitrary $n, \mathsf{C}, \mathsf{IM}, \mathsf{S}, \mathcal{R}_1, \mathcal{R}_2, \mathcal{G}, Q, I$. It then suffices to show: $\mathsf{safe}_0(\mathsf{C}^{\mathsf{sp}}, \mathsf{IM}, \mathsf{S}, \mathcal{R}_1 \cap \mathcal{R}_2, \mathcal{G}, Q, I)$, which follows from the definition of safe_0.

Inductive case n=m+1

$$\forall \mathsf{C}, \mathsf{IM}, \mathsf{S}, \mathcal{R}_1, \mathcal{R}_2, \mathcal{G}, Q.\ \mathsf{safe}_m(\mathsf{C}^{\mathsf{sp}}, \mathsf{IM}, \mathsf{S}, \mathcal{R}_1, \mathcal{G}, Q, I) \Rightarrow \mathsf{safe}_m(\mathsf{C}^{\mathsf{sp}}, \mathsf{IM}, \mathsf{S}, \mathcal{R}_1 \cap \mathcal{R}_2, \mathcal{G}, Q, I) \quad \text{(I.H.)}$$

Pick arbitrary $\mathsf{C}, \mathsf{IM}, \mathsf{S}, \mathcal{R}_1, \mathcal{R}_2, \mathcal{G}, Q, I$ such that:

$$\mathsf{safe}_n(\mathsf{C}^{\mathsf{sp}}, \mathsf{IM}, \mathsf{S}, \mathcal{R}_1, \mathcal{G}, Q, I) \tag{5}$$

We then need to show: $\mathsf{safe}_n(\mathsf{C}^{\mathsf{sp}}, \mathsf{IM}, \mathsf{S}, \mathcal{R}_1 \cap \mathcal{R}_2, \mathcal{G}, Q, I)$. Property (1) follows immediately from (5). To show property (4), pick arbitrary $\tau, \mathsf{IM}_1, \mathsf{IM}_2, \mathsf{B}_1, \mathsf{B}_2, \mathsf{C}', \mathsf{IM}', \mathsf{S}', l \neq \lightning$ such that $\mathsf{IM} = \Downarrow(\mathsf{IM}_1, \mathsf{B}_1, \tau)$, $\mathsf{IM}' = \Downarrow(\mathsf{IM}_2, \mathsf{B}_2, \tau)$ and $\mathsf{C}, \mathsf{S}, \mathsf{IM}_1, \mathsf{B}_1 \xRightarrow{\tau:l} \mathsf{C}', \mathsf{S}', \mathsf{IM}_2, \mathsf{B}_2$. From (5) we then have $((\mathsf{IM}, \mathsf{S}), (\mathsf{IM}', \mathsf{S})) \in \mathcal{G} \cup \mathsf{A}_{\mathsf{sp}}$ and $\mathsf{safe}_m(\mathsf{C}^{\mathsf{sp}}, \mathsf{IM}', \mathsf{S}', \mathcal{R}_1, \mathcal{G}, Q, I)$. As such, from (I.H.) we have $((\mathsf{IM}, \mathsf{S}), (\mathsf{IM}', \mathsf{S})) \in \mathcal{G} \cup \mathsf{A}_{\mathsf{sp}}$ and $\mathsf{safe}_m(\mathsf{C}^{\mathsf{sp}}, \mathsf{IM}', \mathsf{S}', \mathcal{R}_1 \cap \mathcal{R}_2, \mathcal{G}, Q, I)$, as required.

To show property (3), pick arbitrary $\mathsf{IM}', \mathsf{S}'$ such that $((\mathsf{IM}, \mathsf{S}), (\mathsf{IM}', \mathsf{S}')) \in (\mathcal{R}_1 \cap \mathcal{R}_2) \cup \mathsf{A}_{\mathsf{sp}}$. We then know that $((\mathsf{IM}, \mathsf{S}), (\mathsf{IM}', \mathsf{S}')) \in \mathcal{R}_1 \cup \mathsf{A}_{\mathsf{sp}}$ and thus from (5) we have $\mathsf{safe}_m(\mathsf{C}^{\mathsf{sp}}, \mathsf{IM}', \mathsf{S}', \mathcal{R}_1, \mathcal{G}, Q, I)$. As such, from (I.H.) we have $\mathsf{safe}_m(\mathsf{C}^{\mathsf{sp}}, \mathsf{IM}', \mathsf{S}', \mathcal{R}_1 \cap \mathcal{R}_2, \mathcal{G}, Q, I)$, as required. □

Lemma 13 (Guarantee extension). *For all* $n, \mathsf{C}, \mathsf{IM}, \mathsf{S}, \mathcal{R}, \mathcal{G}_1, \mathcal{G}_2, Q, I$*: if* $\mathsf{safe}_n(\mathsf{C}^{\mathsf{sp}}, \mathsf{IM}, \mathsf{S}, \mathcal{R}, \mathcal{G}_1, Q, I)$*, then* $\mathsf{safe}_n(\mathsf{C}^{\mathsf{sp}}, \mathsf{IM}, \mathsf{S}, \mathcal{R}, \mathcal{G}_1 \cup \mathcal{G}_2, Q, I)$.

Proof. By induction on n.

Base case n=0
Pick arbitrary $n, \mathsf{C}, \mathsf{IM}, \mathsf{S}, \mathcal{R}, \mathcal{G}_1, \mathcal{G}_2, QI$. It then suffices to show: $\mathsf{safe}_0(\mathsf{C}^{\mathsf{sp}}, \mathsf{IM}, \mathsf{S}, \mathcal{R}, \mathcal{G}_1 \cup \mathcal{G}_2, Q, I)$, which follows from the definition of safe_0.

Inductive case n=m+1

$$\forall \mathsf{C}, \mathsf{IM}, \mathsf{S}, \mathcal{R}, \mathcal{G}_1, \mathcal{G}_2, Q, I.\ \mathsf{safe}_m(\mathsf{C}^{\mathsf{sp}}, \mathsf{IM}, \mathsf{S}, \mathcal{R}, \mathcal{G}_1, Q, I) \Rightarrow \mathsf{safe}_m(\mathsf{C}^{\mathsf{sp}}, \mathsf{IM}, \mathsf{S}, \mathcal{R}, \mathcal{G}_1 \cup \mathcal{G}_2, Q, I) \quad \text{(I.H.)}$$

Pick arbitrary $\mathsf{C}, \mathsf{IM}, \mathsf{S}, \mathcal{R}, \mathcal{G}_1, \mathcal{G}_2, Q, I$ such that:

$$\mathsf{safe}_n(\mathsf{C}^{\mathsf{sp}}, \mathsf{IM}, \mathsf{S}, \mathcal{R}, \mathcal{G}_1, Q, I) \tag{6}$$

We need to show: $\mathsf{safe}_n(\mathsf{C}^{\mathsf{sp}}, \mathsf{IM}, \mathsf{S}, \mathcal{R}, \mathcal{G}_1 \cup \mathcal{G}_2, Q, I)$. Property (1) follows immediately from (6).

To show property (4), pick arbitrary $\tau, \mathsf{IM}_1, \mathsf{IM}_2, \mathsf{B}_1, \mathsf{B}_2, \mathsf{C}', \mathsf{IM}', \mathsf{S}', l \neq \lightning$ such that $\mathsf{IM} = \Downarrow(\mathsf{IM}_1, \mathsf{B}_1, \tau)$, $\mathsf{IM}' = \Downarrow(\mathsf{IM}_2, \mathsf{B}_2, \tau)$ and $\mathsf{C}, \mathsf{S}, \mathsf{IM}_1, \mathsf{B}_1 \xRightarrow{\tau:l} \mathsf{C}', \mathsf{S}', \mathsf{IM}_2, \mathsf{B}_2$. From (6) we then have $((\mathsf{IM}, \mathsf{S}), (\mathsf{IM}', \mathsf{S})) \in \mathcal{G}_1 \cup \mathsf{A}_{\mathsf{sp}}$ and $\mathsf{safe}_m(\mathsf{C}^{\mathsf{sp}}, \mathsf{IM}', \mathsf{S}', \mathcal{R}, \mathcal{G}_1, Q, I)$. As such, from (I.H.) we have $((\mathsf{IM}, \mathsf{S}), (\mathsf{IM}', \mathsf{S})) \in \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathsf{A}_{\mathsf{sp}}$ and $\mathsf{safe}_m(\mathsf{C}^{\mathsf{sp}}, \mathsf{IM}', \mathsf{S}', \mathcal{R}, \mathcal{G}_1 \cup \mathcal{G}_2, Q, I)$, as required.

To show property (3), pick arbitrary $\mathsf{IM}', \mathsf{S}'$ such that $((\mathsf{IM}, \mathsf{S}), (\mathsf{IM}', \mathsf{S}')) \in \mathcal{R} \cup \mathsf{A}_{\mathsf{sp}}$. From (6) we then have $\mathsf{safe}_m(\mathsf{C}^{\mathsf{sp}}, \mathsf{IM}', \mathsf{S}', \mathcal{R}, \mathcal{G}_1, Q,)$, and thus from (I.H.) we have $\mathsf{safe}_m(\mathsf{C}^{\mathsf{sp}}, \mathsf{IM}', \mathsf{S}', \mathcal{R}, \mathcal{G}_1 \cup \mathcal{G}_2, Q,)$, as required. □

Lemma 14 (Post weakening)**.** *For all* $n, \mathsf{C}, \mathsf{IM}, \mathsf{S}, \mathcal{R}, \mathcal{G}, Q, Q', I, I'$*: if* $\mathsf{safe}_n(\mathsf{C}^{\mathsf{sp}}, \mathsf{IM}, \mathsf{S}, \mathcal{R}, \mathcal{G}, Q, I)$ *then* $\mathsf{safe}_n(\mathsf{C}^{\mathsf{sp}}, \mathsf{IM}, \mathsf{S}, \mathcal{R}, \mathcal{G}, Q \cup Q', I \cup I')$.

Proof. By induction on n.

Base case n=0
Pick arbitrary $n, \mathsf{C}, \mathsf{IM}, \mathsf{S}, \mathcal{R}, \mathcal{G}, Q, Q', I, I'$. It then suffices to show: $\mathsf{safe}_0(\mathsf{C}^{\mathsf{sp}}, \mathsf{IM}, \mathsf{S}, \mathcal{R}, \mathcal{G}, Q \cup Q', I \cup I')$, which follows from the definition of safe_0.

Inductive case n=m+1

$$\forall \mathsf{C}, \mathsf{IM}, \mathsf{S}, \mathcal{R}, \mathcal{G}, Q, Q', I, I'.\ \mathsf{safe}_m(\mathsf{C}^{\mathsf{sp}}, \mathsf{IM}, \mathsf{S}, \mathcal{R}, \mathcal{G}, Q, I) \Rightarrow \mathsf{safe}_m(\mathsf{C}^{\mathsf{sp}}, \mathsf{IM}, \mathsf{S}, \mathcal{R}, \mathcal{G}, Q \cup Q', I \cup I') \quad \text{(I.H.)}$$

Pick arbitrary $\mathsf{C}, \mathsf{IM}, \mathsf{S}, \mathcal{R}, \mathcal{G}, Q, Q', I, I'$ such that:

$$\mathsf{safe}_n(\mathsf{C}^{\mathsf{sp}}, \mathsf{IM}, \mathsf{S}, \mathcal{R}, \mathcal{G}, Q, I) \quad (7)$$

We then need to show: $\mathsf{safe}_n(\mathsf{C}^{\mathsf{sp}}, \mathsf{IM}, \mathsf{S}, \mathcal{R}, \mathcal{G}, Q \cup Q', I \cup I')$. Property (1) follows from (7) and the facts that $Q \subseteq Q \cup Q'$ and $I \subseteq I \cup I'$.

To show property (4), pick arbitrary $\tau, \mathsf{IM}_1, \mathsf{IM}_2, \mathsf{B}_1, \mathsf{B}_2, \mathsf{C}', \mathsf{IM}', \mathsf{S}', l \neq \lightning$ such that $\mathsf{IM} = \Downarrow(\mathsf{IM}_1, \mathsf{B}_1, \tau)$, $\mathsf{IM}' = \Downarrow(\mathsf{IM}_2, \mathsf{B}_2, \tau)$ and $\mathsf{C}, \mathsf{S}, \mathsf{IM}_1, \mathsf{B}_1 \xRightarrow{\tau:l} \mathsf{C}', \mathsf{S}', \mathsf{IM}_2, \mathsf{B}_2$. From (7) we then have $((\mathsf{IM}, \mathsf{S}), (\mathsf{IM}', \mathsf{S})) \in \mathcal{G} \cup \mathsf{A}_{\mathsf{sp}}$ and $\mathsf{safe}_m(\mathsf{C}^{\mathsf{sp}}, \mathsf{IM}', \mathsf{S}', \mathcal{R}, \mathcal{G}, Q, I)$. As such, from (I.H.) we have $((\mathsf{IM}, \mathsf{S}), (\mathsf{IM}', \mathsf{S})) \in \mathcal{G} \cup \mathsf{A}_{\mathsf{sp}}$ and $\mathsf{safe}_m(\mathsf{C}^{\mathsf{sp}}, \mathsf{IM}', \mathsf{S}', \mathcal{R}, \mathcal{G}, Q \cup Q', I \cup I')$, as required.

To show property (3), pick arbitrary $\mathsf{IM}', \mathsf{S}'$ such that $((\mathsf{IM}, \mathsf{S}), (\mathsf{IM}', \mathsf{S}')) \in \mathcal{R} \cup \mathsf{A}_{\mathsf{sp}}$. From (7) we have $\mathsf{safe}_m(\mathsf{C}^{\mathsf{sp}}, \mathsf{IM}', \mathsf{S}', \mathcal{R}, \mathcal{G}, Q, I)$. As such, from (I.H.) we have $\mathsf{safe}_m(\mathsf{C}^{\mathsf{sp}}, \mathsf{IM}', \mathsf{S}', \mathcal{R}, \mathcal{G}, Q \cup Q', I \cup I')$, as required. □

Lemma 15 (Sequential safety)**.** *For all* $n, \mathsf{c}_1, \mathsf{c}_2, \mathsf{IM}_1, \mathsf{S}_1, \mathcal{R}_1, \mathcal{R}_2, \mathcal{G}_1, \mathcal{G}_2, R, Q, I$*: if* $\mathsf{safe}_n(\mathsf{c}_1^{\mathsf{sp}}, \mathsf{IM}_1, \mathsf{S}_1, \mathcal{R}_1, \mathcal{G}_1, R, I)$, $\forall (\mathsf{IM}_2, \mathsf{S}_2) \in R.\ \mathsf{safe}_n(\mathsf{c}_2^{\mathsf{sp}}, \mathsf{IM}_2, \mathsf{S}_2, \mathcal{R}_2, \mathcal{G}_2, Q,)$ *and* $\mathsf{wf}(\mathcal{R}_1 \cap \mathcal{R}_2, \mathcal{G}_1 \cup \mathcal{G}_2, Q, I)$*, then* $\mathsf{safe}_n((\mathsf{c}_1; \mathsf{c}_2)^{\mathsf{sp}}, \mathsf{IM}_1, \mathsf{S}_1, \mathcal{R}_1 \cap \mathcal{R}_2, \mathcal{G}_1 \cup \mathcal{G}_2, Q, I)$.

Proof. By induction on n.

Base case n=0
Pick arbitrary $n, \mathsf{c}_1, \mathsf{c}_2, \mathsf{IM}_1, \mathsf{S}_1, \mathcal{R}_1, \mathcal{R}_2, \mathcal{G}_1, \mathcal{G}_2, R, Q, I$. It then suffices to show $\mathsf{safe}_0((\mathsf{c}_1; \mathsf{c}_2)^{\mathsf{sp}}, \mathsf{IM}_1, \mathsf{S}_1, \mathcal{R}_1 \cap \mathcal{R}_2, \mathcal{G}_1 \cup \mathcal{G}_2, Q, I)$, which follows from the definition of safe_0.

Inductive case n=m+1

$$\begin{aligned}&\forall \mathsf{c}_1, \mathsf{c}_2, \mathsf{IM}_1, \mathsf{S}_1, \mathcal{R}_1, \mathcal{R}_2, \mathcal{G}_1, \mathcal{G}_2, R, Q, I, I'.\\ &\quad \mathsf{safe}_m(\mathsf{c}_1^{\mathsf{sp}}, \mathsf{IM}_1, \mathsf{S}_1, \mathcal{R}_1, \mathcal{G}_1, R, I') \wedge \forall (\mathsf{IM}_2, \mathsf{S}_2) \in R.\ \mathsf{safe}_m(\mathsf{c}_2^{\mathsf{sp}}, \mathsf{IM}_2, \mathsf{S}_2, \mathcal{R}_2, \mathcal{G}_2, Q, I)\\ &\quad \wedge\ I' \subseteq I \wedge \mathsf{wf}(\mathcal{R}_1 \cap \mathcal{R}_2, \mathcal{G}_1 \cup \mathcal{G}_2, Q, I) \Rightarrow\\ &\qquad \mathsf{safe}_m((\mathsf{c}_1; \mathsf{c}_2)^{\mathsf{sp}}, \mathsf{IM}_1, \mathsf{S}_1, \mathcal{R}_1 \cap \mathcal{R}_2, \mathcal{G}_1 \cup \mathcal{G}_2, Q, I) \quad \text{(I.H.)}\end{aligned}$$

Pick arbitrary $n, \mathsf{c}_1, \mathsf{c}_2, \mathsf{IM}_1, \mathsf{S}_1, \mathcal{R}_1, \mathcal{R}_2, \mathcal{G}_1, \mathcal{G}_2, R, Q, I, I'$ such that:

$$\mathsf{safe}_n(\mathsf{c}_1^{\mathsf{sp}}, \mathsf{IM}_1, \mathsf{S}_1, \mathcal{R}_1, \mathcal{G}_1, R, I') \quad (8)$$

$$\forall(\mathsf{IM}_2, \mathsf{S}_2) \in R.\ \mathsf{safe}_n(\mathsf{c}_2^{\mathsf{sp}}, \mathsf{IM}_2, \mathsf{S}_2, \mathcal{R}_2, \mathcal{G}_2, Q, I) \tag{9}$$

$$I' \subseteq I \wedge \mathsf{wf}(\mathcal{R}_1 \cap \mathcal{R}_2, \mathcal{G}_1 \cup \mathcal{G}_2, Q, I) \tag{10}$$

We then need to show: $\mathsf{safe}_n(\mathsf{c}_1; \mathsf{c}_2^{\mathsf{sp}}, \mathsf{IM}_1, \mathsf{S}_1, \mathcal{R}_1 \cap \mathcal{R}_2, \mathcal{G}_1 \cup \mathcal{G}_2, Q, I)$. For property (1) note that $\mathsf{c}_1; \mathsf{c}_2 \neq \mathsf{C}_{\mathbf{skip}}$. Moreover, from (8) we have $(\mathsf{IM}_1, \mathsf{S}_1) \in I'$ and thus from (10) we have $(\mathsf{IM}_1, \mathsf{S}_1) \in I$, as required.

To show property (4), pick arbitrary $\tau, \mathsf{IM}_3, \mathsf{IM}_4, \mathsf{B}_1, \mathsf{B}_2, \mathsf{c}', \mathsf{IM}_2, \mathsf{S}_2, l \neq \not{\epsilon}$ such that $\mathsf{IM}_1 = \Downarrow(\mathsf{IM}_3, \mathsf{B}_1, \tau)$, $\mathsf{IM}_2 = \Downarrow(\mathsf{IM}_4, \mathsf{B}_2, \tau)$ and $\mathsf{c}_1; \mathsf{c}_2, \mathsf{IM}_3, \mathsf{B}_1, \mathsf{S}_1 \overset{\tau:l}{\Longrightarrow} \mathsf{c}', \mathsf{IM}_4, \mathsf{B}_2, \mathsf{S}_2$. From the operational semantics there are then four cases to consider:
1) $l = \epsilon$, $\mathsf{c}_1 = \mathbf{skip}$, $\mathsf{c}' = \mathsf{c}_2$, $\mathsf{S}_2 = \mathsf{S}_1$, $\mathsf{IM}_3 = \mathsf{IM}_4$ and $\mathsf{B}_1 = \mathsf{B}_2$; or
2) $l \neq \epsilon$, $\mathsf{c}_1, \mathsf{S}_1 \xrightarrow{\tau:l} \mathsf{c}_3, \mathsf{S}_2$, $\mathsf{IM}_3, \mathsf{B}_1 \xrightarrow{\tau:l} \mathsf{IM}_4, \mathsf{B}_2$ and $\mathsf{c}' = \mathsf{c}_3; \mathsf{c}_2$; or
3) $l = \epsilon$, $\mathsf{c}' = \mathsf{c}_1; \mathsf{c}_2$, $\mathsf{S}_2 = \mathsf{S}_1$, $\mathsf{IM}_3, \mathsf{B}_1 \xrightarrow{\tau:\epsilon} \mathsf{IM}_4, \mathsf{B}_2$; or
4) $l = \epsilon$, $\mathsf{c}_1, \mathsf{S}_1 \xrightarrow{\tau:l} \mathsf{c}_3, \mathsf{S}_2$, $\mathsf{c}' = \mathsf{c}_3; \mathsf{c}_2$, $\mathsf{IM}_4 = \mathsf{IM}_3$, $\mathsf{B}_2 = \mathsf{B}_1$.

In case (1), we then have $\mathsf{IM}_1 = \mathsf{IM}_2$, and thus since $\mathcal{G}_1 \cup \mathcal{G}_2$ is reflexive ($\mathsf{wf}(\mathcal{R}_1 \cap \mathcal{R}_2, \mathcal{G}_1 \cup \mathcal{G}_2, Q, I)$ holds), we know that $((\mathsf{IM}_1, \mathsf{S}_1), (\mathsf{IM}_2, \mathsf{S}_1)) \in \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathsf{A}_{\mathsf{sp}}$, as required. Moreover, since $\mathsf{c}_1 = \mathbf{skip}$, from (9) we have $(\mathsf{IM}_1, \mathsf{S}_1) \in R$. As such, since $\mathsf{IM}_2 = \mathsf{IM}_1$, $\mathsf{S}_1 = \mathsf{S}_2$, $\mathsf{c}' = \mathsf{c}_2$ and $(\mathsf{IM}_1, \mathsf{S}_1) \in R$, from (9) and Lemma 9 we know that $\mathsf{safe}_m((\mathsf{c}')^{\mathsf{sp}}, \mathsf{IM}_2, \mathsf{S}_2, \mathcal{R}_2, \mathcal{G}_2, Q, I)$ holds. As such, from Lemma 13 and Lemma 12 we have $\mathsf{safe}_m((\mathsf{c}')^{\mathsf{sp}}, \mathsf{IM}_2, \mathsf{S}_2, \mathcal{R}_1 \cap \mathcal{R}_2, \mathcal{G}_1 \cup \mathcal{G}_2, Q, I)$, as required.

In cases (2, 3, 4) from the operational semantics we then have $\mathsf{c}_1, \mathsf{IM}_3, \mathsf{B}_1, \mathsf{S}_1 \overset{\tau:l}{\Longrightarrow} \mathsf{c}_3, \mathsf{IM}_4, \mathsf{B}_2, \mathsf{S}_2$ with $\mathsf{c}_3 = \mathsf{c}_1$ in case (3). As such, from (8) we have $((\mathsf{IM}_1, \mathsf{S}_1), (\mathsf{IM}_2, \mathsf{S}_1)) \in \mathcal{G}_1 \cup \mathsf{A}_{\mathsf{sp}} \subseteq \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathsf{A}_{\mathsf{sp}}$. Moreover, from (8) we have $\mathsf{safe}_m(\mathsf{c}_3^{\mathsf{sp}}, \mathsf{IM}_2, \mathsf{S}_2, \mathcal{R}_1, \mathcal{G}_1, R, I)$. On the other hand, from (9) and Lemma 9 we have $\forall(\mathsf{IM}_2, \mathsf{S}_2) \in R.\ \mathsf{safe}_m(\mathsf{c}_2^{\mathsf{sp}}, \mathsf{IM}_2, \mathsf{S}_2, \mathcal{R}_2, \mathcal{G}_2, Q, I)$. As such, since $\mathsf{c}' = \mathsf{c}_3; \mathsf{c}_2$, from (I.H.) we have $\mathsf{safe}_m((\mathsf{c}')^{\mathsf{sp}}, \mathsf{IM}_2, \mathsf{S}_2, \mathcal{R}_1 \cap \mathcal{R}_2, \mathcal{G}_1 \cup \mathcal{G}_2, Q, I)$, as required.

To show property (3), pick arbitrary $\mathsf{IM}', \mathsf{S}'$ such that $((\mathsf{IM}_1, \mathsf{S}_1), (\mathsf{IM}', \mathsf{S}')) \in (\mathcal{R}_1 \cap \mathcal{R}_2) \cup \mathsf{A}_{\mathsf{sp}}$; i.e. $(\mathsf{IM}_1, \mathsf{S}_1), (\mathsf{IM}', \mathsf{S}')) \in \mathcal{R}_1 \cup \mathsf{A}_{\mathsf{sp}}$. As such, from (8) we have $\mathsf{safe}_m(\mathsf{c}_1^{\mathsf{sp}}, \mathsf{IM}', \mathsf{S}', \mathcal{R}_1, \mathcal{G}_1, R, I)$. On the other hand, from (9) and Lemma 9 we have $\forall(\mathsf{IM}_2, \mathsf{S}_2) \in R.\ \mathsf{safe}_m(\mathsf{c}_2^{\mathsf{sp}}, \mathsf{IM}_2, \mathsf{S}_2, \mathcal{R}_2, \mathcal{G}_2, Q, I)$. As such, from (I.H.) we have $\mathsf{safe}_m((\mathsf{c}_1; \mathsf{c}_2)^{\mathsf{sp}}, \mathsf{IM}', \mathsf{S}', \mathcal{R}_1 \cap \mathcal{R}_2, \mathcal{G}_1 \cup \mathcal{G}_2, Q, I)$, as required. □

Lemma 16 (Parallel safety). *For all* $n, \mathsf{c}, \mathsf{C}, \mathsf{IM}, \mathsf{S}, \mathcal{R}_1, \mathcal{R}_2, \mathcal{G}_1, \mathcal{G}_2, Q_1, Q_2, Q, I_1, I_2, I, A_1, A_2$: *if* $\mathsf{safe}_n(\mathsf{c}^{\mathsf{sp}}, \mathsf{IM}, \mathsf{S}, \mathcal{R}_1, \mathcal{G}_1, Q_1, I_1)$, $\mathsf{safe}_n(\mathsf{C}^{\mathsf{sp}}, \mathsf{IM}, \mathsf{S}, \mathcal{R}_2, \mathcal{G}_2, Q_2, I_2)$, $\langle \mathcal{R}_1, \mathcal{G}_1 \rangle$ *and* $\langle \mathcal{R}_2, \mathcal{G}_2 \rangle$ *are semantically non-interfering,* $\mathtt{fr}(\mathsf{c}) \subseteq A_1$, $\mathsf{s\text{-}stable}(\mathcal{R}_1, A_1)$, $\mathsf{s\text{-}stable}(Q_1, A_1)$, $\mathtt{wr}(C_2) \cap A_1 = \emptyset$, $\mathtt{fr}(\mathsf{C}) \subseteq A_2$, $\mathsf{s\text{-}stable}(\mathcal{R}_2, A_2)$, $\mathsf{s\text{-}stable}(Q_2, A_2)$, $\mathtt{wr}(C_1) \cap A_2 = \emptyset$, $\mathsf{wf}(\mathcal{R}, \mathcal{G}_1 \cup \mathcal{G}_2, Q, I)$, $\mathcal{R} \subseteq \mathcal{R}_1 \cap \mathcal{R}_2$, $Q_1 \cap Q_2 \subseteq Q$ *and* $I_1 \cap I_2 \subseteq I$ *then* $\mathsf{safe}_n((\mathsf{c} \,||\, \mathsf{C})^{\mathsf{sp}}, \mathsf{IM}, \mathsf{S}, \mathcal{R}, \mathcal{G}_1 \cup \mathcal{G}_2, Q, I)$.

Proof. By induction on n.

Base case $n = 0$
Pick arbitrary $n, \mathsf{c}, \mathsf{C}, \mathsf{IM}, \mathsf{S}, \mathcal{R}_1, \mathcal{R}_2, \mathcal{G}_1, \mathcal{G}_2, R, Q_1, Q_2, Q, I_1, I_2, I, A_1, A_2$. It then suffices to show $\mathsf{safe}_0((\mathsf{c} \,||\, \mathsf{C})^{\mathsf{sp}}, \mathsf{IM}, \mathsf{S}, \mathcal{R}, \mathcal{G}_1 \cup \mathcal{G}_2, Q, I)$, which follows from the definition of safe_0.

Inductive case $n{=}m{+}1$

$$\begin{aligned}
&\forall \mathsf{c}, \mathsf{C}, \mathsf{IM}, \mathsf{S}, \mathcal{R}_1, \mathcal{R}_2, \mathcal{G}_1, \mathcal{G}_2, Q_1, Q_2, Q, I_1, I_2, I, A_1, A_2.\\
&\quad \mathsf{safe}_m(\mathsf{c}^{\mathsf{sp}}, \mathsf{IM}, \mathsf{S}, \mathcal{R}_1, \mathcal{G}_1, Q_1, I_1) \land \mathsf{safe}_m(\mathsf{C}^{\mathsf{sp}}, \mathsf{IM}, \mathsf{S}, \mathcal{R}_2, \mathcal{G}_2, Q_2, I_2)\\
&\quad \land\ \langle \mathcal{R}_1, \mathcal{G}_1 \rangle \text{ and } \langle \mathcal{R}_2, \mathcal{G}_2 \rangle \text{ are semantically non-interfering}\\
&\quad \land\ \mathtt{fr}(\mathsf{c}) \subseteq A_1 \land \text{s-stable}(\mathcal{R}_1, A_1) \land \text{s-stable}(Q_1, A_1) \land \mathtt{wr}(C_2) \cap A_1{=}\emptyset\\
&\quad \land\ \mathtt{fr}(\mathsf{C}) \subseteq A_2 \land \text{s-stable}(\mathcal{R}_2, A_2) \land \text{s-stable}(Q_2, A_2) \land \mathtt{wr}(C_1) \cap A_2{=}\emptyset\\
&\quad \land\ \mathsf{wf}(\mathcal{R}, \mathcal{G}_1 \cup \mathcal{G}_2, Q, I)\\
&\quad \land\ \mathcal{R} \subseteq \mathcal{R}_1 \cap \mathcal{R}_2 \land Q_1 \cap Q_2 \subseteq Q \land I_1 \cap I_2 \subseteq I \Rightarrow\\
&\qquad \mathsf{safe}_m((\mathsf{c} \,||\, \mathsf{C})^{\mathsf{sp}}, \mathsf{IM}, \mathsf{S}, \mathcal{R}, \mathcal{G}_1 \cup \mathcal{G}_2, Q, I) \qquad \text{(I.H.)}
\end{aligned}$$

Pick arbitrary $n, \mathsf{c}, \mathsf{C}, \mathsf{IM}, \mathsf{S}, \mathcal{R}_1, \mathcal{R}_2, \mathcal{R}, \mathcal{G}_1, \mathcal{G}_2, Q_1, Q_2, Q, A_1, A_2$ such that:

$$\mathsf{safe}_n(\mathsf{c}^{\mathsf{sp}}, \mathsf{IM}, \mathsf{S}, \mathcal{R}_1, \mathcal{G}_1, Q_1, I_1) \tag{11}$$
$$\mathsf{safe}_n(\mathsf{C}^{\mathsf{sp}}, \mathsf{IM}, \mathsf{S}, \mathcal{R}_2, \mathcal{G}_2, Q_2, I_2) \tag{12}$$
$$\langle \mathcal{R}_1, \mathcal{G}_1 \rangle \text{ and } \langle \mathcal{R}_2, \mathcal{G}_2 \rangle \text{ are semantically non-interfering} \tag{13}$$
$$\mathtt{fr}(\mathsf{c}) \subseteq A_1 \land \text{s-stable}(\mathcal{R}_1, A_1) \land \text{s-stable}(Q_1, A_1) \land \mathtt{wr}(C_2) \cap A_1{=}\emptyset \tag{14}$$
$$\mathtt{fr}(\mathsf{C}) \subseteq A_2 \land \text{s-stable}(\mathcal{R}_2, A_2) \land \text{s-stable}(Q_2, A_2) \land \mathtt{wr}(C_1) \cap A_2{=}\emptyset \tag{15}$$
$$\mathsf{wf}(\mathcal{R}, \mathcal{G}_1 \cup \mathcal{G}_2, Q, I) \tag{16}$$
$$\mathcal{R} \subseteq \mathcal{R}_1 \cap \mathcal{R}_2 \land Q_1 \cap Q_2 \subseteq Q \land I_1 \cap I_2 \subseteq I \tag{17}$$

We then need to show: $\mathsf{safe}_n((\mathsf{c} \,||\, \mathsf{C})^{\mathsf{sp}}, \mathsf{IM}, \mathsf{S}, \mathcal{R}, \mathcal{G}_1 \cup \mathcal{G}_2, Q, I)$. For property (1), note that $\mathsf{c} \,||\, \mathsf{C} \neq \mathsf{C}_{\mathbf{skip}}$.Moreover, from (11) and (12) we know $(\mathsf{IM}, \mathsf{S}) \in I_1$ and $(\mathsf{IM}, \mathsf{S}) \in I_2$ and thus $(\mathsf{IM}, \mathsf{S}) \in I_1 \cap I_2$. As such, from (17) we have $(\mathsf{IM}, \mathsf{S}) \in I$, as required.

To show property (4), pick arbitrary $\tau, \mathsf{IM}_1, \mathsf{IM}_2, \mathsf{B}_1, \mathsf{B}_2, \mathsf{C}', \mathsf{IM}_2, \mathsf{S}', l{\neq}\lightning$ such that $\mathsf{IM}{=}\Downarrow(\mathsf{IM}_1, \mathsf{B}_1, \tau)$, $\mathsf{IM}'{=}\Downarrow(\mathsf{IM}_2, \mathsf{B}_2, \tau)$ and $\mathsf{c} \,||\, \mathsf{C}, \mathsf{IM}_1, \mathsf{B}_1, \mathsf{S} \overset{\tau:l}{\Longrightarrow} \mathsf{C}', \mathsf{IM}_2, \mathsf{B}_2, \mathsf{S}'$. Let τ_c denote the thread executing c. From the operational semantics there are then six cases to consider:

1) $\tau{=}\tau_c$, $l{=}\epsilon$, $\mathsf{c}, \mathsf{S} \xrightarrow{\tau:l} \mathsf{c}', \mathsf{S}'$, $\mathsf{IM}_2{=}\mathsf{IM}_1$, $\mathsf{B}_2{=}\mathsf{B}_1$, and $\mathsf{C}'{=}\mathsf{c}' \,||\, \mathsf{C}$;
2) $\tau{=}\tau_c$, $l{=}\epsilon$, $\mathsf{IM}_1, \mathsf{B}_1 \xrightarrow{\tau:l} \mathsf{IM}_2, \mathsf{B}_2, \mathsf{S}'{=}\mathsf{S}$ and $\mathsf{C}'{=}\mathsf{c} \,||\, \mathsf{C}$;
3) $\tau{=}\tau_c$, $l{\neq}\epsilon$, $\mathsf{c}, \mathsf{S} \xrightarrow{\tau:l} \mathsf{c}', \mathsf{S}'$, $\mathsf{IM}_1, \mathsf{B}_1 \xrightarrow{\tau:l} \mathsf{IM}_2, \mathsf{B}_2$, and $\mathsf{C}'{=}\mathsf{c}' \,||\, \mathsf{C}$;
4) $\tau \in dom(\mathsf{C})$, $l{=}\epsilon$, $\mathsf{C}(\tau), \mathsf{S} \xrightarrow{\tau:l} \mathsf{c}', \mathsf{S}'$, $\mathsf{IM}_2{=}\mathsf{IM}_1$, $\mathsf{B}_2{=}\mathsf{B}_1$, and $\mathsf{C}'{=}\mathsf{c} \,||\, \mathsf{C}[\tau \mapsto \mathsf{c}']$;
5) $\tau \in dom(\mathsf{C})$, $l{=}\epsilon$, $\mathsf{IM}_1, \mathsf{B}_1 \xrightarrow{\tau:l} \mathsf{IM}_2, \mathsf{B}_2, \mathsf{S}'{=}\mathsf{S}$ and $\mathsf{C}'{=}\mathsf{c} \,||\, \mathsf{C}$;
6) $\tau \in dom(\mathsf{C})$, $l{\neq}\epsilon$, $\mathsf{C}(\tau), \mathsf{S} \xrightarrow{\tau:l} \mathsf{c}', \mathsf{S}'$, $\mathsf{IM}_1, \mathsf{B}_1 \xrightarrow{\tau:l} \mathsf{IM}_2, \mathsf{B}_2$, and $\mathsf{C}'{=}\mathsf{c} \,||\, \mathsf{C}[\tau \mapsto \mathsf{c}']$.

In cases (1, 2, 3) from the operational semantics we then have $\mathsf{c}, \mathsf{IM}_1, \mathsf{B}_1, \mathsf{S} \overset{\tau:l}{\Longrightarrow} \mathsf{c}', \mathsf{IM}_2, \mathsf{B}_2, \mathsf{S}'$ and $\mathsf{C}'{=}\mathsf{c}' \,||\, \mathsf{C}$ with $\mathsf{c}'{=}\mathsf{c}$ in case (2). As such, from (11) we have $((\mathsf{IM}, \mathsf{S}), (\mathsf{IM}', \mathsf{S})) \in \mathcal{G}_1 \cup \mathsf{A}_{\mathsf{sp}} \subseteq \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathsf{A}_{\mathsf{sp}}$. Moreover, from (11) we have $\mathsf{safe}_m((\mathsf{c}')^{\mathsf{sp}}, \mathsf{IM}', \mathsf{S}', \mathcal{R}_1, \mathcal{G}_1, Q_1,)$. On the other hand, since $((\mathsf{IM}, \mathsf{S}), (\mathsf{IM}', \mathsf{S})) \in \mathcal{G}_1 \cup \mathsf{A}_{\mathsf{sp}}$, from (13) we know $((\mathsf{IM}, \mathsf{S}), (\mathsf{IM}', \mathsf{S})) \in \mathcal{R}_2 \cup \mathsf{A}_{\mathsf{sp}}$. As such,from (12) we have $\mathsf{safe}_m(\mathsf{C}^{\mathsf{sp}}, \mathsf{IM}', \mathsf{S}, \mathcal{R}_2, \mathcal{G}_2, Q_2,)$. Moreover, from Prop. 1 we know $\mathsf{S} \sim_{\overline{\mathtt{wr}(\mathsf{c})}} \mathsf{S}'$, and thus since from (15) we have $\mathtt{wr}(\mathsf{c}) \cap A_2{=}\emptyset$, we have $\mathsf{S} \sim_{A_2} \mathsf{S}'$. As such, since $\mathsf{safe}_m(\mathsf{C}^{\mathsf{sp}}, \mathsf{IM}', \mathsf{S}, \mathcal{R}_2, \mathcal{G}_2, Q_2,)$, from (15) and Prop. 1 we have $\mathsf{safe}_m(\mathsf{C}^{\mathsf{sp}}, \mathsf{IM}', \mathsf{S}', \mathcal{R}_2, \mathcal{G}_2, Q_2,)$. Furthermore, from (14) and Prop. 1 we have $\mathtt{fr}(\mathsf{c}') \subseteq A_1$. Consequently, since $\mathsf{C}'{=}\mathsf{c}' \,||\, \mathsf{C}$, $\mathsf{safe}_m(\mathsf{c}'^{\mathsf{sp}}, \mathsf{IM}', \mathsf{S}', \mathcal{R}_1, \mathcal{G}_1, Q_1,)$, $\mathsf{safe}_m(\mathsf{C}^{\mathsf{sp}}, \mathsf{IM}', \mathsf{S}', \mathcal{R}_2, \mathcal{G}_2, Q_2,)$ and $\mathtt{fr}(\mathsf{c}') \subseteq A_1$, from (13), (14), (15), (16), (17) and (I.H.) we have $\mathsf{safe}_m((\mathsf{C}')^{\mathsf{sp}}, \mathsf{IM}', \mathsf{S}', \mathcal{R}, \mathcal{G}_1 \cup \mathcal{G}_2, Q,)$, as required.

In cases (4, 5, 6) from the operational semantics we have $\mathsf{C}, \mathsf{IM}_1, \mathsf{B}_1, \mathsf{S} \overset{\tau:l}{\Longrightarrow} \mathsf{C}[\tau \mapsto \mathsf{c}'], \mathsf{IM}_2, \mathsf{B}_2, \mathsf{S}'$ and $\mathsf{C}'{=}\mathsf{c}\,||\,\mathsf{C}[\tau \mapsto \mathsf{c}']$ with $\mathsf{c}'{=}\mathsf{C}(\tau)$ in case 5. The remainder of the proof is symmetric to that in cases (1, 2, 3) and is omitted here.

To show property (3), pick arbitrary $\mathsf{IM}', \mathsf{S}'$ such that $((\mathsf{IM}, \mathsf{S}), (\mathsf{IM}', \mathsf{S}')) \in \mathcal{R} \cup \mathsf{A}_{\mathsf{sp}}$. That is, from (17) we have $((\mathsf{IM}, \mathsf{S}), (\mathsf{IM}', \mathsf{S}')) \in \mathcal{R}_1 \cup \mathsf{A}_{\mathsf{sp}}$ and $((\mathsf{IM}, \mathsf{S}), (\mathsf{IM}', \mathsf{S}')) \in \mathcal{R}_2 \cup \mathsf{A}_{\mathsf{sp}}$. As such, from (11) we have $\mathsf{safe}_m(\mathsf{c}^{\mathsf{sp}}, \mathsf{IM}', \mathsf{S}', \mathcal{R}_1, \mathcal{G}_1, Q_1,)$. Similarly, from (12) we have $\mathsf{safe}_m(\mathsf{C}^{\mathsf{sp}}, \mathsf{IM}', \mathsf{S}', \mathcal{R}_2, \mathcal{G}_2, Q_2,)$. As such, from (I.H.) we have $\mathsf{safe}_m((\mathsf{c}\,||\,\mathsf{C})^{\mathsf{sp}}, \mathsf{IM}', \mathsf{S}', \mathcal{R}, \mathcal{G}_1 \cup \mathcal{G}_2, Q,)$, as required. □

Lemma 17. *For all* $\Delta, \mathsf{C}, \mathsf{S}, \mathsf{IM}, \mathsf{IM}', \mathsf{B}_1, \mathsf{B}_2, \mathsf{IM}_1, \mathsf{IM}_2, \tau$*:*
if $\Delta \vdash \mathsf{C}, \mathsf{S}, \mathsf{IM}_1, \mathsf{B}_1 \overset{\tau:\epsilon}{\Longrightarrow} \mathsf{C}, \mathsf{S}, \mathsf{IM}_2, \mathsf{B}_2$, $\mathsf{IM}{=}\Downarrow(\mathsf{IM}_1, \mathsf{B}_1, \tau)$ *and* $\mathsf{IM}'{=}\Downarrow(\mathsf{IM}_2, \mathsf{B}_2, \tau)$
then $\mathsf{IM}'{=}\mathsf{IM}$.

Proof. Pick an arbitrary $\Delta, \mathsf{C}, \mathsf{S}, \mathsf{IM}, \mathsf{IM}', \mathsf{B}_1, \mathsf{B}_2, \mathsf{IM}_1, \mathsf{IM}_2, \tau$ such that:

$$\Delta \vdash \mathsf{C}, \mathsf{S}, \mathsf{IM}_1, \mathsf{B}_1 \overset{\tau:\epsilon}{\Longrightarrow} \mathsf{C}, \mathsf{S}, \mathsf{IM}_2, \mathsf{B}_2 \tag{18}$$

$$\mathsf{IM}{=}\Downarrow(\mathsf{IM}_1, \mathsf{B}_1, \tau) \wedge \mathsf{IM}'{=}\Downarrow(\mathsf{IM}_2, \mathsf{B}_2, \tau) \tag{19}$$

Given the operational semantics, there are now three cases to consider: 1) $\mathsf{IM}_2{=}\mathsf{IM}_1$ and $\mathsf{B}_2{=}\mathsf{B}_1$; or 2) there exists $\langle y_o, v\rangle, \mathsf{b}$ such that $o \in \{\mathsf{v}, \mathsf{s}, \mathsf{p}\}$, $\mathsf{B}_1(\tau){=}\langle y_o, v\rangle.\mathsf{b}$, $\mathsf{B}_2{=}\mathsf{B}_1[\tau \mapsto \mathsf{b}]$ and $\mathsf{IM}_2{=}\mathsf{IM}_1[y_o \mapsto v]$; or 3) there exists $X, \mathsf{b}, \overline{x^i}$ such that $X{=}\overline{x^i}$, $\mathsf{B}_1(\tau){=}X.\mathsf{b}$, $\mathsf{B}_2{=}\mathsf{B}_1[\tau \mapsto \mathsf{b}]$ and $\mathsf{IM}_2{=}\mathsf{IM}_1[\overline{x^i_{\mathsf{s}} \mapsto \mathsf{IM}_1(x^i_{\mathsf{v}})}]$. In case (1) we simply have $\mathsf{IM}{=}\Downarrow(\mathsf{IM}_1, \mathsf{B}_1, \tau){=}\Downarrow(\mathsf{IM}_2, \mathsf{B}_2, \tau){=}\mathsf{IM}'$. In case (2, 3) by definition we have $\mathsf{IM}_1, \mathsf{B}_1 \overset{\tau:\epsilon}{\longrightarrow} \mathsf{IM}_2, \mathsf{B}_2$ and thus from Lemma 8 $\mathsf{IM}{=}\Downarrow(\mathsf{IM}_1, \mathsf{B}_1, \tau){=}\Downarrow(\mathsf{IM}_2, \mathsf{B}_2, \tau){=}\mathsf{IM}'$, as required. □

Theorem 7 (POG soundness). *For all* $\mathcal{R}, \mathcal{G}, P, \mathsf{C}, Q$*:*

$$\langle \mathcal{R}; \mathcal{G}\rangle \vdash \{P\}\ \mathsf{C}\ \{Q\} \Rightarrow \langle \mathcal{R}; \mathcal{G}\rangle \Vdash \{P\}\ \mathsf{C}\ \{Q\}$$

Proof. Pick arbitrary $\mathcal{R}, \mathcal{G}, P, \mathsf{C}, Q$ such that $\langle \mathcal{R}; \mathcal{G}\rangle \vdash \{P\}\ \mathsf{C}\ \{Q\}$ holds. We proceed by induction on the structure of $\langle \mathcal{R}; \mathcal{G}\rangle \vdash \{P\}\ \mathsf{C}\ \{Q\}$.

Case (SKIP)
We then have $[\![\mathsf{C}]\!]{=}\mathbf{skip}^{\mathsf{sp}}$, $Q{=}P$, $\mathcal{R}{=}\{P\}$ and $\mathcal{G}{=}\emptyset$. Pick an arbitrary n and $(\mathsf{IM}, \mathsf{S}) \in [\![P]\!]$. Given the interpretations of $\mathcal{R}$ and $\mathcal{G}$ and since P is stable, we know that $\mathsf{wf}([\![\mathcal{R}]\!]^{\mathsf{r}}, [\![\mathcal{G}]\!]^{\mathsf{g}}, [\![P]\!], [\![P_{\mathsf{p}}]\!])$ holds. As such, from Lemma 11 we have $\mathsf{safe}_n([\![\mathsf{C}]\!], \mathsf{IM}, \mathsf{S}, [\![\mathcal{R}]\!]^{\mathsf{r}}, [\![\mathcal{G}]\!]^{\mathsf{g}}, [\![P]\!], [\![P_{\mathsf{p}}]\!])$, as required.

Case (ASSIGN)
We then have $[\![\mathsf{C}]\!]{=}\mathsf{c}_t^{\mathsf{sp}}$ where $\mathsf{c}_t{=}\mathsf{a} := \mathsf{e}$ for some a, e; $\mathcal{R}{=}\{P, Q\}$ and $\mathcal{G}{=}\emptyset$ and $P \Rightarrow Q[\mathsf{e}/\mathsf{a}]$. Pick an arbitrary n. We proceed by induction on n.

Base case $n{=}0$
Pick an arbitrary $(\mathsf{IM}, \mathsf{S}) \in [\![P]\!]$. We must show $\mathsf{safe}_0([\![\mathsf{C}]\!], \mathsf{IM}, \mathsf{S}, [\![\mathcal{R}]\!]^{\mathsf{r}}, [\![\mathcal{G}]\!]^{\mathsf{g}}, [\![Q]\!], [\![Q_{\mathsf{p}}]\!])$, which follows trivially from the definition of safe_0.

Inductive case $n{=}m{+}1$

$$\forall (\mathsf{IM}, \mathsf{S}) \in [\![P]\!].\ \mathsf{safe}_m([\![\mathsf{C}]\!], \mathsf{IM}, \mathsf{S}, [\![\mathcal{R}]\!]^{\mathsf{r}}, [\![\mathcal{G}]\!]^{\mathsf{g}}, [\![Q]\!], [\![Q_{\mathsf{p}}]\!]) \tag{I.H.}$$

Pick an arbitrary $(\mathsf{IM},\mathsf{S}) \in [\![P]\!]$. For property (1) note that $\mathsf{a} := \mathsf{e} \neq \mathsf{C}_{\mathbf{skip}}$. Moreover, as $(\mathsf{IM},\mathsf{S}) \in [\![P]\!]$ and $P \Rightarrow Q[\mathsf{e}/\mathsf{a}]$ we also have $(\mathsf{IM},\mathsf{S}) \in [\![Q[\mathsf{e}/\mathsf{a}]]\!]$. As such, from Prop. 2 we have $(\mathsf{IM},\mathsf{S}) \in [\![Q_{\mathsf{p}}]\!]$, as required.

To show property (4), pick arbitrary $\tau, \mathsf{IM}_1, \mathsf{IM}_2, \mathsf{B}_1, \mathsf{B}_2, \mathsf{C}', \mathsf{S}', l \neq \lightning$ such that $\mathsf{IM}{=}\Downarrow(\mathsf{IM}_1, \mathsf{B}_1, \tau)$, $\mathsf{IM}'{=}\Downarrow(\mathsf{IM}_2, \mathsf{B}_2, \tau)$ and $\mathsf{c}_t, \mathsf{IM}_1, \mathsf{B}_1, \mathsf{S} \overset{\tau:l}{\Longrightarrow} \mathsf{C}', \mathsf{IM}_2, \mathsf{B}_2, \mathsf{S}'$. Given the operational semantics, there are now two cases to consider: 1) $l{=}\epsilon$, $\mathsf{C}'{=}\mathsf{c}_t$, $\mathsf{IM}_1, \mathsf{B}_1 \xrightarrow{\tau:l} \mathsf{IM}_2, \mathsf{B}_2, \mathsf{S}{=}\mathsf{S}'$; or 2) $\mathsf{C}'{=}\mathbf{skip}$, $\mathsf{IM}_2{=}\mathsf{IM}_1$, $\mathsf{B}_2{=}\mathsf{B}_1$, $\mathsf{S}'{=}\mathsf{S}[\mathsf{a} \mapsto \mathsf{S}(\mathsf{e})]$ and thus $\mathsf{IM}'{=}\mathsf{IM}$.

In case (1) from Lemma 8 we have $\mathsf{IM}{=}\mathsf{IM}'$. As such, since by definition $[\![\mathcal{G}]\!]^{\mathsf{g}}$ is reflexive, we have $((\mathsf{IM},\mathsf{S}),(\mathsf{IM}',\mathsf{S})) \in \mathcal{G}$. Moreover, since $\mathsf{C}'{=}\mathsf{c}_t$, $\mathsf{IM}'{=}\mathsf{IM}$ and $\mathsf{S}'{=}\mathsf{S}$, from (I.H.) we have $\mathsf{safe}_m((\mathsf{C}')^{\mathsf{sp}}, \mathsf{IM}', \mathsf{S}', [\![\mathcal{R}]\!]^{\mathsf{r}}, [\![\mathcal{G}]\!]^{\mathsf{g}}, [\![Q]\!], [\![Q_{\mathsf{p}}]\!])$, as required.

In case (2), since by definition $[\![\mathcal{G}]\!]^{\mathsf{g}}$ is reflexive and $\mathsf{IM}'{=}\mathsf{IM}$, we have $((\mathsf{IM},\mathsf{S}),(\mathsf{IM}',\mathsf{S})) \in \mathcal{G}$. Moreover, since $(\mathsf{IM},\mathsf{S}) \in [\![P]\!]$, $\mathsf{IM}{=}\mathsf{IM}'$ and $P \Rightarrow Q[\mathsf{e}/\mathsf{a}]$, from the S' definition we know $(\mathsf{IM}',\mathsf{S}') \in [\![Q]\!]$. On the other hand, since Q is stable, by definition we know $[\![Q]\!]$ is stable. Furthermore, from the interpretation of the triple and by definitions of $[\![.]\!]^{\mathsf{r}}$ and $[\![.]\!]^{\mathsf{g}}$ we know that $\mathsf{wf}([\![\mathcal{R}]\!]^{\mathsf{r}}, [\![\mathcal{G}]\!]^{\mathsf{g}}, [\![Q]\!], [\![Q_{\mathsf{p}}]\!])$ holds. As such, from Lemma 11 we have $\mathsf{safe}_m(\mathbf{skip}^{\mathsf{sp}}, \mathsf{IM}', \mathsf{S}', [\![\mathcal{R}]\!]^{\mathsf{r}}, [\![\mathcal{G}]\!]^{\mathsf{g}}, [\![Q]\!], [\![Q_{\mathsf{p}}]\!])$, as required.

To show property (3), pick arbitrary $\mathsf{IM}', \mathsf{S}'$ such that $((\mathsf{IM},\mathsf{S}),(\mathsf{IM}',\mathsf{S}')) \in [\![\{P,Q\}]\!]^{\mathsf{r}}$. From the definition of $[\![\mathcal{R}]\!]^{\mathsf{r}}$ and as $(\mathsf{IM},\mathsf{S}) \in [\![P]\!]$ we know $(\mathsf{IM}',\mathsf{S}') \in [\![P]\!]$. As such, from (I.H.) we have $\mathsf{safe}_m(\mathsf{C}^{\mathsf{sp}}, \mathsf{IM}', \mathsf{S}', [\![\mathcal{R}]\!]^{\mathsf{r}}, [\![\mathcal{G}]\!]^{\mathsf{g}}, [\![Q]\!], [\![Q_{\mathsf{p}}]\!])$, as required.

Similarly, let $\mathsf{IM}_a{=}\mathsf{IM}[\overline{x_{\mathsf{p}} \mapsto \mathsf{IM}(x_{\mathsf{s}})}]$, and let $\mathsf{IM}_s{=}\mathsf{IM}[x_{\mathsf{s}} \mapsto \mathsf{IM}(x_{\mathsf{v}})]$ for an arbitrary x. As P is stable and $(\mathsf{IM},\mathsf{S}) \in [\![P]\!]$, from the definition of stability we then know $(\mathsf{IM}_s,\mathsf{S}),(\mathsf{IM}_a,\mathsf{S}) \in [\![P]\!]$, and thus from (I.H.) we have $\mathsf{safe}_m(\mathsf{C}^{\mathsf{sp}}, \mathsf{IM}_s, \mathsf{S}, [\![\mathcal{R}]\!]^{\mathsf{r}}, [\![\mathcal{G}]\!]^{\mathsf{g}}, [\![Q]\!], [\![Q_{\mathsf{p}}]\!])$ and $\mathsf{safe}_m(\mathsf{C}^{\mathsf{sp}}, \mathsf{IM}_a, \mathsf{S}, [\![\mathcal{R}]\!]^{\mathsf{r}}, [\![\mathcal{G}]\!]^{\mathsf{g}}, [\![Q]\!], [\![Q_{\mathsf{p}}]\!])$, as required.

Case (WRITE)
We then have $[\![\mathsf{C}]\!]{=}\mathsf{c}_t^{\mathsf{sp}}$ where $\mathsf{c}_t{=}x_{\mathsf{v}} := \mathsf{e}$ for some $x_{\mathsf{v}}, \mathsf{e}$; $\mathcal{R}{=}\{P,Q\}$ and $\mathcal{G}{=}\{\langle x_{\mathsf{v}}, \mathsf{e}, P\rangle\}$ and $P \Rightarrow Q[\mathsf{e}/x_{\mathsf{v}}]$. Pick an arbitrary n. We proceed by induction on n.

Base case $n{=}0$
Pick an arbitrary $(\mathsf{IM},\mathsf{S}) \in [\![P]\!]$. We must show $\mathsf{safe}_0([\![\mathsf{C}]\!], \mathsf{IM}, \mathsf{S}, [\![\mathcal{R}]\!]^{\mathsf{r}}, [\![\mathcal{G}]\!]^{\mathsf{g}}, [\![Q]\!], [\![Q_{\mathsf{p}}]\!])$, which follows trivially from the definition of safe_0.

Inductive case $n{=}m{+}1$

$$\forall (\mathsf{IM},\mathsf{S}) \in [\![P]\!].\ \mathsf{safe}_m([\![\mathsf{C}]\!], \mathsf{IM}, \mathsf{S}, [\![\mathcal{R}]\!]^{\mathsf{r}}, [\![\mathcal{G}]\!]^{\mathsf{g}}, [\![Q]\!], [\![Q_{\mathsf{p}}]\!]) \tag{I.H.}$$

Pick an arbitrary $(\mathsf{IM},\mathsf{S}) \in [\![P]\!]$. For property (1) note that $x_{\mathsf{v}} := \mathsf{a} \neq \mathsf{C}_{\mathbf{skip}}$. Moreover, as $(\mathsf{IM},\mathsf{S}) \in [\![P]\!]$ and $P \Rightarrow Q[\mathsf{e}/x_{\mathsf{v}}]$ we also have $(\mathsf{IM},\mathsf{S}) \in [\![Q[\mathsf{e}/x_{\mathsf{v}}]]\!]$. As such, from Prop. 2 we have $(\mathsf{IM},\mathsf{S}) \in [\![Q_{\mathsf{p}}]\!]$, as required.

To show property (4), pick arbitrary $\tau, \mathsf{IM}_1, \mathsf{IM}_2, \mathsf{B}_1, \mathsf{B}_2, \mathsf{c}', \mathsf{S}', l \neq \lightning$ such that $\mathsf{IM}{=}\Downarrow(\mathsf{IM}_1, \mathsf{B}_1, \tau)$, $\mathsf{IM}'{=}\Downarrow(\mathsf{IM}_2, \mathsf{B}_2, \tau)$ and $\mathsf{c}_t, \mathsf{IM}_1, \mathsf{B}_1, \mathsf{S} \overset{\tau:l}{\Longrightarrow} \mathsf{c}', \mathsf{IM}_2, \mathsf{B}_2, \mathsf{S}'$. Given the operational semantics, there are now two cases to consider: 1) $l{=}\epsilon$, $\mathsf{C}'{=}\mathsf{c}_t$, $\mathsf{IM}_1, \mathsf{B}_1 \xrightarrow{\tau:l} \mathsf{IM}_2, \mathsf{B}_2, \mathsf{S}{=}\mathsf{S}'$; or 2) $\mathsf{C}'{=}\mathbf{skip}$, $\mathsf{IM}_2{=}\mathsf{IM}_1$, $\mathsf{B}_2{=}\mathsf{B}_1[\tau \mapsto \langle x_{\mathsf{v}}, \mathsf{S}(\mathsf{e})\rangle]$, $\mathsf{S}'{=}\mathsf{S}$.

In case (1) from Lemma 8 we have $\mathsf{IM}{=}\mathsf{IM}'$. As such, since by definition $[\![\mathcal{G}]\!]^{\mathsf{g}}$ is reflexive, we have $((\mathsf{IM},\mathsf{S}),(\mathsf{IM}',\mathsf{S})) \in \mathcal{G}$. Moreover, since $\mathsf{C}'{=}\mathsf{c}_t$, $\mathsf{IM}'{=}\mathsf{IM}$ and $\mathsf{S}'{=}\mathsf{S}$, from (I.H.) we have $\mathsf{safe}_m((\mathsf{C}')^{\mathsf{sp}}, \mathsf{IM}', \mathsf{S}', [\![\mathcal{R}]\!]^{\mathsf{r}}, [\![\mathcal{G}]\!]^{\mathsf{g}}, [\![Q]\!], [\![Q_{\mathsf{p}}]\!])$, as required.

In case (2), from the definition of $\Downarrow(.)$ we know that $\mathsf{IM'}{=}\mathsf{IM}[x_\mathsf{v} \mapsto \mathsf{S(e)}]$. As such, since $(\mathsf{IM},\mathsf{S}) \in [\![P]\!]$, from the definition of $\mathcal{G}$ we have $((\mathsf{IM},\mathsf{S}),(\mathsf{IM'},\mathsf{S'})) \in [\![\mathcal{G}]\!]^\mathsf{g}$, as required. Moreover, since $\mathsf{S'}{=}\mathsf{S}$, $(\mathsf{IM},\mathsf{S}) \in [\![P]\!]$ and $P \Rightarrow Q[\mathsf{e}/x_\mathsf{v}]$, from the definition of $\mathsf{IM'}$ we know $(\mathsf{IM'},\mathsf{S'}) \in [\![Q]\!]$. On the other hand, since Q is stable, by definition we know $[\![Q]\!]$ is stable. Furthermore, from the interpretation of the triple and by definitions of $[\![.]\!]^\mathsf{r}$ and $[\![.]\!]^\mathsf{g}$ we know that $\mathsf{wf}([\![\mathcal{R}]\!]^\mathsf{r}, [\![\mathcal{G}]\!]^\mathsf{g}, [\![Q]\!], [\![Q_\mathsf{p}]\!])$ holds. As such, from Lemma 11 we have $\mathsf{safe}_m(\mathbf{skip}^\mathsf{sp}, \mathsf{IM'}, \mathsf{S'}, [\![\mathcal{R}]\!]^\mathsf{r}, [\![\mathcal{G}]\!]^\mathsf{g}, [\![Q]\!], [\![Q_\mathsf{p}]\!])$, as required.

To show property (3), pick arbitrary $\mathsf{IM'}, \mathsf{S'}$ such that $((\mathsf{IM},\mathsf{S}),(\mathsf{IM'},\mathsf{S'})) \in [\![\{P,Q\}]\!]^\mathsf{r}$. From the definition of $[\![\mathcal{R}]\!]^\mathsf{r}$ and as $(\mathsf{IM},\mathsf{S}) \in [\![P]\!]$ we know $(\mathsf{IM'},\mathsf{S'}) \in [\![P]\!]$. As such, from (I.H.) we have $\mathsf{safe}_m(\mathsf{C}^\mathsf{sp}, \mathsf{IM'}, \mathsf{S'}, [\![\mathcal{R}]\!]^\mathsf{r}, [\![\mathcal{G}]\!]^\mathsf{g}, [\![Q]\!], [\![Q_\mathsf{p}]\!])$, as required.

Similarly, let $\mathsf{IM}_a{=}\mathsf{IM}[\overline{x_\mathsf{p} \mapsto \mathsf{IM}(x_\mathsf{s})}]$, and let $\mathsf{IM}_s{=}\mathsf{IM}[x_\mathsf{s} \mapsto \mathsf{IM}(x_\mathsf{v})]$ for an arbitrary x. As P is stable and $(\mathsf{IM},\mathsf{S}) \in [\![P]\!]$, from the definition of stability we then know $(\mathsf{IM}_s,\mathsf{S}), (\mathsf{IM}_a,\mathsf{S}) \in [\![P]\!]$, and thus from (I.H.) we have $\mathsf{safe}_m(\mathsf{C}^\mathsf{sp}, \mathsf{IM}_s, \mathsf{S}, [\![\mathcal{R}]\!]^\mathsf{r}, [\![\mathcal{G}]\!]^\mathsf{g}, [\![Q]\!], [\![Q_\mathsf{p}]\!])$ and $\mathsf{safe}_m(\mathsf{C}^\mathsf{sp}, \mathsf{IM}_a, \mathsf{S}, [\![\mathcal{R}]\!]^\mathsf{r}, [\![\mathcal{G}]\!]^\mathsf{g}, [\![Q]\!], [\![Q_\mathsf{p}]\!])$, as required.

Case (READ)
We then have $[\![\mathsf{C}]\!]{=}\mathsf{c}_t^\mathsf{sp}$ where $\mathsf{c}_t{=}\mathsf{a} := x_\mathsf{v}$ for some x_v, a; $\mathcal{R}{=}\{P,Q\}$ and $\mathcal{G}{=}\emptyset$ and $P \Rightarrow Q[x_\mathsf{v}/\mathsf{a}]$. Pick an arbitrary n. We proceed by induction on n.

Base case $n{=}0$
Pick an arbitrary $(\mathsf{IM},\mathsf{S}) \in [\![P]\!]$. We must show $\mathsf{safe}_0([\![\mathsf{C}]\!], \mathsf{IM}, \mathsf{S}, [\![\mathcal{R}]\!]^\mathsf{r}, [\![\mathcal{G}]\!]^\mathsf{g}, [\![Q]\!], [\![Q_\mathsf{p}]\!])$, which follows trivially from the definition of safe_0.

Inductive case $n{=}m{+}1$

$$\forall(\mathsf{IM},\mathsf{S}) \in [\![P]\!].\ \mathsf{safe}_m([\![\mathsf{C}]\!], \mathsf{IM}, \mathsf{S}, [\![\mathcal{R}]\!]^\mathsf{r}, [\![\mathcal{G}]\!]^\mathsf{g}, [\![Q]\!], [\![Q_\mathsf{p}]\!]) \tag{I.H.}$$

Pick an arbitrary $(\mathsf{IM},\mathsf{S}) \in [\![P]\!]$. For property (1) note that $\mathsf{a} := x_\mathsf{v} \neq \mathsf{C}_\mathbf{skip}$. Moreover, as $(\mathsf{IM},\mathsf{S}) \in [\![P]\!]$ and $P \Rightarrow Q[x_\mathsf{v}/\mathsf{a}]$ we also have $(\mathsf{IM},\mathsf{S}) \in [\![Q[x_\mathsf{v}/\mathsf{a}]]\!]$. As such, from Prop. 2 we have $(\mathsf{IM},\mathsf{S}) \in [\![Q_\mathsf{p}]\!]$, as required.

To show property (4), pick arbitrary $\tau, \mathsf{IM}_1, \mathsf{IM}_2, \mathsf{B}_1, \mathsf{B}_2, \mathsf{C'}, \mathsf{S'}, l{\neq}\lightning$ such that $\mathsf{IM}{=}\Downarrow(\mathsf{IM}_1, \mathsf{B}_1, \tau)$, $\mathsf{IM'}{=}\Downarrow(\mathsf{IM}_2, \mathsf{B}_2, \tau)$ and $\mathsf{c}_t, \mathsf{IM}_1, \mathsf{B}_1, \mathsf{S} \overset{\tau:l}{\Longrightarrow} \mathsf{C'}, \mathsf{IM}_2, \mathsf{B}_2, \mathsf{S'}$. Given the operational semantics, there are now two cases to consider: 1) $l{=}\epsilon$, $\mathsf{C'}{=}\mathsf{c}_t$, $\mathsf{IM}_1, \mathsf{B}_1 \xrightarrow{\tau:l} \mathsf{IM}_2, \mathsf{B}_2$, $\mathsf{S}{=}\mathsf{S'}$; or 2) $\mathsf{C'}{=}\mathbf{skip}$, $\mathsf{IM}_2{=}\mathsf{IM}_1$, $\mathsf{B}_2{=}\mathsf{B}_1$, $\mathsf{S'}{=}\mathsf{S}[\tau \mapsto \mathsf{S}(\tau)[\mathsf{a} \mapsto \mathsf{IM}(x_\mathsf{v})]]$ and thus $\mathsf{IM'}{=}\mathsf{IM}$.

In case (1) from Lemma 8 we have $\mathsf{IM}{=}\mathsf{IM'}$. As such, since by definition $[\![\mathcal{G}]\!]^\mathsf{g}$ is reflexive, we have $((\mathsf{IM},\mathsf{S}),(\mathsf{IM'},\mathsf{S})) \in \mathcal{G}$. Moreover, since $\mathsf{C'}{=}\mathsf{c}_t$, $\mathsf{IM'}{=}\mathsf{IM}$ and $\mathsf{S'}{=}\mathsf{S}$, from (I.H.) we have $\mathsf{safe}_m((\mathsf{C'})^\mathsf{sp}, \mathsf{IM'}, \mathsf{S'}, [\![\mathcal{R}]\!]^\mathsf{r}, [\![\mathcal{G}]\!]^\mathsf{g}, [\![Q]\!], [\![Q_\mathsf{p}]\!])$, as required.

In case (2), as by definition $[\![\mathcal{G}]\!]^\mathsf{g}$ is reflexive and $\mathsf{IM'}{=}\mathsf{IM}$, we have $((\mathsf{IM},\mathsf{S}),(\mathsf{IM'},\mathsf{S})) \in \mathcal{G}$. Moreover, as $\mathsf{IM'}{=}\mathsf{IM}$, $(\mathsf{IM},\mathsf{S}) \in [\![P]\!]$ and $P \Rightarrow Q[x_\mathsf{v}/\mathsf{a}]$, from the definition of $\mathsf{S'}$ we know $(\mathsf{IM'},\mathsf{S'}) \in [\![Q]\!]$. On the other hand, since Q is stable, by definition we know $[\![Q]\!]$ is stable. Furthermore, from the interpretation of the triple and by definitions of $[\![.]\!]^\mathsf{r}$ and $[\![.]\!]^\mathsf{g}$ we know that $\mathsf{wf}([\![\mathcal{R}]\!]^\mathsf{r}, [\![\mathcal{G}]\!]^\mathsf{g}, [\![Q]\!])$ holds. As such, from Lemma 11 we have $\mathsf{safe}_m(\mathbf{skip}^\mathsf{sp}, \mathsf{IM'}, \mathsf{S'}, [\![\mathcal{R}]\!]^\mathsf{r}, [\![\mathcal{G}]\!]^\mathsf{g}, [\![Q]\!], [\![Q_\mathsf{p}]\!])$, as required.

To show property (3), pick arbitrary $\mathsf{IM'}, \mathsf{S'}$ such that $((\mathsf{IM},\mathsf{S}),(\mathsf{IM'},\mathsf{S'})) \in [\![\{P,Q\}]\!]^\mathsf{r}$. From the definition of $[\![\mathcal{R}]\!]^\mathsf{r}$ and as $(\mathsf{IM},\mathsf{S}) \in [\![P]\!]$ we know $(\mathsf{IM'},\mathsf{S'}) \in [\![P]\!]$. As such, from (I.H.) we have $\mathsf{safe}_m(\mathsf{C}^\mathsf{sp}, \mathsf{IM'}, \mathsf{S'}, [\![\mathcal{R}]\!]^\mathsf{r}, [\![\mathcal{G}]\!]^\mathsf{g}, [\![Q]\!], [\![Q_\mathsf{p}]\!])$, as required.

Similarly, let $\mathsf{IM}_a{=}\mathsf{IM}[\overline{x_\mathsf{p} \mapsto \mathsf{IM}(x_\mathsf{s})}]$, and let $\mathsf{IM}_s{=}\mathsf{IM}[x_\mathsf{s} \mapsto \mathsf{IM}(x_\mathsf{v})]$ for an arbitrary x. As P is stable and $(\mathsf{IM},\mathsf{S}) \in [\![P]\!]$, from the definition of stability we then know $(\mathsf{IM}_s,\mathsf{S}), (\mathsf{IM}_a,\mathsf{S}) \in [\![P]\!]$, and thus from (I.H.) we have $\mathsf{safe}_m(\mathsf{C}^\mathsf{sp}, \mathsf{IM}_s, \mathsf{S}, [\![\mathcal{R}]\!]^\mathsf{r}, [\![\mathcal{G}]\!]^\mathsf{g}, [\![Q]\!], [\![Q_\mathsf{p}]\!])$ and $\mathsf{safe}_m(\mathsf{C}^\mathsf{sp}, \mathsf{IM}_a, \mathsf{S}, [\![\mathcal{R}]\!]^\mathsf{r}$,

$[\![\mathcal{G}]\!]^{\mathsf{g}}, [\![Q]\!], [\![Q_{\mathsf{p}}]\!])$, as required.

Case (FLUSH)
We then have $[\![\mathsf{C}]\!] = \mathsf{c}_t^{\mathsf{sp}}$ where c_t=$\langle$**persist** $X\rangle$ when $x \in X$; $\mathcal{R}$=$\{P, Q\}$; $\mathcal{G}$=$\emptyset$ and $P \Rightarrow Q[X_{\mathsf{v}}/X_{\mathsf{s}}]$. Pick an arbitrary n. We proceed by induction on n.

Base case n=0
Pick an arbitrary $(\mathsf{IM}, \mathsf{S}) \in [\![P]\!]$. We need to show: $\mathsf{safe}_0([\![\mathsf{C}]\!], \mathsf{IM}, \mathsf{S}, [\![\mathcal{R}]\!]^{\mathsf{r}}, [\![\mathcal{G}]\!]^{\mathsf{g}}, [\![Q]\!], [\![Q_{\mathsf{p}}]\!])$, which follows trivially from the definition of safe_0.

Inductive case n=m+1

$$\forall(\mathsf{IM}, \mathsf{S}) \in [\![P]\!].\ \mathsf{safe}_m([\![\mathsf{C}]\!], \mathsf{IM}, \mathsf{S}, [\![\mathcal{R}]\!]^{\mathsf{r}}, [\![\mathcal{G}]\!]^{\mathsf{g}}, [\![Q]\!], [\![Q_{\mathsf{p}}]\!]) \tag{I.H.}$$

Pick an arbitrary $(\mathsf{IM}, \mathsf{S}) \in [\![P]\!]$. For property (1) note that $\mathsf{c}_t \neq \mathsf{C}_{\mathbf{skip}}$. Moreover, as $(\mathsf{IM}, \mathsf{S}) \in [\![P]\!]$ and $P \Rightarrow Q[x_{\mathsf{v}}/\mathsf{a}]$ we also have $(\mathsf{IM}, \mathsf{S}) \in [\![Q[X_{\mathsf{v}}/X_{\mathsf{s}}]]\!]$. As such, from Prop. 2 we have $(\mathsf{IM}, \mathsf{S}) \in [\![Q_{\mathsf{p}}]\!]$, as required.

To show property (4), pick arbitrary $\tau, \mathsf{IM}_1, \mathsf{IM}_2, \mathsf{B}_1, \mathsf{B}_2, \mathsf{C}', \mathsf{S}', l \neq \lightning$ such that IM=$\Downarrow(\mathsf{IM}_1, \mathsf{B}_1, \tau)$, IM'=$\Downarrow(\mathsf{IM}_2, \mathsf{B}_2, \tau)$ and $\mathsf{c}_t, \mathsf{IM}_1, \mathsf{B}_1, \mathsf{S} \overset{\tau:l}{\Longrightarrow} \mathsf{C}', \mathsf{IM}_2, \mathsf{B}_2, \mathsf{S}'$. Given the operational semantics, there are now two cases to consider: 1) l=ϵ, C'=c_t, $\mathsf{IM}_1, \mathsf{B}_1 \xrightarrow{\tau:l} \mathsf{IM}_2, \mathsf{B}_2$, S=S'; or 2) C'=**skip**, IM_2=IM_1, B_2=$\mathsf{B}_1[\tau \mapsto \mathsf{b}.X]$ when $\mathsf{B}_1(\tau)$=b, and S'=S.

In case (1) from Lemma 8 we have IM=IM'. As such, since by definition $[\![\mathcal{G}]\!]^{\mathsf{g}}$ is reflexive, we have $((\mathsf{IM}, \mathsf{S}), (\mathsf{IM}', \mathsf{S})) \in \mathcal{G}$. Moreover, since C'=c_t, IM'=IM and S'=S, from (I.H.) we have $\mathsf{safe}_m((\mathsf{C}')^{\mathsf{sp}}, \mathsf{IM}', \mathsf{S}', [\![\mathcal{R}]\!]^{\mathsf{r}}, [\![\mathcal{G}]\!]^{\mathsf{g}}, [\![Q]\!], [\![Q_{\mathsf{p}}]\!])$, as required.

Let X=$\overline{x^i}$. In case (2), from the definition of $\Downarrow(.)$ we know that IM'=$\mathsf{IM}[\overline{x_{\mathsf{s}}^i \mapsto \mathsf{IM}(x_{\mathsf{v}}^i)}]$. As such, from the definition of A_{sp} we have $((\mathsf{IM}, \mathsf{S}), (\mathsf{IM}', \mathsf{S})) \in \mathsf{A}_{\mathsf{sp}}$, as required. Moreover, since S'=S, $(\mathsf{IM}, \mathsf{S}) \in [\![P]\!]$ and $P \Rightarrow Q[X_{\mathsf{v}}/X_{\mathsf{s}}]$, from the definition of IM' we know $(\mathsf{IM}', \mathsf{S}') \in [\![Q]\!]$. On the other hand, since Q is stable, by definition we know $[\![Q]\!]$ is stable. Furthermore, from the interpretation of the triple and by definitions of $[\![.]\!]^{\mathsf{r}}$ and $[\![.]\!]^{\mathsf{g}}$ we know that $\mathsf{wf}([\![\mathcal{R}]\!]^{\mathsf{r}}, [\![\mathcal{G}]\!]^{\mathsf{g}}, [\![Q]\!])$ holds. As such, from Lemma 11 we have $\mathsf{safe}_m(\mathbf{skip}^{\mathsf{sp}}, \mathsf{IM}', \mathsf{S}', [\![\mathcal{R}]\!]^{\mathsf{r}}, [\![\mathcal{G}]\!]^{\mathsf{g}}, [\![Q]\!], [\![Q_{\mathsf{p}}]\!])$, as required.

To show property (3), pick arbitrary $\mathsf{IM}', \mathsf{S}'$ such that $((\mathsf{IM}, \mathsf{S}), (\mathsf{IM}', \mathsf{S}')) \in [\![\{P, Q\}]\!]^{\mathsf{r}}$. From the definition of $[\![\mathcal{R}]\!]^{\mathsf{r}}$ and as $(\mathsf{IM}, \mathsf{S}) \in [\![P]\!]$ we know $(\mathsf{IM}', \mathsf{S}') \in [\![P]\!]$. As such, from (I.H.) we have $\mathsf{safe}_m(\mathsf{C}^{\mathsf{sp}}, \mathsf{IM}', \mathsf{S}', [\![\mathcal{R}]\!]^{\mathsf{r}}, [\![\mathcal{G}]\!]^{\mathsf{g}}, [\![Q]\!], [\![Q_{\mathsf{p}}]\!])$, as required.

Similarly, let IM_a=$\mathsf{IM}[\overline{x_{\mathsf{p}} \mapsto \mathsf{IM}(x_{\mathsf{s}})}]$, and let IM_s=$\mathsf{IM}[x_{\mathsf{s}} \mapsto \mathsf{IM}(x_{\mathsf{v}})]$ for an arbitrary x. As P is stable and $(\mathsf{IM}, \mathsf{S}) \in [\![P]\!]$, from the definition of stability we then know $(\mathsf{IM}_s, \mathsf{S}), (\mathsf{IM}_a, \mathsf{S}) \in [\![P]\!]$, and thus from (I.H.) we have $\mathsf{safe}_m(\mathsf{C}^{\mathsf{sp}}, \mathsf{IM}_s, \mathsf{S}, [\![\mathcal{R}]\!]^{\mathsf{r}}, [\![\mathcal{G}]\!]^{\mathsf{g}}, [\![Q]\!], [\![Q_{\mathsf{p}}]\!])$ and $\mathsf{safe}_m(\mathsf{C}^{\mathsf{sp}}, \mathsf{IM}_a, \mathsf{S}, [\![\mathcal{R}]\!]^{\mathsf{r}}, [\![\mathcal{G}]\!]^{\mathsf{g}}, [\![Q]\!], [\![Q_{\mathsf{p}}]\!])$, as required.

Case (CAS)
We then have $[\![\mathsf{C}]\!]$=$\mathsf{c}_t^{\mathsf{sp}}$ where c_t=$\mathbf{CAS}(x_{\mathsf{v}}, \mathsf{e}_1, \mathsf{e}_2)$ for some $x_{\mathsf{v}}, \mathsf{e}_1, \mathsf{e}_2$; $\mathcal{R}$=$\{P, Q\}$ and $\mathcal{G}$=$\{\langle x_{\mathsf{v}}, \mathsf{e}_2, P \wedge x_{\mathsf{v}}{=}\mathsf{e}_1\rangle\}$, $P \wedge x_{\mathsf{v}}{=}\mathsf{e}_1 \Rightarrow Q[\mathsf{e}_1/\mathsf{a}][\mathsf{e}_2/x_{\mathsf{v}}]$ and $P \wedge x_{\mathsf{v}} \neq \mathsf{e}_1 \Rightarrow Q[x_{\mathsf{v}}/\mathsf{a}]$. Pick an arbitrary n. We proceed by induction on n.

Base case n=0
Pick an arbitrary $(\mathsf{IM}, \mathsf{S}) \in [\![P]\!]$. We must show $\mathsf{safe}_0([\![\mathsf{C}]\!], \mathsf{IM}, \mathsf{S}, [\![\mathcal{R}]\!]^{\mathsf{r}}, [\![\mathcal{G}]\!]^{\mathsf{g}}, [\![Q]\!], [\![Q_{\mathsf{p}}]\!])$, which follows trivially from the definition of safe_0.

Inductive case $n{=}m{+}1$

$$\forall(\mathsf{IM},\mathsf{S}) \in [\![P]\!].\ \mathsf{safe}_m([\![\mathsf{C}]\!],\mathsf{IM},\mathsf{S},[\![\mathcal{R}]\!]^{\mathsf{r}},[\![\mathcal{G}]\!]^{\mathsf{g}},[\![Q]\!],[\![Q_{\mathsf{p}}]\!]) \tag{I.H.}$$

Pick an arbitrary $(\mathsf{IM},\mathsf{S}) \in [\![P]\!]$. For property (1) note that $\mathbf{CAS}(x_{\mathsf{v}},\mathsf{e}_1,\mathsf{e}_2) \neq \mathsf{C}_{\mathbf{skip}}$. Moreover, as $(\mathsf{IM},\mathsf{S}) \in [\![P]\!]$, $P \wedge x_{\mathsf{v}}{=}\mathsf{e}_1 \Rightarrow Q[\mathsf{e}_1/\mathsf{a}][\mathsf{e}_2/x_{\mathsf{v}}]$ and $P \wedge x_{\mathsf{v}} \neq \mathsf{e}_1 \Rightarrow Q[x_{\mathsf{v}}/\mathsf{a}]$, we also have $(\mathsf{IM},\mathsf{S}) \in [\![Q[\mathsf{e}_1/\mathsf{a}][\mathsf{e}_2/x_{\mathsf{v}}]]\!] \cup [\![Q[x_{\mathsf{v}}/\mathsf{a}]]\!]$. As such, from Prop. 2 we have $(\mathsf{IM},\mathsf{S}) \in [\![Q_{\mathsf{p}}]\!]$, as required.

To show property (4), pick arbitrary $\tau,\mathsf{IM}_1,\mathsf{IM}_2,\mathsf{B}_1,\mathsf{B}_2,\mathsf{C}',\mathsf{S}',l{\neq}\lightning$ such that $\mathsf{IM}{=}\Downarrow(\mathsf{IM}_1,\mathsf{B}_1,\tau)$, $\mathsf{IM}'{=}\Downarrow(\mathsf{IM}_2,\mathsf{B}_2,\tau)$ and $\mathsf{c}_t,\mathsf{IM}_1,\mathsf{B}_1,\mathsf{S} \overset{\tau:l}{\Longrightarrow} \mathsf{C}',\mathsf{IM}_2,\mathsf{B}_2,\mathsf{S}'$. Given the operational semantics there are now three cases to consider: 1) $l{=}\epsilon$, $\mathsf{C}'{=}\mathsf{c}_t$, $\mathsf{IM}_1,\mathsf{B}_1 \xrightarrow{\tau:l} \mathsf{IM}_2,\mathsf{B}_2$, $\mathsf{S}{=}\mathsf{S}'$; or 2) $\mathsf{B}_1(\tau){=}\mathsf{B}_2(\tau){=}\epsilon$, $\mathsf{C}'{=}\mathbf{skip}$, $\mathsf{IM}_1(x_{\mathsf{v}}) \neq \mathsf{S}(\mathsf{e}_1)$, $\mathsf{IM}_2{=}\mathsf{IM}_1$, $\mathsf{S}'{=}\mathsf{S}[\tau \mapsto \mathsf{S}(\tau)[\mathsf{a} \mapsto \mathsf{IM}(x_{\mathsf{v}})]]$; or 3) $\mathsf{B}_1(\tau){=}\mathsf{B}_2(\tau){=}\epsilon$, $\mathsf{C}'{=}\mathbf{skip}$, $\mathsf{IM}_1(x_{\mathsf{v}}) = \mathsf{S}(\mathsf{e}_1)$, $\mathsf{IM}_2{=}\mathsf{IM}_1[x_{\mathsf{v}} \mapsto \mathsf{S}(\mathsf{e}_2)]$, $\mathsf{S}'{=}\mathsf{S}[\mathsf{a} \mapsto \mathsf{S}(\mathsf{e}_1)]$.

In case (1) from Lemma 8 we have $\mathsf{IM}{=}\mathsf{IM}'$. As such, since by definition $[\![\mathcal{G}]\!]^{\mathsf{g}}$ is reflexive, we have $((\mathsf{IM},\mathsf{S}),(\mathsf{IM}',\mathsf{S})) \in \mathcal{G}$. Moreover, since $\mathsf{C}'{=}\mathsf{c}_t$, $\mathsf{IM}'{=}\mathsf{IM}$ and $\mathsf{S}'{=}\mathsf{S}$, from (I.H.) we have $\mathsf{safe}_m((\mathsf{C}')^{\mathsf{sp}},\mathsf{IM}',\mathsf{S}',[\![\mathcal{R}]\!]^{\mathsf{r}},[\![\mathcal{G}]\!]^{\mathsf{g}},[\![Q]\!],[\![Q_{\mathsf{p}}]\!])$, as required.

In case (2), as $\mathsf{B}_1(\tau){=}\mathsf{B}_2(\tau){=}\epsilon$ and $\mathsf{IM}_2{=}\mathsf{IM}_1$, from the definition of $\Downarrow(.)$ we have $\mathsf{IM}{=}\Downarrow(\mathsf{IM}_1,\mathsf{B}_1,\tau){=}\Downarrow(\mathsf{IM}_2,\mathsf{B}_2,\tau){=}\mathsf{IM}'$. As such, as by definition $[\![\mathcal{G}]\!]^{\mathsf{g}}$ is reflexive, we have $((\mathsf{IM},\mathsf{S}),(\mathsf{IM}',\mathsf{S})) \in \mathcal{G}$. Moreover, since $(\mathsf{IM},\mathsf{S}) \in [\![P]\!]$ and $P \wedge x_{\mathsf{v}} \neq \mathsf{e}_1 \Rightarrow Q[x_{\mathsf{v}}/\mathsf{a}]$, from the definitions of IM', S' we know $(\mathsf{IM}',\mathsf{S}') \in [\![Q]\!]$. On the other hand, since Q is stable, by definition we know $[\![Q]\!]$ is stable. Furthermore, from the interpretation of the triple and by definitions of $[\![.]\!]^{\mathsf{r}}$ and $[\![.]\!]^{\mathsf{g}}$ we know that $\mathsf{wf}([\![\mathcal{R}]\!]^{\mathsf{r}},[\![\mathcal{G}]\!]^{\mathsf{g}},[\![Q]\!])$ holds. As such, from Lemma 11 we have $\mathsf{safe}_m(\mathbf{skip}^{\mathsf{sp}},\mathsf{IM}',\mathsf{S}',[\![\mathcal{R}]\!]^{\mathsf{r}},[\![\mathcal{G}]\!]^{\mathsf{g}},[\![Q]\!],[\![Q_{\mathsf{p}}]\!])$, as required.

In case (3), from the definition of $\Downarrow(.)$ we know $\mathsf{IM}'{=}\mathsf{IM}[x_{\mathsf{v}} \mapsto \mathsf{S}(\mathsf{e}_2)]$. As such, as $(\mathsf{IM},\mathsf{S}) \in [\![P]\!]$, from the definition of $\mathcal{G}$ we have $((\mathsf{IM},\mathsf{S}),(\mathsf{IM}',\mathsf{S})) \in [\![\mathcal{G}]\!]^{\mathsf{g}}$, as required. Moreover, since $(\mathsf{IM},\mathsf{S}) \in [\![P]\!]$ and $P \wedge x_{\mathsf{v}}{=}\mathsf{e}_1 \Rightarrow Q[\mathsf{e}_1/\mathsf{a}][\mathsf{e}_2/x_{\mathsf{v}}]$, from the definitions of IM', S' we know $(\mathsf{IM}',\mathsf{S}') \in [\![Q]\!]$. On the other hand, since Q is stable, by definition we know $[\![Q]\!]$ is stable. Furthermore, from the interpretation of the triple and by definitions of $[\![.]\!]^{\mathsf{r}}$ and $[\![.]\!]^{\mathsf{g}}$ we know that $\mathsf{wf}([\![\mathcal{R}]\!]^{\mathsf{r}},[\![\mathcal{G}]\!]^{\mathsf{g}},[\![Q]\!])$ holds. As such, from Lemma 11 we have $\mathsf{safe}_m(\mathbf{skip}^{\mathsf{sp}},\mathsf{IM}',\mathsf{S}',[\![\mathcal{R}]\!]^{\mathsf{r}},[\![\mathcal{G}]\!]^{\mathsf{g}},[\![Q]\!],[\![Q_{\mathsf{p}}]\!])$, as required.

To show property (3), pick arbitrary IM',S' such that $((\mathsf{IM},\mathsf{S}),(\mathsf{IM}',\mathsf{S}')) \in [\![\{P,Q\}]\!]^{\mathsf{r}}$. From the definition of $[\![\mathcal{R}]\!]^{\mathsf{r}}$ and as $(\mathsf{IM},\mathsf{S}) \in [\![P]\!]$ we know $(\mathsf{IM}',\mathsf{S}') \in [\![P]\!]$. As such, from (I.H.) we have $\mathsf{safe}_m(\mathsf{C}^{\mathsf{sp}},\mathsf{IM}',\mathsf{S}',[\![\mathcal{R}]\!]^{\mathsf{r}},[\![\mathcal{G}]\!]^{\mathsf{g}},[\![Q]\!],[\![Q_{\mathsf{p}}]\!])$, as required.

Similarly, let $\mathsf{IM}_a{=}\mathsf{IM}[x_{\mathsf{p}} \mapsto \mathsf{IM}(x_{\mathsf{s}})]$, and let $\mathsf{IM}_s{=}\mathsf{IM}[x_{\mathsf{s}} \mapsto \mathsf{IM}(x_{\mathsf{v}})]$ for an arbitrary x. As P is stable and $(\mathsf{IM},\mathsf{S}) \in [\![P]\!]$, from the definition of stability we then know $(\mathsf{IM}_s,\mathsf{S}),(\mathsf{IM}_a,\mathsf{S}) \in [\![P]\!]$, and thus from (I.H.) we have $\mathsf{safe}_m(\mathsf{C}^{\mathsf{sp}},\mathsf{IM}_s,\mathsf{S},[\![\mathcal{R}]\!]^{\mathsf{r}},[\![\mathcal{G}]\!]^{\mathsf{g}},[\![Q]\!],[\![Q_{\mathsf{p}}]\!])$ and $\mathsf{safe}_m(\mathsf{C}^{\mathsf{sp}},\mathsf{IM}_a,\mathsf{S},[\![\mathcal{R}]\!]^{\mathsf{r}},[\![\mathcal{G}]\!]^{\mathsf{g}},[\![Q]\!],[\![Q_{\mathsf{p}}]\!])$, as required.

Case (FAA)
The proof of this case is analogous to that of (CAS) and is thus omitted here.

Case (SEQ)
We then have $\mathsf{C} = \mathsf{c}_1;\mathsf{c}_2$ for some $\mathsf{c}_1,\mathsf{c}_2$, $\mathcal{R}{=}\mathcal{R}_1 \cup \mathcal{R}_2$ and $\mathcal{G}{=}\mathcal{G}_1 \cup \mathcal{G}_2$, $\langle \mathcal{R}_1;\mathcal{G}_1\rangle \vdash \{P\}\ \mathsf{c}_1\ \{R\}$ and $\langle \mathcal{R}_2;\mathcal{G}_2\rangle \vdash \{R\}\ \mathsf{c}_1\ \{Q\}$. Given the interpretations of $\mathcal{R}$ and $\mathcal{G}$ and since Q is stable, we know that $\mathsf{wf}([\![\mathcal{R}_1]\!]^{\mathsf{r}} \cap [\![\mathcal{R}_2]\!]^{\mathsf{r}},[\![\mathcal{G}_1]\!]^{\mathsf{g}} \cup [\![\mathcal{G}_2]\!]^{\mathsf{g}},[\![Q]\!],[\![Q_{\mathsf{p}}]\!])$ holds. Pick an arbitrary n and $(\mathsf{IM},\mathsf{S}) \in [\![P]\!]$. Given the interpretations of $\mathcal{R}$ and $\mathcal{G}$, it suffices to show $\mathsf{safe}_n([\![\mathsf{c}_1;\mathsf{c}_2]\!],\mathsf{IM},\mathsf{S},[\![\mathcal{R}_1]\!]^{\mathsf{r}} \cap [\![\mathcal{R}_2]\!]^{\mathsf{r}},[\![\mathcal{G}_1]\!]^{\mathsf{g}} \cup [\![\mathcal{G}_2]\!]^{\mathsf{g}},[\![Q]\!],[\![Q_{\mathsf{p}}]\!])$. On the other hand, from the inductive hypothesis and since $\langle \mathcal{R}_1;\mathcal{G}_1\rangle \vdash \{P\}\ \mathsf{c}_1\ \{R\}$ and $\langle \mathcal{R}_2;\mathcal{G}_2\rangle \vdash \{R\}\ \mathsf{c}_1\ \{Q\}$ hold, we know that $\mathsf{safe}_n([\![\mathsf{c}_1]\!],\mathsf{IM},\mathsf{S},[\![\mathcal{R}_1]\!]^{\mathsf{r}},[\![\mathcal{G}_1]\!]^{\mathsf{g}},[\![R]\!],[\![R_{\mathsf{p}}]\!])$ holds and

that $\forall(\mathsf{IM}',\mathsf{S}') \in [\![R]\!].\ \mathsf{safe}_n([\![\mathsf{c}_2]\!],\mathsf{IM}',\mathsf{S}',[\![\mathcal{R}_2]\!]^{\mathsf{r}},[\![\mathcal{G}_2]\!]^{\mathsf{g}},[\![Q]\!],[\![Q_{\mathsf{p}}]\!])$ holds. Moreover, as from the inductive hypothesis we know $\langle\mathcal{R}_2;\mathcal{G}_2\rangle \Vdash \{R\}\ \mathsf{c}_1\ \{Q\}$ is a valid judgement, from Prop. 2 we have $[\![R_{\mathsf{p}}]\!] \subseteq [\![Q_{\mathsf{p}}]\!]$. As such, given the definition of $[\![.]\!]$, from Lemma 15 we have $\mathsf{safe}_n([\![\mathsf{c}_1;\mathsf{c}_2]\!],\mathsf{IM},\mathsf{S},[\![\mathcal{R}_1]\!]^{\mathsf{r}}\cap[\![\mathcal{R}_2]\!]^{\mathsf{r}},[\![\mathcal{G}_1]\!]^{\mathsf{g}}\cup[\![\mathcal{G}_2]\!]^{\mathsf{g}},[\![Q]\!],[\![Q_{\mathsf{p}}]\!])$, as required.

Case (ITE)
We then have C=**if** (e) **then** c_1 **else** c_2 for some $\mathsf{e},\mathsf{c}_1,\mathsf{c}_2$; $\mathcal{R}$=$\mathcal{R}'\cup\{P\}$ for some $\mathcal{R}'$, $\langle\mathcal{R}';\mathcal{G}\rangle \vdash \{P\wedge \mathsf{e}{\neq}0\}\ \mathsf{c}_1\ \{Q\}$ and $\langle\mathcal{R}';\mathcal{G}\rangle \vdash \{P\wedge \mathsf{e}{=}0\}\ \mathsf{c}_2\ \{Q\}$. Note that by definition we have $[\![\mathsf{C}]\!]{=}\mathsf{c}_t^{\mathsf{sp}}$ where c_t=**if** $((\!|\mathsf{e}|\!))$ **then** $(\!|\mathsf{c}_1|\!)$ **else** $(\!|\mathsf{c}_2|\!)$. Pick an arbitrary n. We proceed by induction on n.

Base case n=0
Pick an arbitrary $(\mathsf{IM},\mathsf{S}) \in [\![P]\!]$. We must show $\mathsf{safe}_0([\![\mathsf{C}]\!],\mathsf{IM},\mathsf{S},[\![\mathcal{R}]\!]^{\mathsf{r}},[\![\mathcal{G}]\!]^{\mathsf{g}},[\![Q]\!],[\![Q_{\mathsf{p}}]\!])$, which follows trivially from the definition of safe_0.

Inductive case n=m+1

$$\forall(\mathsf{IM},\mathsf{S}) \in [\![P]\!].\ \mathsf{safe}_m([\![\mathsf{C}]\!],\mathsf{IM},\mathsf{S},[\![\mathcal{R}]\!]^{\mathsf{r}},[\![\mathcal{G}]\!]^{\mathsf{g}},[\![Q]\!],[\![Q_{\mathsf{p}}]\!]) \tag{I.H.}$$

Pick an arbitrary $(\mathsf{IM},\mathsf{S}) \in [\![P]\!]$. Property (1) follows trivially from (I.H.) and since $\mathsf{c}_t \neq \mathsf{C}_{\mathbf{skip}}$.

To show property (4), pick arbitrary $\tau,\mathsf{IM}_1,\mathsf{IM}_2,\mathsf{B}_1,\mathsf{B}_2,\mathsf{C}',\mathsf{S}',l{\neq}\lightning$ such that $\mathsf{IM}{=}\Downarrow(\mathsf{IM}_1,\mathsf{B}_1,\tau)$, $\mathsf{IM}'{=}\Downarrow(\mathsf{IM}_2,\mathsf{B}_2,\tau)$ and $\mathsf{c}_t,\mathsf{IM}_1,\mathsf{B}_1,\mathsf{S} \overset{\tau:l}{\Longrightarrow} \mathsf{C}',\mathsf{IM}_2,\mathsf{B}_2,\mathsf{S}'$. Given the operational semantics, there are now three cases to consider:
1) $l{=}\epsilon$, $\mathsf{C}'{=}\mathsf{c}_t$, $\mathsf{IM}_1,\mathsf{B}_1 \xrightarrow{\tau:l} \mathsf{IM}_2,\mathsf{B}_2$, $\mathsf{S}{=}\mathsf{S}'$; or
2) $\mathsf{C}'{=}(\!|\mathsf{c}_1|\!)$, $\mathsf{S}(\mathsf{e}){\neq}0$, $\mathsf{IM}_2{=}\mathsf{IM}_1$, $\mathsf{B}_2{=}\mathsf{B}_1$, $\mathsf{S}'{=}\mathsf{S}$ and thus $\mathsf{IM}'{=}\mathsf{IM}$; or
3) $\mathsf{C}'{=}(\!|\mathsf{c}_2|\!)$, $\mathsf{S}(\mathsf{e}){=}0$, $\mathsf{IM}_2{=}\mathsf{IM}_1$, $\mathsf{B}_2{=}\mathsf{B}_1$, $\mathsf{S}'{=}\mathsf{S}$ and thus $\mathsf{IM}'{=}\mathsf{IM}$.

In case (1) from Lemma 8 we have $\mathsf{IM}{=}\mathsf{IM}'$. As such, since by definition $[\![\mathcal{G}]\!]^{\mathsf{g}}$ is reflexive, we have $((\mathsf{IM},\mathsf{S}),(\mathsf{IM}',\mathsf{S}')) \in \mathcal{G}$. Moreover, since $\mathsf{C}'{=}\mathsf{c}_t$, $\mathsf{IM}'{=}\mathsf{IM}$ and $\mathsf{S}'{=}\mathsf{S}$, from (I.H.) we have $\mathsf{safe}_m((\mathsf{C}')^{\mathsf{sp}},\mathsf{IM}',\mathsf{S}',[\![\mathcal{R}]\!]^{\mathsf{r}},[\![\mathcal{G}]\!]^{\mathsf{g}},[\![Q]\!],)$, as required.

In case (2), since by definition $[\![\mathcal{G}]\!]^{\mathsf{g}}$ is reflexive and $\mathsf{IM}'{=}\mathsf{IM}$, we have $((\mathsf{IM},\mathsf{S}),(\mathsf{IM}',\mathsf{S}')) \in \mathcal{G}$. Moreover, since $(\mathsf{IM},\mathsf{S}) \in [\![P]\!]$ and $\mathsf{S}(\mathsf{e}){\neq}0$, we also have $(\mathsf{IM},\mathsf{S}) \in [\![P\wedge \mathsf{e}{\neq}0]\!]$. That is, $(\mathsf{IM}',\mathsf{S}') \in [\![P\wedge \mathsf{e}{\neq}0]\!]$. As such, since $\langle\mathcal{R}';\mathcal{G}\rangle \vdash \{P\wedge \mathsf{e}{\neq}0\}\ \mathsf{c}_1\ \{Q\}$ holds, from the inductive hypothesis we have $\mathsf{safe}_m((\!|\mathsf{c}_1|\!),\mathsf{IM}',\mathsf{S}',[\![\mathcal{R}']\!]^{\mathsf{r}},[\![\mathcal{G}]\!]^{\mathsf{g}},[\![Q]\!],)$. Moreover, since $\mathcal{R}' \subseteq \mathcal{R}$, from the definition of $[\![.]\!]^{\mathsf{r}}$ we know that $[\![\mathcal{R}]\!]^{\mathsf{r}} \subseteq [\![\mathcal{R}']\!]^{\mathsf{r}}$. As such, since $\mathsf{C}'{=}(\!|\mathsf{c}_1|\!)$, from Lemma 12 we have $\mathsf{safe}_m((\mathsf{C}')^{\mathsf{sp}},\mathsf{IM}',\mathsf{S}',[\![\mathcal{R}]\!]^{\mathsf{r}},[\![\mathcal{G}]\!]^{\mathsf{g}},[\![Q]\!],)$, as required.

The proof of case (3) is analogous to that of case (2) and is omitted here.

To show property (3), pick arbitrary IM',S' such that $((\mathsf{IM},\mathsf{S}),(\mathsf{IM}',\mathsf{S}')) \in [\![\mathcal{R}]\!]^{\mathsf{r}}$. From the definition of $[\![\mathcal{R}]\!]^{\mathsf{r}}$ and as $(\mathsf{IM},\mathsf{S}) \in [\![P]\!]$ we know $(\mathsf{IM}',\mathsf{S}') \in [\![P]\!]$. As such, from (I.H.) we have $\mathsf{safe}_m(\mathsf{C}^{\mathsf{sp}},\mathsf{IM}',\mathsf{S}',[\![\mathcal{R}]\!]^{\mathsf{r}},[\![\mathcal{G}]\!]^{\mathsf{g}},[\![P]\!],)$, as required.

Similarly, let $\mathsf{IM}_a{=}\mathsf{IM}[\overline{x_{\mathsf{p}} \mapsto \mathsf{IM}(x_{\mathsf{s}})}^{\mathsf{sp}}]$, and let $\mathsf{IM}_s{=}\mathsf{IM}[x_{\mathsf{s}} \mapsto \mathsf{IM}(x_{\mathsf{v}})]$ for an arbitrary x. As P is stable and $(\mathsf{IM},\mathsf{S}) \in [\![P]\!]$, from the definition of stability we then know $(\mathsf{IM}_s,\mathsf{S}),(\mathsf{IM}_a,\mathsf{S}) \in [\![P]\!]$, and thus from (I.H.) we have $\mathsf{safe}_m(\mathsf{C}^{\mathsf{sp}},\mathsf{IM}_s,\mathsf{S},[\![\mathcal{R}]\!]^{\mathsf{r}},[\![\mathcal{G}]\!]^{\mathsf{g}},[\![Q]\!],)$ and $\mathsf{safe}_m(\mathsf{C}^{\mathsf{sp}},\mathsf{IM}_a,\mathsf{S},[\![\mathcal{R}]\!]^{\mathsf{r}},[\![\mathcal{G}]\!]^{\mathsf{g}},[\![Q]\!],)$, as required.

Case (WHILE)
We then have C=**while**(e) c_1 for some e,c_1, $\mathcal{R}$=$\mathcal{R}'\{Q\}$ for some $\mathcal{R}'$, $\langle\mathcal{R}';\mathcal{G}\rangle \vdash \{P\wedge \mathsf{e}{\neq}0\}\ \mathsf{c}_1\ \{P\}$ and $P\wedge \mathsf{e}{=}0 \Rightarrow Q$. Note that by definition we have $[\![\mathsf{C}]\!]{=}\mathsf{c}_t^{\mathsf{sp}}$ where c_t=**while**$((\!|\mathsf{e}|\!))(\!|\mathsf{c}_1|\!)$. Pick an arbitrary n. We proceed by induction on n.

Base case n=0
Pick an arbitrary $(\mathsf{IM},\mathsf{S}) \in [\![P]\!]$. We must show $\mathsf{safe}_0([\![\mathsf{C}]\!],\mathsf{IM},\mathsf{S},[\![\mathcal{R}]\!]^{\mathsf{r}},[\![\mathcal{G}]\!]^{\mathsf{g}},[\![Q]\!],[\![Q_{\mathsf{p}}]\!])$, which follows trivially from the definition of safe_0.

Inductive case n=m+1

$$\forall(\mathsf{IM},\mathsf{S}) \in [\![P]\!].\ \mathsf{safe}_m([\![\mathsf{C}]\!],\mathsf{IM},\mathsf{S},[\![\mathcal{R}]\!]^{\mathsf{r}},[\![\mathcal{G}]\!]^{\mathsf{g}},[\![Q]\!],[\![Q_{\mathsf{p}}]\!]) \tag{I.H.}$$

Pick an arbitrary $(\mathsf{IM},\mathsf{S}) \in [\![P]\!]$. Property (1) follows trivially from (I.H.) and since $\mathsf{c}_t \neq \mathsf{C}_{\mathbf{skip}}$.

To show property (4), pick arbitrary $\tau, \mathsf{IM}_1, \mathsf{IM}_2, \mathsf{B}_1, \mathsf{B}_2, \mathsf{C}', \mathsf{S}', l{\neq}\lightning$ such that $\mathsf{IM}{=}\Downarrow(\mathsf{IM}_1,\mathsf{B}_1,\tau)$, $\mathsf{IM}'{=}\Downarrow(\mathsf{IM}_2,\mathsf{B}_2,\tau)$ and $\mathsf{c}_t,\mathsf{IM}_1,\mathsf{B}_1,\mathsf{S} \xRightarrow{\tau:l} \mathsf{C}',\mathsf{IM}_2,\mathsf{B}_2,\mathsf{S}'$. Given the operational semantics, there are now three cases to consider:
1) $l{=}\epsilon$, $\mathsf{C}'{=}\mathsf{c}_t$, $\mathsf{IM}_1,\mathsf{B}_1 \xrightarrow{\tau:l} \mathsf{IM}_2,\mathsf{B}_2$, $\mathsf{S}{=}\mathsf{S}'$; or
2) $\mathsf{C}'{=}(\!|\mathsf{c}_1|\!)$, $\mathsf{S}(\mathsf{e}){\neq}0$, $\mathsf{IM}_2{=}\mathsf{IM}_1$, $\mathsf{B}_2{=}\mathsf{B}_1$, $\mathsf{S}'{=}\mathsf{S}$ and thus $\mathsf{IM}'{=}\mathsf{IM}$; or
3) $\mathsf{C}'{=}\mathbf{skip}$, $\mathsf{S}(\mathsf{e}){=}0$, $\mathsf{IM}_2{=}\mathsf{IM}_1$, $\mathsf{B}_2{=}\mathsf{B}_1$, $\mathsf{S}'{=}\mathsf{S}$ and thus $\mathsf{IM}'{=}\mathsf{IM}$.

In case (1) from Lemma 8 we have $\mathsf{IM}{=}\mathsf{IM}'$. As such, since by definition $[\![\mathcal{G}]\!]^{\mathsf{g}}$ is reflexive, we have $((\mathsf{IM},\mathsf{S}),(\mathsf{IM}',\mathsf{S})) \in \mathcal{G}$. Moreover, since $\mathsf{C}'{=}\mathsf{c}_t$, $\mathsf{IM}'{=}\mathsf{IM}$ and $\mathsf{S}'{=}\mathsf{S}$, from (I.H.) we have $\mathsf{safe}_m((\mathsf{C}')^{\mathsf{sp}},\mathsf{IM}',\mathsf{S}',[\![\mathcal{R}]\!]^{\mathsf{r}},[\![\mathcal{G}]\!]^{\mathsf{g}},[\![Q]\!],)$, as required.

In case (2), since by definition $[\![\mathcal{G}]\!]^{\mathsf{g}}$ is reflexive and $\mathsf{IM}'{=}\mathsf{IM}$, we have $((\mathsf{IM},\mathsf{S}),(\mathsf{IM}',\mathsf{S})) \in \mathcal{G}$. Moreover, since $(\mathsf{IM},\mathsf{S}) \in [\![P]\!]$ and $\mathsf{S}(\mathsf{e}){\neq}0$, we also have $(\mathsf{IM},\mathsf{S}) \in [\![P \wedge \mathsf{e}{\neq}0]\!]$. That is, $(\mathsf{IM}',\mathsf{S}') \in [\![P \wedge \mathsf{e}{\neq}0]\!]$. As such, since $\langle\mathcal{R}';\mathcal{G}\rangle \vdash \{P \wedge \mathsf{e}{\neq}0\}\ \mathsf{c}_1\ \{Q\}$ holds, from the inductive hypothesis we have $\mathsf{safe}_m((\!|\mathsf{c}_1|\!),\mathsf{IM}',\mathsf{S}',[\![\mathcal{R}']\!]^{\mathsf{r}},[\![\mathcal{G}]\!]^{\mathsf{g}},[\![Q]\!],)$. Moreover, since $\mathcal{R}' \subseteq \mathcal{R}$, from the definition of $[\![.]\!]^{\mathsf{r}}$ we know that $[\![\mathcal{R}]\!]^{\mathsf{r}} \subseteq [\![\mathcal{R}']\!]^{\mathsf{r}}$. As such, since $\mathsf{C}'{=}(\!|\mathsf{c}_1|\!)$, from Lemma 12 we have $\mathsf{safe}_m((\mathsf{c}')^{\mathsf{sp}},\mathsf{IM}',\mathsf{S}',[\![\mathcal{R}]\!]^{\mathsf{r}},[\![\mathcal{G}]\!]^{\mathsf{g}},[\![Q]\!],)$, as required.

In case (3) since by definition $[\![\mathcal{G}]\!]^{\mathsf{g}}$ is reflexive and $\mathsf{IM}'{=}\mathsf{IM}$, we have $((\mathsf{IM},\mathsf{S}),(\mathsf{IM}',\mathsf{S})) \in \mathcal{G}$. Moreover, since $(\mathsf{IM},\mathsf{S}) \in [\![P]\!]$ and $\mathsf{S}(\mathsf{e}){=}0$, we also have $(\mathsf{IM},\mathsf{S}) \in [\![P \wedge \mathsf{e}{=}0]\!]$. That is, $(\mathsf{IM}',\mathsf{S}') \in [\![P \wedge \mathsf{e}{=}0]\!]$. Consequently, since $P \wedge \mathsf{e}{=}0 \Rightarrow Q$, we have $(\mathsf{IM}',\mathsf{S}') \in [\![\in]\!]Q$. Furthermore, from the interpretation of the triple and by definitions of $[\![.]\!]^{\mathsf{r}}$ and $[\![.]\!]^{\mathsf{g}}$ we know that $\mathsf{wf}([\![\mathcal{R}]\!]^{\mathsf{r}},[\![\mathcal{G}]\!]^{\mathsf{g}},[\![Q]\!])$ holds. As such, from Lemma 11 we have $\mathsf{safe}_m(\mathbf{skip}^{\mathsf{sp}},\mathsf{IM}',\mathsf{S}',[\![\mathcal{R}]\!]^{\mathsf{r}},[\![\mathcal{G}]\!]^{\mathsf{g}},[\![Q]\!],)$. That is, we have $\mathsf{safe}_m((\mathsf{C}')^{\mathsf{sp}},\mathsf{IM}',\mathsf{S}',[\![\mathcal{R}]\!]^{\mathsf{r}},[\![\mathcal{G}]\!]^{\mathsf{g}},[\![Q]\!],)$, as required.

To show property (3), pick arbitrary IM',S' such that $((\mathsf{IM},\mathsf{S}),(\mathsf{IM}',\mathsf{S}')) \in [\![\mathcal{R}]\!]^{\mathsf{r}}$. From the definition of $[\![\mathcal{R}]\!]^{\mathsf{r}}$ and as $(\mathsf{IM},\mathsf{S}) \in [\![P]\!]$ we know $(\mathsf{IM}',\mathsf{S}') \in [\![P]\!]$. As such, from (I.H.) we have $\mathsf{safe}_m(\mathsf{C}^{\mathsf{sp}},\mathsf{IM}',\mathsf{S}',[\![\mathcal{R}]\!]^{\mathsf{r}},[\![\mathcal{G}]\!]^{\mathsf{g}},[\![P]\!],)$, as required.

Similarly, let $\mathsf{IM}_a{=}\mathsf{IM}[\overline{x_{\mathsf{p}} \mapsto \mathsf{IM}(x_{\mathsf{s}})}^{\mathsf{sp}}]$, and let $\mathsf{IM}_s{=}\mathsf{IM}[x_{\mathsf{s}} \mapsto \mathsf{IM}(x_{\mathsf{v}})]$ for an arbitrary x. As P is stable and $(\mathsf{IM},\mathsf{S}) \in [\![P]\!]$, from the definition of stability we then know $(\mathsf{IM}_s,\mathsf{S}),(\mathsf{IM}_a,\mathsf{S}) \in [\![P]\!]$, and thus from (I.H.) we have $\mathsf{safe}_m(\mathsf{C}^{\mathsf{sp}},\mathsf{IM}_s,\mathsf{S},[\![\mathcal{R}]\!]^{\mathsf{r}},[\![\mathcal{G}]\!]^{\mathsf{g}},[\![Q]\!],)$ and $\mathsf{safe}_m(\mathsf{C}^{\mathsf{sp}},\mathsf{IM}_a,\mathsf{S},[\![\mathcal{R}]\!]^{\mathsf{r}},[\![\mathcal{G}]\!]^{\mathsf{g}},[\![Q]\!],)$, as required.

Case (PAR)
We then have $\mathsf{C} = \mathsf{c}\,||\,\mathsf{C}'$ for some c,C', $\mathcal{R}{=}\mathcal{R}_1 \cup \mathcal{R}_2 \cup \{Q\}$, $\mathcal{G}{=}\mathcal{G}_1 \cup \mathcal{G}_2$, $\langle\mathcal{R}_1;\mathcal{G}_1\rangle \vdash \{P_1\}\ \mathsf{c}\ \{Q_1\}$, $\langle\mathcal{R}_2;\mathcal{G}_2\rangle \vdash \{P_2\}\ \mathsf{C}'\ \{Q_2\}$, $Q_1 \wedge Q_2 \Rightarrow Q$, $\langle\mathcal{R}_1,\mathcal{G}_1\rangle$ and $\langle\mathcal{R}_2,\mathcal{G}_2\rangle$ are non-interfering, $A_1 \cap \mathsf{wr}(\mathsf{C}'){=}\emptyset$, where $A_1{=}\mathsf{fr}(\mathsf{c},\mathcal{R}_1)$ and $A_2 \cap \mathsf{wr}(\mathsf{c}){=}\emptyset$, where $A_2{=}\mathsf{fr}(\mathsf{C}',\mathcal{R}_2)$.

Pick an arbitrary n and $(\mathsf{IM},\mathsf{S}) \in [\![P_1 \wedge P_2]\!]$. Given the interpretations of $\mathcal{R}$ and $\mathcal{G}$, it suffices to show $\mathsf{safe}_n([\![\mathsf{c}\,||\,\mathsf{C}']\!],\mathsf{IM},\mathsf{S},[\![\mathcal{R}]\!]^{\mathsf{r}},[\![\mathcal{G}_1]\!]^{\mathsf{g}} \cup [\![\mathcal{G}_2]\!]^{\mathsf{g}},[\![Q]\!],[\![Q_{\mathsf{p}}]\!])$.

It is straightforward to demonstrate that in all our triples the free registers of the postcondition are included in that of rely, and thus that $\mathsf{fr}(Q_1) \subseteq \mathsf{fr}(\mathcal{R}_1) \subseteq A_1$ and $\mathsf{fr}(Q_2) \subseteq \mathsf{fr}(\mathcal{R}_2) \subseteq A_2$. From Prop. 1 we then know that $\text{s-stable}([\![Q_1]\!],A_1)$ and $\text{s-stable}([\![Q_2]\!],A_2)$ hold. Similarly, from the $[\![.]\!]^{\mathsf{r}}$ definition, we know that $\text{s-stable}([\![\mathcal{R}_1]\!],A_1)$ and $\text{s-stable}([\![\mathcal{R}_2]\!],A_2)$ hold. Given the definitions of $\mathcal{R}$

and $\mathcal{G}$, the interpretation functions $[\![.]\!]^{\mathsf{r}}$, $[\![.]\!]^{\mathsf{g}}$, $[\![.]\!]$, and since Q is stable and $Q_1 \wedge Q_2 \Rightarrow Q$, we know that $\mathsf{wf}([\![\mathcal{R}]\!]^{\mathsf{r}}, [\![\mathcal{G}_1]\!]^{\mathsf{g}} \cup [\![\mathcal{G}_2]\!]^{\mathsf{g}}, [\![Q]\!])$ holds, $[\![\mathcal{R}]\!]^{\mathsf{r}} \subseteq [\![\mathcal{R}_1 \cup \mathcal{R}_2]\!]^{\mathsf{r}} = [\![\mathcal{R}_1]\!]^{\mathsf{r}} \cap [\![\mathcal{R}_2]\!]^{\mathsf{r}}$, $[\![Q_1]\!] \cap [\![Q_2]\!] \subseteq [\![Q]\!]$ and $[\![(Q_1)_{\mathsf{p}}]\!] \cap [\![(Q_2)_{\mathsf{p}}]\!] \subseteq [\![Q_{\mathsf{p}}]\!]$. Moreover, since $\langle \mathcal{R}_1, \mathcal{G}_1 \rangle$ and $\langle \mathcal{R}_2, \mathcal{G}_2 \rangle$ are non-interfering, from Lemma 10 we know $\langle [\![\mathcal{R}_1]\!]^{\mathsf{r}}, [\![\mathcal{G}_1]\!]^{\mathsf{g}} \rangle$ and $\langle [\![\mathcal{R}_2]\!]^{\mathsf{r}}, [\![\mathcal{G}_2]\!]^{\mathsf{g}} \rangle$ are semantically non-interfering. On the other hand, from the inductive hypothesis and since $\langle \mathcal{R}_1; \mathcal{G}_1 \rangle \vdash \{P\}\ \mathsf{c}\ \{R\}$ and $\langle \mathcal{R}_2; \mathcal{G}_2 \rangle \vdash \{R\}\ \mathsf{C}'\ \{Q\}$ hold, we know $\mathsf{safe}_n([\![\mathsf{c}]\!], \mathsf{IM}, \mathsf{S}_1, [\![\mathcal{R}_1]\!]^{\mathsf{r}}, [\![\mathcal{G}_1]\!]^{\mathsf{g}}, [\![Q_1]\!],)$ and $\mathsf{safe}_n([\![\mathsf{C}']\!], \mathsf{IM}, \mathsf{S}_2, [\![\mathcal{R}_2]\!]^{\mathsf{r}}, [\![\mathcal{G}_2]\!]^{\mathsf{g}}, [\![Q_2]\!],)$ hold. As such, from Lemma 16 we have $\mathsf{safe}_n([\![\mathsf{c} \,||\, \mathsf{C}']\!], \mathsf{IM}, \mathsf{S}, [\![\mathcal{R}]\!]^{\mathsf{r}}, [\![\mathcal{G}_1]\!]^{\mathsf{g}} \cup [\![\mathcal{G}_2]\!]^{\mathsf{g}}, [\![Q]\!],)$, as required.

Case (CONSEQ)
Let us assume there exists $P, P', Q, Q', \mathcal{R}', \mathcal{R}_1, \mathcal{G}, \mathcal{G}', \mathsf{C}$ such that $P \Rightarrow P'$, $\langle \mathcal{R}'; \mathcal{G}' \rangle \vdash \{P'\}\ \mathsf{C}\ \{Q'\}$, $\mathcal{R}' \subseteq \mathcal{R}_1$, $\mathcal{G}' \subseteq \mathcal{G}$ and $Q' \Rightarrow Q$. Let $\mathcal{R}{=}\mathcal{R}_1 \cup \{P, Q\}$ and $[\![\mathsf{C}]\!]{=}\mathsf{C}_t^{\mathsf{sp}}$ for some C_t. As $\mathcal{G}' \subseteq \mathcal{G}$, from the definition of $[\![.]\!]^{\mathsf{g}}$ we have $[\![\mathcal{G}']\!]^{\mathsf{g}} \subseteq [\![\mathcal{G}]\!]^{\mathsf{g}}$. Similarly, as $\mathcal{R}' \subseteq \mathcal{R}_1 \subseteq \mathcal{R}$, from the definition of $[\![.]\!]^{\mathsf{r}}$ we have $[\![\mathcal{R}]\!]^{\mathsf{r}} \subseteq [\![\mathcal{R}']\!]^{\mathsf{r}}$. Furthermore, since $Q' \Rightarrow Q$, we have $[\![Q']\!] \subseteq [\![Q]\!]$ and $[\![Q'_{\mathsf{p}}]\!] \subseteq [\![Q_{\mathsf{p}}]\!]$. Moreover, from the inductive hypothesis and since $\langle \mathcal{R}'; \mathcal{G}' \rangle \vdash \{P'\}\ \mathsf{C}\ \{Q'\}$, we know that:

$$\forall n.\ \forall (\mathsf{IM}, \mathsf{S}) \in [\![P']\!].\ \mathsf{safe}_n([\![\mathsf{C}]\!], \mathsf{IM}, \mathsf{S}, [\![\mathcal{R}']\!]^{\mathsf{r}}, [\![\mathcal{G}']\!]^{\mathsf{g}}, [\![Q']\!], [\![Q'_{\mathsf{p}}]\!]) \tag{20}$$

We are then required to show $\langle [\![\mathcal{R}]\!]^{\mathsf{r}}; [\![\mathcal{G}]\!]^{\mathsf{g}} \rangle \Vdash \{[\![P]\!]\}\ [\![\mathsf{C}]\!]\ \{[\![Q]\!]\}$. That is, we must now show that for all n, and for all $(\mathsf{IM}, \mathsf{S}) \in [\![P]\!]$, $\mathsf{safe}_n([\![\mathsf{C}]\!], \mathsf{IM}, \mathsf{S}, [\![\mathcal{R}]\!]^{\mathsf{r}}, [\![\mathcal{G}]\!]^{\mathsf{g}}, Q, [\![Q_{\mathsf{p}}]\!])$ holds. Pick an arbitrary n and $(\mathsf{IM}, \mathsf{S}) \in [\![P]\!]$. As $P \Rightarrow P'$, we then know $(\mathsf{IM}, \mathsf{S}) \in [\![P']\!]$.

To show property (1), let us assume $\mathsf{C}_t{=}\mathsf{C}_{\mathbf{skip}}$. As $(\mathsf{IM}, \mathsf{S}) \in [\![P']\!]$, from (20) we have $(\mathsf{IM}, \mathsf{S}) \in Q'$. Moreover, as $Q' \Rightarrow Q$, we then know $(\mathsf{IM}, \mathsf{S}) \in Q$, as required. Moreover, as $(\mathsf{IM}, \mathsf{S}) \in [\![P']\!]$, from (20) we have $(\mathsf{IM}, \mathsf{S}) \in [\![Q'_{\mathsf{p}}]\!]$. Consequently, as $[\![Q'_{\mathsf{p}}]\!] \subseteq [\![Q_{\mathsf{p}}]\!]$, we have $(\mathsf{IM}, \mathsf{S}) \in [\![Q_{\mathsf{p}}]\!]$, as required.

To show property (4), pick arbitrary $\tau, \mathsf{IM}_1, \mathsf{IM}_2, \mathsf{B}_1, \mathsf{B}_2, \mathsf{C}', \mathsf{IM}', \mathsf{S}', l$ such that $\mathsf{IM}{=}\Downarrow(\mathsf{IM}_1, \mathsf{B}_1, \tau)$, $\mathsf{IM}' = \Downarrow(\mathsf{IM}_2, \mathsf{B}_2, \tau)$ and $\mathsf{C}_t, \mathsf{IM}_1, \mathsf{B}_1, \mathsf{S} \xRightarrow{\tau:l} \mathsf{C}', \mathsf{IM}_2, \mathsf{B}_2, \mathsf{S}'$. As $(\mathsf{IM}, \mathsf{S}) \in [\![P']\!]$, from (20) we have $((\mathsf{IM}, \mathsf{S}), (\mathsf{IM}', \mathsf{S})) \in [\![\mathcal{G}']\!]^{\mathsf{g}} \subseteq [\![\mathcal{G}]\!]^{\mathsf{g}}$, as required. Similarly, from (20) we have $\mathsf{safe}_m((\mathsf{C}')^{\mathsf{sp}}, \mathsf{IM}', \mathsf{S}', [\![\mathcal{R}']\!]^{\mathsf{r}}, [\![\mathcal{G}']\!]^{\mathsf{g}}, [\![Q']\!], [\![Q'_{\mathsf{p}}]\!])$. As such, as $[\![\mathcal{R}]\!]^{\mathsf{r}} \subseteq [\![\mathcal{R}']\!]^{\mathsf{r}}$, $[\![\mathcal{G}']\!]^{\mathsf{g}} \subseteq [\![\mathcal{G}]\!]^{\mathsf{g}}$, $[\![Q']\!] \subseteq [\![Q]\!]$, $[\![Q'_{\mathsf{p}}]\!] \subseteq [\![Q_{\mathsf{p}}]\!]$, from Lemmas 12 to 14 we have $\mathsf{safe}_m((\mathsf{C}')^{\mathsf{sp}}, \mathsf{IM}', \mathsf{S}', [\![\mathcal{R}]\!]^{\mathsf{r}}, [\![\mathcal{G}]\!]^{\mathsf{g}}, [\![Q]\!], [\![Q_{\mathsf{p}}]\!])$, as required.

To show property (3), pick arbitrary $\mathsf{IM}', \mathsf{S}'$ such that $((\mathsf{IM}, \mathsf{S}), (\mathsf{IM}', \mathsf{S}')) \in ([\![\mathcal{R}]\!]^{\mathsf{r}}) \cup \mathsf{A}_{\mathsf{sp}}$. As $[\![\mathcal{R}]\!]^{\mathsf{r}} \subseteq [\![\mathcal{R}']\!]^{\mathsf{r}}$, we then have $((\mathsf{IM}, \mathsf{S}), (\mathsf{IM}', \mathsf{S}')) \in ([\![\mathcal{R}']\!]^{\mathsf{r}}) \cup \mathsf{A}_{\mathsf{sp}}$. As such, from (20) we have $\mathsf{safe}_m([\![\mathsf{C}]\!], \mathsf{IM}', \mathsf{S}', [\![\mathcal{R}']\!]^{\mathsf{r}}, [\![\mathcal{G}']\!]^{\mathsf{g}}, [\![Q']\!], [\![Q'_{\mathsf{p}}]\!])$. Consequently, since $[\![\mathcal{R}]\!]^{\mathsf{r}} \subseteq [\![\mathcal{R}']\!]^{\mathsf{r}}$, $[\![\mathcal{G}']\!]^{\mathsf{g}} \subseteq [\![\mathcal{G}]\!]^{\mathsf{g}}$, $[\![Q']\!] \subseteq [\![Q]\!]$ and $[\![Q'_{\mathsf{p}}]\!] \subseteq [\![Q_{\mathsf{p}}]\!]$, from Lemmas 12 to 14 we have $\mathsf{safe}_m([\![\mathsf{C}]\!], \mathsf{IM}', \mathsf{S}', [\![\mathcal{R}]\!]^{\mathsf{r}}, [\![\mathcal{G}]\!]^{\mathsf{g}}, [\![Q]\!], [\![Q_{\mathsf{p}}]\!])$, as required.

Case (REC)
We are required to show that if $P \Rightarrow P'$, $Q' \Rightarrow Q$, $\langle \top; \top \rangle \vdash \{P'\}\ \mathsf{C}\ \{Q'\}$, $\langle \top; \top \rangle \vdash \{R\}\ \mathsf{C}_{\mathsf{rec}}\ \{Q'\}$ and $Q' \Rightarrow R[\overline{x_{\mathsf{p}}/x_{\mathsf{s}}}][\overline{x_{\mathsf{p}}/x_{\mathsf{v}}}]$, then $\mathsf{C}_{\mathsf{rec}} \vdash \{P\}\ \mathsf{C}\ \{Q\}$ holds. That is, we must show for all $\mathsf{IM}, \mathsf{S}, \mathsf{IM}', \mathsf{S}'$, if $(\mathsf{IM}, \mathsf{S}) \in [\![P]\!]$ and $\mathsf{C}_{\mathsf{rec}} \vdash \mathsf{C}, \mathsf{S}, \mathsf{IM}, \mathsf{B}_0 \Rightarrow^* \mathsf{C}_{\mathbf{skip}}, \mathsf{S}', \mathsf{IM}', \mathsf{B}_0$, then $(\mathsf{IM}', \mathsf{S}') \in [\![Q]\!]$. From the inductive hypothesis we know $\langle \top; \top \rangle \Vdash \{P'\}\ \mathsf{C}\ \{Q'\}$, $\langle \top; \top \rangle \Vdash \{R\}\ \mathsf{C}_{\mathsf{rec}}\ \{Q'\}$ are valid judgements.

Let us write $\underset{\mathsf{nc}}{\Rightarrow}$ for non-crashing $\Rightarrow$ transitions (i.e. not involving the (CRASH) transition $\xRightarrow{-:\lightning}$); similarly, let us write $\underset{\lightning}{\Rightarrow}$ for crashing transitions (of the form $\xRightarrow{-:\lightning}$). Let us write $(\underset{\mathsf{nc}}{\Rightarrow}^* \underset{\lightning}{\Rightarrow})^0$ for the identity relation, and write $(\underset{\mathsf{nc}}{\Rightarrow}^* \underset{\lightning}{\Rightarrow})^{n+1}$ for $\underset{\mathsf{nc}}{\Rightarrow}^* \underset{\lightning}{\Rightarrow} (\underset{\mathsf{nc}}{\Rightarrow}^* \underset{\lightning}{\Rightarrow})^n$, for all $n \in \mathbb{N}$.

Pick arbitrary $\mathsf{IM}, \mathsf{S}, \mathsf{IM}', \mathsf{S}'$ such that $(\mathsf{IM}, \mathsf{S}) \in [\![P]\!]$ and $\mathsf{C_{rec}} \vdash \mathsf{C}, \mathsf{S}, \mathsf{IM}, \mathsf{B}_0 \Rightarrow^* \mathsf{C_{skip}}, \mathsf{S}', \mathsf{IM}', \mathsf{B}_0$. Note that $\Rightarrow^*$ can be split to segments involving non-crashing transitions and crashing transition. That is, there exists n such that $\mathsf{C_{rec}} \vdash \mathsf{C}, \mathsf{S}, \mathsf{IM}, \mathsf{B}_0 \; (\underset{\mathsf{nc}}{\Rightarrow^*} \underset{\lightning}{\Rightarrow})^n \underset{\mathsf{nc}}{\Rightarrow^*} \; \mathsf{C_{skip}}, \mathsf{S}', \mathsf{IM}', \mathsf{B}_0$.

As $(\mathsf{IM}, \mathsf{S}) \in [\![P]\!]$ and $P \Rightarrow P'$, we also have $(\mathsf{IM}, \mathsf{S}) \in [\![P']\!]$. In what follows we show that the following hold, and thus we have $(\mathsf{IM}', \mathsf{S}') \in [\![Q]\!]$, as required.

$$\begin{array}{l}\forall n.\ \forall \mathsf{IM}, \mathsf{S}, \mathsf{IM}', \mathsf{S}'. \\ \quad (\mathsf{IM}, \mathsf{S}) \in [\![P']\!] \wedge \mathsf{C_{rec}} \vdash \mathsf{C}, \mathsf{S}, \mathsf{IM}, \mathsf{B}_0 \; (\underset{\mathsf{nc}}{\Rightarrow^*} \underset{\lightning}{\Rightarrow})^n \underset{\mathsf{nc}}{\Rightarrow^*} \; \mathsf{C_{skip}}, \mathsf{S}', \mathsf{IM}', \mathsf{B}_0 \Rightarrow (\mathsf{IM}', \mathsf{S}') \in [\![Q]\!] \\ \quad (\mathsf{IM}, \mathsf{S}) \in [\![R]\!] \wedge \mathsf{C_{rec}} \vdash \mathsf{C_{rec}}, \mathsf{S}, \mathsf{IM}, \mathsf{B}_0 \; (\underset{\mathsf{nc}}{\Rightarrow^*} \underset{\lightning}{\Rightarrow})^n \underset{\mathsf{nc}}{\Rightarrow^*} \; \mathsf{C_{skip}}, \mathsf{S}', \mathsf{IM}', \mathsf{B}_0 \Rightarrow (\mathsf{IM}', \mathsf{S}') \in [\![Q]\!]\end{array} \tag{21}$$

RTS. (21) We proceed by induction on n.

Base case n=0
Pick arbitrary $\mathsf{IM}, \mathsf{S}, \mathsf{IM}', \mathsf{S}'$ such that $(\mathsf{IM}, \mathsf{S}) \in [\![P']\!]$ and $\mathsf{C_{rec}} \vdash \mathsf{C}, \mathsf{S}, \mathsf{IM}, \mathsf{B}_0 \; (\underset{\mathsf{nc}}{\Rightarrow^*} \underset{\lightning}{\Rightarrow})^0 \underset{\mathsf{nc}}{\Rightarrow^*} \mathsf{C_{skip}}, \mathsf{S}', \mathsf{IM}', \mathsf{B}_0$. That is, $\mathsf{C_{rec}} \vdash \mathsf{C}, \mathsf{S}, \mathsf{IM}, \mathsf{B}_0 \underset{\mathsf{nc}}{\Rightarrow^*} \mathsf{C_{skip}}, \mathsf{S}', \mathsf{IM}', \mathsf{B}_0$. Consequently, as $(\mathsf{IM}, \mathsf{S}) \in [\![P']\!]$ from the validity of the $\langle \mathsf{T}; \mathsf{T} \rangle \vdash \{P'\}\ \mathsf{C}\ \{Q'\}$ judgement we know $(\mathsf{IM}', \mathsf{S}') \in Q'$. Finally, as $Q' \Rightarrow Q$, we also have $(\mathsf{IM}', \mathsf{S}') \in Q'$, as required.

Similarly, pick arbitrary $\mathsf{IM}, \mathsf{S}, \mathsf{IM}', \mathsf{S}'$ such that $(\mathsf{IM}, \mathsf{S}) \in [\![R]\!]$ and $\mathsf{C_{rec}} \vdash \mathsf{C_{rec}}, \mathsf{S}, \mathsf{IM}, \mathsf{B}_0 \; (\underset{\mathsf{nc}}{\Rightarrow^*} \underset{\lightning}{\Rightarrow})^0 \underset{\mathsf{nc}}{\Rightarrow^*} \mathsf{C_{skip}}, \mathsf{S}', \mathsf{IM}', \mathsf{B}_0$. That is, $\mathsf{C_{rec}} \vdash \mathsf{C_{rec}}, \mathsf{S}, \mathsf{IM}, \mathsf{B}_0 \underset{\mathsf{nc}}{\Rightarrow^*} \mathsf{C_{skip}}, \mathsf{S}', \mathsf{IM}', \mathsf{B}_0$. Consequently, as $(\mathsf{IM}, \mathsf{S}) \in [\![R]\!]$ from the validity of the $\langle \mathsf{T}; \mathsf{T} \rangle \vdash \{R\}\ \mathsf{C_{rec}}\ \{Q'\}$ judgement we know $(\mathsf{IM}', \mathsf{S}') \in Q'$. Finally, as $Q' \Rightarrow Q$, we also have $(\mathsf{IM}', \mathsf{S}') \in Q'$, as required.

Inductive case n=m+1

$$\begin{array}{l}\forall \mathsf{IM}, \mathsf{S}, \mathsf{IM}', \mathsf{S}'. \\ \quad (\mathsf{IM}, \mathsf{S}) \in [\![P']\!] \wedge \mathsf{C_{rec}} \vdash \mathsf{C}, \mathsf{S}, \mathsf{IM}, \mathsf{B}_0 \; (\underset{\mathsf{nc}}{\Rightarrow^*} \underset{\lightning}{\Rightarrow})^m \underset{\mathsf{nc}}{\Rightarrow^*} \; \mathsf{C_{skip}}, \mathsf{S}', \mathsf{IM}', \mathsf{B}_0 \Rightarrow (\mathsf{IM}', \mathsf{S}') \in [\![Q]\!] \\ \quad (\mathsf{IM}, \mathsf{S}) \in [\![R]\!] \wedge \mathsf{C_{rec}} \vdash \mathsf{C_{rec}}, \mathsf{S}, \mathsf{IM}, \mathsf{B}_0 \; (\underset{\mathsf{nc}}{\Rightarrow^*} \underset{\lightning}{\Rightarrow})^m \underset{\mathsf{nc}}{\Rightarrow^*} \; \mathsf{C_{skip}}, \mathsf{S}', \mathsf{IM}', \mathsf{B}_0 \Rightarrow (\mathsf{IM}', \mathsf{S}') \in [\![Q]\!]\end{array} \tag{I.H}$$

Pick arbitrary $\mathsf{IM}, \mathsf{S}, \mathsf{IM}', \mathsf{S}'$ such that $(\mathsf{IM}, \mathsf{S}) \in [\![P']\!]$ and $\mathsf{C_{rec}} \vdash \mathsf{C}, \mathsf{S}, \mathsf{IM}, \mathsf{B}_0 \; (\underset{\mathsf{nc}}{\Rightarrow^*} \underset{\lightning}{\Rightarrow})^n \Rightarrow^*_{\mathsf{nc}} \mathsf{C_{skip}}, \mathsf{S}', \mathsf{IM}', \mathsf{B}_0$. That is, there exist $\mathsf{S}_1, \mathsf{B}_1, \mathsf{IM}_1, \mathsf{IM}_2, \mathsf{C}_1$ such that:
$\mathsf{C_{rec}} \vdash \mathsf{C}, \mathsf{S}, \mathsf{IM}, \mathsf{B}_0 \underset{\mathsf{nc}}{\Rightarrow^*} \mathsf{C}_1, \mathsf{S}_1, \mathsf{IM}_1, \mathsf{B}_1$
$\mathsf{C_{rec}} \vdash \mathsf{C}_1, \mathsf{S}_1, \mathsf{IM}_1, \mathsf{B}_1 \underset{\lightning}{\Rightarrow} \mathsf{C_{rec}}, \mathsf{S}_0, \mathsf{IM}_2, \mathsf{B}_0$ where $\mathsf{IM}_2 = \mathsf{IM}_1[\overline{x_\mathsf{s} \mapsto \mathsf{IM}_1(x_\mathsf{p})}][\overline{x_\mathsf{v} \mapsto \mathsf{IM}_1(x_\mathsf{p})}]$
$\mathsf{C_{rec}} \vdash \mathsf{C_{rec}}, \mathsf{S}_0, \mathsf{IM}_2, \mathsf{B}_0 \; (\underset{\mathsf{nc}}{\Rightarrow^*} \underset{\lightning}{\Rightarrow})^m \underset{\mathsf{nc}}{\Rightarrow^*} \mathsf{C_{skip}}, \mathsf{S}', \mathsf{IM}', \mathsf{B}_0$

As $(\mathsf{IM}, \mathsf{S}) \in [\![P']\!]$ and $\mathsf{C_{rec}} \vdash \mathsf{C}, \mathsf{S}, \mathsf{IM}, \mathsf{B}_0 \underset{\mathsf{nc}}{\Rightarrow^*} \mathsf{C}_1, \mathsf{S}_1, \mathsf{IM}_1, \mathsf{B}_1$, from the validity of the $\langle \mathsf{T}; \mathsf{T} \rangle \vdash \{P'\}\ \mathsf{C}\ \{Q'\}$ judgement and the definition of safe we know that $(\mathsf{IM}_1, \mathsf{S}_1) \in [\![Q'_\mathsf{p}]\!]$. As such, given the definition of Q'_p and since $\mathsf{IM}_2 = \mathsf{IM}_1[\overline{x_\mathsf{s} \mapsto \mathsf{IM}_1(x_\mathsf{p})}][\overline{x_\mathsf{v} \mapsto \mathsf{IM}_1(x_\mathsf{p})}]$, we also have $(\mathsf{IM}_2, \mathsf{S}_0) \in [\![Q'_\mathsf{p}]\!]$. On the other hand, as $Q' \Rightarrow R[\overline{x_\mathsf{p}/x_\mathsf{s}}][\overline{x_\mathsf{p}/x_\mathsf{v}}]$, from Prop. 2 we have $Q'_\mathsf{p} \Rightarrow R[\overline{x_\mathsf{p}/x_\mathsf{s}}][\overline{x_\mathsf{p}/x_\mathsf{v}}]$. As such, since $(\mathsf{IM}_2, \mathsf{S}_0) \in [\![Q'_\mathsf{p}]\!]$, we also have $(\mathsf{IM}_2, \mathsf{S}_0) \in [\![R[\overline{x_\mathsf{p}/x_\mathsf{s}}][\overline{x_\mathsf{p}/x_\mathsf{v}}]]\!]$. Consequently, as $\mathsf{IM}_2 = \mathsf{IM}_1[\overline{x_\mathsf{s} \mapsto \mathsf{IM}_1(x_\mathsf{p})}][\overline{x_\mathsf{v} \mapsto \mathsf{IM}_1(x_\mathsf{p})}]$ and thus for all x: $\mathsf{IM}_2(x_\mathsf{v}) = \mathsf{IM}_2(x_\mathsf{s}) = \mathsf{IM}_2(x_\mathsf{p})$, we also have $(\mathsf{IM}_2, \mathsf{S}_0) \in [\![R]\!]$. As a result, since $\mathsf{C_{rec}} \vdash \mathsf{C_{rec}}, \mathsf{S}_0, \mathsf{IM}_2, \mathsf{B}_0 \; (\underset{\mathsf{nc}}{\Rightarrow^*} \underset{\lightning}{\Rightarrow})^m \underset{\mathsf{nc}}{\Rightarrow^*} \mathsf{C_{skip}}, \mathsf{S}', \mathsf{IM}', \mathsf{B}_0$,

from (I.H) we have $(\mathsf{IM}', \mathsf{S}') \in [\![Q]\!]$, as required.

Similarly, pick $\mathsf{IM}, \mathsf{S}, \mathsf{IM}', \mathsf{S}'$ such that $(\mathsf{IM}, \mathsf{S}) \in [\![R]\!]$ and $\mathsf{C}_{\mathsf{rec}} \vdash \mathsf{C}, \mathsf{S}, \mathsf{IM}, \mathsf{B}_0 \ (\Rightarrow^*_{\mathsf{nc}} \Rightarrow_{\lightning})^n \Rightarrow^*_{\mathsf{nc}} \mathsf{C}_{\mathbf{skip}}, \mathsf{S}', \mathsf{IM}', \mathsf{B}_0$. That is, there exist $\mathsf{S}_1, \mathsf{B}_1, \mathsf{IM}_1, \mathsf{IM}_2, \mathsf{C}_1$ such that:
$\mathsf{C}_{\mathsf{rec}} \vdash \mathsf{C}, \mathsf{S}, \mathsf{IM}, \mathsf{B}_0 \Rightarrow^*_{\mathsf{nc}} \mathsf{C}_1, \mathsf{S}_1, \mathsf{IM}_1, \mathsf{B}_1$
$\mathsf{C}_{\mathsf{rec}} \vdash \mathsf{C}_1, \mathsf{S}_1, \mathsf{IM}_1, \mathsf{B}_1 \Rightarrow_{\lightning} \mathsf{C}_{\mathsf{rec}}, \mathsf{S}_0, \mathsf{IM}_2, \mathsf{B}_0$ where $\mathsf{IM}_2{=}\mathsf{IM}_1[\overline{x_{\mathsf{s}} \mapsto \mathsf{IM}_1(x_{\mathsf{p}})}][\overline{x_{\mathsf{v}} \mapsto \mathsf{IM}_1(x_{\mathsf{p}})}]$
$\mathsf{C}_{\mathsf{rec}} \vdash \mathsf{C}_{\mathsf{rec}}, \mathsf{S}_0, \mathsf{IM}_2, \mathsf{B}_0 \ (\Rightarrow^*_{\mathsf{nc}} \Rightarrow_{\lightning})^m \Rightarrow^*_{\mathsf{nc}} \mathsf{C}_{\mathbf{skip}}, \mathsf{S}', \mathsf{IM}', \mathsf{B}_0$

As $(\mathsf{IM}, \mathsf{S}) \in [\![R]\!]$ and $\mathsf{C}_{\mathsf{rec}} \vdash \mathsf{C}, \mathsf{S}, \mathsf{IM}, \mathsf{B}_0 \Rightarrow^*_{\mathsf{nc}} \mathsf{C}_1, \mathsf{S}_1, \mathsf{IM}_1, \mathsf{B}_1$, from the validity of the $\langle \top; \top \rangle \vdash \{R\}\ \mathsf{C}\ \{Q'\}$ judgement and the definition of safe we know that $(\mathsf{IM}_1, \mathsf{S}_1) \in [\![Q'_{\mathsf{p}}]\!]$. As such, given the definition of Q'_{p} and since $\mathsf{IM}_2{=}\mathsf{IM}_1[\overline{x_{\mathsf{s}} \mapsto \mathsf{IM}_1(x_{\mathsf{p}})}][\overline{x_{\mathsf{v}} \mapsto \mathsf{IM}_1(x_{\mathsf{p}})}]$, we also have $(\mathsf{IM}_2, \mathsf{S}_0) \in [\![Q'_{\mathsf{p}}]\!]$. On the other hand, as $Q' \Rightarrow R[\overline{x_{\mathsf{p}}/x_{\mathsf{s}}}][\overline{x_{\mathsf{p}}/x_{\mathsf{v}}}]$, from Prop. 2 we have $Q'_{\mathsf{p}} \Rightarrow R[\overline{x_{\mathsf{p}}/x_{\mathsf{s}}}][\overline{x_{\mathsf{p}}/x_{\mathsf{v}}}]$. As such, since $(\mathsf{IM}_2, \mathsf{S}_0) \in [\![Q'_{\mathsf{p}}]\!]$, we also have $(\mathsf{IM}_2, \mathsf{S}_0) \in [\![R[\overline{x_{\mathsf{p}}/x_{\mathsf{s}}}][\overline{x_{\mathsf{p}}/x_{\mathsf{v}}}]]\!]$. Consequently, as $\mathsf{IM}_2{=}\mathsf{IM}_1[\overline{x_{\mathsf{s}} \mapsto \mathsf{IM}_1(x_{\mathsf{p}})}][\overline{x_{\mathsf{v}} \mapsto \mathsf{IM}_1(x_{\mathsf{p}})}]$ and thus for all x: $\mathsf{IM}_2(x_{\mathsf{v}}){=}\mathsf{IM}_2(x_{\mathsf{s}}){=}\mathsf{IM}_2(x_{\mathsf{p}})$, we also have $(\mathsf{IM}_2, \mathsf{S}_0) \in [\![R]\!]$. As a result, since $\mathsf{C}_{\mathsf{rec}} \vdash \mathsf{C}_{\mathsf{rec}}, \mathsf{S}_0, \mathsf{IM}_2, \mathsf{B}_0 \ (\Rightarrow^*_{\mathsf{nc}} \Rightarrow_{\lightning})^m \Rightarrow^*_{\mathsf{nc}} \mathsf{C}_{\mathbf{skip}}, \mathsf{S}', \mathsf{IM}', \mathsf{B}_0$, from (I.H) we have $(\mathsf{IM}', \mathsf{S}') \in [\![Q]\!]$, as required. □

Theorem 8 (Adequacy). *For all closed triples $\langle \mathcal{R}; \mathcal{G} \rangle \vdash \{P\}\ \mathsf{C}\ \{Q\}$ that can be derived using POG rules in Fig. 3, and all $\mathsf{M}, \mathsf{S}, \mathsf{M}', \mathsf{S}', \mathsf{IM}$, such that $(\mathsf{IM}, \mathsf{S}) \models P$ and $\forall x.\ \mathsf{M}(x){=}\mathsf{IM}(x_{\mathsf{v}}){=}\mathsf{IM}(x_{\mathsf{s}}){=}\mathsf{IM}(x_{\mathsf{p}})$: if $\mathsf{C}, \mathsf{S}, \mathsf{M}, \mathsf{PB}_0, \mathsf{B}_0 \rightarrow^* \mathsf{C}_{\mathbf{skip}}, \mathsf{S}', \mathsf{M}', -, -$, then there exists IM' such that $\forall x.\ \mathsf{M}'(x){=}\mathsf{IM}'(x_{\mathsf{p}})$ and $(\mathsf{IM}', \mathsf{S}') \models Q$*

Proof. Pick an arbitrary closed triple $\langle \mathcal{R}; \mathcal{G} \rangle \vdash \{P\}\ \mathsf{C}\ \{Q\}$ derived using POG rules in Fig. 3. Pick arbitrary $\mathsf{M}, \mathsf{S}, \mathsf{M}', \mathsf{S}', \mathsf{IM}$ such that $(\mathsf{IM}, \mathsf{S}) \models P$, $\forall x.\ \mathsf{M}(x){=}\mathsf{IM}(x_{\mathsf{v}}){=}\mathsf{IM}(x_{\mathsf{s}}){=}\mathsf{IM}(x_{\mathsf{p}})$, and $\mathsf{C}, \mathsf{S}, \mathsf{M}, \mathsf{PB}_0, \mathsf{B}_0 \rightarrow^* \mathsf{C}_{\mathbf{skip}}, \mathsf{S}', \mathsf{M}', -, -$. From Thm. 6 we know there exists IM' such that $\Delta \vdash [\![\mathsf{C}]\!], \mathsf{S}, \mathsf{IM}, \mathsf{B}_0 \Rightarrow^* [\![\mathsf{C}_{\mathbf{skip}}]\!], \mathsf{S}', \mathsf{IM}', -$ and $\forall x.\ \mathsf{M}'(x){=}\mathsf{IM}'(x_{\mathsf{p}})$. Moreover, since $\langle \mathcal{R}; \mathcal{G} \rangle \vdash \{P\}\ \mathsf{C}\ \{Q\}$ can be derived using POG rules in Fig. 3, from Thm. 2 we know for all n, $\mathsf{safe}_n([\![\mathsf{C}]\!], (\mathsf{IM}, \mathsf{S}), [\![\mathcal{R}]\!]^{\mathsf{r}}, [\![\mathcal{G}]\!]^{\mathsf{g}}, [\![Q]\!])$ holds, and thus from the first conjunct of safe we have $(\mathsf{IM}', \mathsf{S}') \in [\![Q]\!]$, i.e. $(\mathsf{IM}', \mathsf{S}') \models Q$. □

C EQUIVALENT DECLARATIVE Px86$_{\mathsf{sim}}$ SPECIFICATION

Definition 13. An execution $(E, I, P, \mathsf{po}, \mathsf{rf}, \mathsf{mo}, \mathsf{nvo})$ is *Px86$_{sim}$-consistent* iff there exists a strict order, $\mathsf{tso} \subseteq E \times E$, such that:

- $I \times (E \setminus I) \subseteq \mathsf{tso}$ (TSO-MO)
- $\mathsf{mo} \subseteq \mathsf{tso}$ (TSO-MO)
- tso is total on $E \setminus R$ (TSO-TOTAL)
- $\mathsf{rf} \subseteq \mathsf{tso} \cup \mathsf{po}$ (TSO-RF1)
- $\forall x \in \mathsf{Loc}.\ \forall (w, r) \in \mathsf{rf}_x.\ \forall w' \in W_x \cup U_x.\ (w', r) \in \mathsf{tso} \cup \mathsf{po} \Rightarrow (w, w') \notin \mathsf{tso}$ (TSO-RF2)
- $([W \cup U \cup R]; \mathsf{po}; [W \cup U \cup R]) \setminus (W \times R) \subseteq \mathsf{tso}$ (TSO-PO)
- $([E]; \mathsf{po}; [MF]) \cup ([MF]; \mathsf{po}; [E]) \subseteq \mathsf{tso}$ (TSO-MF)
- $([E \setminus R]; \mathsf{po}; [SF]) \cup ([SF]; \mathsf{po}; [E \setminus R]) \subseteq \mathsf{tso}$ (TSO-SF)
- $([W \cup U \cup FL]; \mathsf{po}; [FL]) \cup ([FL]; \mathsf{po}; [W \cup U \cup FL]) \subseteq \mathsf{tso}$ (TSO-FL-WUFL)
- $\forall X \in \mathsf{CL}.\ ([FL_X]; \mathsf{po}; [FO_X]) \cup ([FO_X]; \mathsf{po}; [FL_X]) \subseteq \mathsf{tso}$ (TSO-FL-FO)
- $([U]; \mathsf{po}; [FO]) \cup ([FO]; \mathsf{po}; [U]) \subseteq \mathsf{tso}$ (TSO-FO-U)
- $\forall X \in \mathsf{CL}.\ ([W_X]; \mathsf{po}; [FO_X]) \subseteq \mathsf{tso}$ (TSO-W-FO)

- $[R]; \mathrm{po}; [SF] \subseteq \mathrm{tso}$ (TSO-R-SF)
- $[R]; \mathrm{po}; [FO \cup FL] \subseteq \mathrm{tso}$ (TSO-SIM)
- $\forall x \in \mathrm{Loc}.\ \mathrm{tso}|_{D_x} \subseteq \mathrm{nvo}$ (NVO-LOC)
- $\forall X \in \mathrm{CL}.\ [W_X \cup U_X]; \mathrm{tso}; [FO_X \cup FL_X] \subseteq \mathrm{nvo}$ (NVO-WU-FOFL)
- $[FO \cup FL]; \mathrm{tso}; [D] \subseteq \mathrm{nvo}$ (NVO-FOFL-D)

Definition 14. A tuple $G = (E, I, P, \mathrm{po}, \mathrm{rf}, \mathrm{mo})$ is *Px86$'_{sim}$-consistent* iff there exist relations $G.\mathrm{tso_p}, G.\mathrm{nvo_p} \subseteq E \times E$ such that:

- G satisfies all conditions of an execution except those on nvo (P-EXEC)
- $G.\mathrm{tso_p}$ is a strict order on E (P-TSO-ORDER)
- $\mathrm{rf} \subseteq G.\mathrm{tso_p} \cup \mathrm{po}$ (P-TSO-RF1)
- $\mathrm{rb} \cup \mathrm{tso_p}$ is acyclic, where $\mathrm{rb} \triangleq \mathrm{rf}^{-1}; \mathrm{mo}$ (P-TSO-RF2)
- $G.\mathrm{nvo_p}$ is a strict order on $G.D$ (P-NVO-ORDER)
- $dom(G.\mathrm{nvo_p}; [P]) \subseteq P$ (P-NVO-P)

where

$$
\begin{aligned}
G.\mathrm{tso_p} \triangleq \Big(& \mathrm{mo} \cup \mathrm{rf}_e \cup (I \times (E \setminus I)) \\
& \cup ([W \cup U \cup R]; \mathrm{po}; [W \cup U \cup R]) \setminus (W \times R) && \text{(P-TSO-PO)} \\
& \cup ([E]; \mathrm{po}; [MF]) \cup ([MF]; \mathrm{po}; [E]) && \text{(P-TSO-MF)} \\
& \cup ([E \setminus R]; \mathrm{po}; [SF]) \cup ([SF]; \mathrm{po}; [E \setminus R]) && \text{(P-TSO-SF)} \\
& \cup ([W \cup U \cup FL]; \mathrm{po}; [FL]) \cup ([FL]; \mathrm{po}; [W \cup U \cup FL]) && \text{(P-TSO-FL-WUFL)} \\
& \cup \textstyle\bigcup_{X \in \mathrm{CL}} ([FL_X]; \mathrm{po}; [FO_X]) \cup ([FO_X]; \mathrm{po}; [FL_X]) && \text{(P-TSO-FL-FO)} \\
& \cup ([U]; \mathrm{po}; [FO]) \cup ([FO]; \mathrm{po}; [U]) && \text{(P-TSO-FO-U)} \\
& \cup \textstyle\bigcup_{X \in \mathrm{CL}} ([W_X]; \mathrm{po}; [FO_X])] && \text{(P-TSO-W-FO)} \\
& \cup [R]; \mathrm{po}; [SF] && \text{(P-TSO-R-SF)} \\
& \cup [R]; \mathrm{po}; [FO \cup FL] \Big)^+ && \text{(P-TSO-SIM)} \\
G.\mathrm{nvo_p} \triangleq \big(& I \times (G.D \setminus I) \\
& \cup \textstyle\bigcup_{x \in \mathrm{Loc}} \mathrm{tso_p}|_{D_x} && \text{(P-NVO-LOC)} \\
& \cup \textstyle\bigcup_{X \in \mathrm{CL}} [W_X \cup U_X]; \mathrm{tso_p}; [FO_X \cup FL_X] && \text{(P-NVO-WU-FOFL)} \\
& \cup [FO \cup FL]; \mathrm{tso_p}; [D] \big)^+ && \text{(P-NVO-FOFL-D)}
\end{aligned}
$$

Given an execution $G = (E, I, P, \mathrm{po}, \mathrm{rf}, \mathrm{mo}, \mathrm{nvo})$, we define $G_\mathrm{p} \triangleq (E, I, P, \mathrm{po}, \mathrm{rf}, \mathrm{mo})$.

Lemma 18. *Given an execution G, if G is Px86$_{sim}$-consistent, then G_p is Px86$'_{sim}$-consistent.*

Proof. Pick an arbitrary execution G such that G is Px86$_{\mathrm{sim}}$-consistent. (P-EXEC) then follows immediately. From the definitions of $G.\mathrm{tso_p}$, $G.\mathrm{nvo_p}$ and the Px86$_{\mathrm{sim}}$-consistency of G we know that $G.\mathrm{tso_p} \subseteq G.\mathrm{tso}$ and $G.\mathrm{nvo_p} \subseteq G.\mathrm{nvo}$. As such (P-TSO-ORDER) and (P-NVO-ORDER) follow immediately.

For (P-TSO-RF1), pick an arbitrary $(a, b) \in G.\mathrm{rf}$. We then know that either 1) $(a, b) \in G.\mathrm{rf}_e$ or 2) $(a, b) \in G.\mathrm{rf}_i$. In case (1) since $G.\mathrm{rf}_e \subseteq G.\mathrm{tso_p}$ we have $(a, b) \in G.\mathrm{tso_p} \subseteq G.\mathrm{tso_p} \cup G.\mathrm{po}$ as required.

In case (2) we then know that either i) $(a, b) \in G.\mathrm{po}$ or ii) $(b, a) \in G.\mathrm{po}$. In case (2.i) we then simply have $(a, b) \in G.\mathrm{po} \subseteq G.\mathrm{tso_p} \cup G.\mathrm{po}$, as required. In case (2.ii) since G is Px86$_{\mathrm{sim}}$-consistent and thus $[G.R]; G.\mathrm{po}; [G.E] \subseteq G.\mathrm{tso}$, and since $b \in R$ we have $(b, a) \in G.\mathrm{tso}$. On the other hand, since G is Px86$_{\mathrm{sim}}$-consistent and thus $(a, b) \in G.\mathrm{rf} \subseteq G.\mathrm{tso} \cup G.\mathrm{po}$, and as $(b, a) \in G.\mathrm{po}$ and so $(a, b) \notin G.po$, we have $(a, b) \in G.tso$. That is, we have $(a, b) \in G.tso$ and $(b, a) \in G.tso$ and thus $(a, a) \in G.tso$, leading to a contradiction as $G.\mathrm{tso}$ is a strict order.

We next demonstrate that for all w, r, if $(w, r) \in G.\mathsf{tso} \cap (WU \times RU)$ and $\mathtt{loc}(w) = \mathtt{loc}(r)$, then $(w, r) \in \mathsf{mo}^?; \mathsf{rf}$. To do this, pick arbitrary w, r such that $(w, r) \in G.\mathsf{tso} \cap (WU \times RU)$ and $\mathtt{loc}(w) = \mathtt{loc}(r)$. We then know there exists w' such that $(w', r) \in \mathsf{rf}$, and either 1) $w = w'$ or 2) $(w, w') \in \mathsf{mo}$; or 3) $(w', w) \in \mathsf{mo}$. In cases (1), (2) we then have $w \xrightarrow{\mathsf{mo}^?} w' \xrightarrow{\mathsf{rf}}$ as required. In case (3) from (TSO-MO) we then have $(w', w) \in G.\mathsf{tso}$. On the other hand, since $(w, r) \in G.\mathsf{tso} \cap (W \times R)$, $\mathtt{loc}(w) = \mathtt{loc}(r)$ and $(w', r) \in \mathsf{rf}$, from (TSO-RF2) we have $(w', w) \notin G.\mathsf{tso}$, leading to a contradiction as we also established $(w', w) \in G.\mathsf{tso}$.

To establish (P-TSO-RF2), we proceed by contradiction and that there is a cycle in $G.\mathsf{tso}_\mathsf{p} \cup \mathsf{rb}$. Given the definition of $G.\mathsf{tso}_\mathsf{p}$ and rb we then know there exist w, r such that $w \xrightarrow{G.\mathsf{tso}_\mathsf{p}} r \xrightarrow{\mathsf{rb}} w$, $w \in WU$, $r \in RU$ and $\mathtt{loc}(w) = \mathtt{loc}(r)$. From the definition of $G.\mathsf{tso}_\mathsf{p}$ we then know that $(w, r) \in G.\mathsf{tso}$ and thus from the definition above we have $(w, r) \in \mathsf{mo}^?; \mathsf{rf}$. As such, from the definition of rb we have $w \xrightarrow{\mathsf{mo}^?;\mathsf{rf}} r \xrightarrow{\mathsf{rf}^{-1};\mathsf{mo}} w$. That is we have $w \xrightarrow{\mathsf{mo}^?;\mathsf{rf};\mathsf{rf}^{-1};\mathsf{mo}} w$, i.e. $w \xrightarrow{\mathsf{mo}^?;\mathsf{mo}} w$ and thus $(w, w) \in \mathsf{mo}$, leading to a contradiction.

For (P-NVO-P), pick an arbitrary a, b such that $(a, b) \in G.\mathsf{nvo}_\mathsf{p}$ and $b \in G.P$. As $G.\mathsf{nvo}_\mathsf{p} \subseteq G.\mathsf{nvo}$, we then have $(a, b) \in G.\mathsf{nvo}$, and since G is an execution, we have $b \in G.P$, as required. □

Lemma 19. *Given a tuple $G = (E, I, P, \mathsf{po}, \mathsf{rf}, \mathsf{mo})$, if G is $Px86^{'}_{sim}$-consistent, then there exists nvo_t such that $G' = (E, I, P, \mathsf{po}, \mathsf{rf}, \mathsf{mo}, \mathsf{nvo}_\mathsf{t})$ is $Px86_{sim}$-consistent.*

Proof. Pick an arbitrary tuple $G = (E, I, P, \mathsf{po}, \mathsf{rf}, \mathsf{mo})$ such that G is $\mathrm{Px86}^{'}_{\mathrm{sim}}$-consistent. Let $\mathsf{tso}_1 \triangleq (G.\mathsf{tso}_\mathsf{p} \cup \mathsf{rb})^+$. Note that from the $\mathrm{Px86}^{'}_{\mathrm{sim}}$-consistency of G (property (P-TSO-RF2)) we know that tso_1 is acyclic. Let tso_t denote an arbitrary extension of tso_1 to a strict total order. Note that we have $G.\mathsf{tso}_\mathsf{p} \subseteq \mathsf{tso}_\mathsf{t}$.

Note that since G is $\mathrm{Px86}^{'}_{\mathrm{sim}}$-consistent, $dom(G.\mathsf{nvo}_\mathsf{p}; [P]) \subseteq P$ and thus $[E \setminus P]; G.\mathsf{nvo}_\mathsf{p}; [P] = \emptyset$. Consequently, given $\mathsf{nvo}_1 \triangleq \left(G.\mathsf{nvo}_\mathsf{p} \cup \left(P \times (G.D \setminus P)\right)\right)^+$, we know $dom(\mathsf{nvo}_1; [P]) \subseteq P$ and that nvo_1 is a strict order on D. Let nvo_t denote an arbitrary extension of nvo_1 to a strict total order on $G.D$. We then have $G.\mathsf{nvo}_\mathsf{p} \subseteq \mathsf{nvo}_\mathsf{t}$. Consequently, from (P-EXEC) we know that $G' = (E, I, P, \mathsf{po}, \mathsf{rf}, \mathsf{mo}, \mathsf{nvo}_\mathsf{t})$ is an execution.

It then suffices to show properties (TSO-MO)–(NVO-FOFL-D) for tso_t and G'. (TSO-MO) follows from the definition of $G.\mathsf{tso}_\mathsf{p}$ and that $G.\mathsf{tso}_\mathsf{p} \subseteq \mathsf{tso}_\mathsf{t}$. (TSO-TOTAL) follows from the construction of tso_t. (TSO-RF1) follows from (P-TSO-RF1) and that $G.\mathsf{tso}_\mathsf{p} \subseteq \mathsf{tso}_\mathsf{t}$. (TSO-RF2) follows from **??**. (TSO-PO)–(TSO-SIM) follow from corresponding tso_p properties and that $G.\mathsf{tso}_\mathsf{p} \subseteq \mathsf{tso}_\mathsf{t}$. (NVO-LOC)–(NVO-FOFL-D) follow from corresponding nvo_p properties and that $G.\mathsf{nvo}_\mathsf{p} \subseteq \mathsf{nvo}_\mathsf{t}$. □

Given a $\mathrm{Px86}^{'}_{\mathrm{sim}}$-consistent tuple $G = (E, I, P, \mathsf{po}, \mathsf{rf}, \mathsf{mo})$, we define the set of $\mathrm{Px86}_{\mathrm{sim}}$-consistent executions:

$$G_\mathsf{T} \triangleq \left\{(E, I, P, \mathsf{po}, \mathsf{rf}, \mathsf{mo}, \mathsf{nvo}_\mathsf{t}) \;\middle|\; \mathsf{nvo}_\mathsf{t} \text{ is an extension of } \mathsf{nvo}_1 \text{ to a strict total order}\right\}$$

with nvo_1 as constructed in the lemma above.

Lemma 20. *For all $Px86_{sim}$-consistent executions G: $G \in (G_\mathsf{p})_\mathsf{T}$.*

Proof. Follows immediately from the definitions of G_p and $(.)_\mathsf{T}$. □

D SOUNDNESS OF OUR TWO-STEP TRANSFORMATION

Definition 15 (Persistency Equivalence)**.** Two executions G and G' are *persistency-equivalent*, written $G \equiv_{\mathsf{p}} G'$, iff:

$$G \equiv_{\mathsf{p}} G' \overset{\text{def}}{\Leftrightarrow} \forall x.\ \mathsf{max}_{\mathsf{p}}(G, x) = \mathsf{max}_{\mathsf{p}}(G', x)$$
$$\text{where} \qquad \mathsf{max}_{\mathsf{p}}(G, x) \triangleq \max\left(G.\mathsf{nvo}|_{G.P \cap G.WU_x}\right)$$

Two programs C and C' are *persistency-equivalent*, written $\mathsf{C} \equiv_{\mathsf{p}} \mathsf{C}'$, iff:

$$\mathsf{C} \equiv_{\mathsf{p}} \mathsf{C}' \overset{\text{def}}{\Leftrightarrow} \forall G \in \{\!|\mathsf{C}|\!\}.\ \exists G' \in \{\!|\mathsf{C}'|\!\}.\ G \equiv_{\mathsf{p}} G' \wedge \forall G' \in \{\!|\mathsf{C}'|\!\}.\ \exists G \in \{\!|\mathsf{C}|\!\}.\ G \equiv_{\mathsf{p}} G'$$

where $\{\!|\mathsf{C}|\!\}$ denotes the set of all Px86$_{\text{sim}}$-consistent executions of C.

Definition 16 (Partial Persistency Equivalence)**.** Two tuples G and G' are *partial-persistency-equivalent*, written $G \equiv_{\mathsf{pp}} G'$, iff:

$$G \equiv_{\mathsf{pp}} G' \overset{\text{def}}{\Leftrightarrow} \forall x.\ \mathsf{max}_{\mathsf{pp}}(G, x) = \mathsf{max}_{\mathsf{pp}}(G', x)$$
$$\text{where} \qquad \mathsf{max}_{\mathsf{pp}}(G, x) \triangleq \max\left(G.\mathsf{nvo}_{\mathsf{p}}|_{G.P \cap G.WU_x}\right)$$

Two programs C and C' are *partial-persistency-equivalent*, written $\mathsf{C} \equiv_{\mathsf{pp}} \mathsf{C}'$, iff:

$$\mathsf{C} \equiv_{\mathsf{pp}} \mathsf{C}' \overset{\text{def}}{\Leftrightarrow} \forall G \in \{\!|\mathsf{C}|\!\}_{\mathsf{pp}}.\ \exists G' \in \{\!|\mathsf{C}'|\!\}_{\mathsf{pp}}.\ G \equiv_{\mathsf{pp}} G' \wedge \forall G' \in \{\!|\mathsf{C}'|\!\}_{\mathsf{pp}}.\ \exists G \in \{\!|\mathsf{C}|\!\}_{\mathsf{pp}}.\ G \equiv_{\mathsf{pp}} G'$$

where $\{\!|\mathsf{C}|\!\}_{\mathsf{pp}} \triangleq \left\{G_{\mathsf{p}} \mid G \in \{\!|\mathsf{C}|\!\}\right\}$ denotes the set of all Px86$_{\text{sim}}^{'}$-consistent tuples of C.

Note that given a Px86$_{\text{sim}}^{'}$-consistent tuple G, since 1) $G.\mathsf{mo}$ is total and $G.\mathsf{mo} \subseteq G.\mathsf{tso}_{\mathsf{p}}$, 2) $G.\mathsf{tso}_{\mathsf{p}}|_{D_x} \subseteq G.\mathsf{nvo}_{\mathsf{p}}$ for each location x, and 3) $G.I \subseteq G.P$, then $\max\left(G.\mathsf{nvo}_{\mathsf{p}}|_{G.P \cap G.WU_x}\right)$ is always (uniquely) defined.

Lemma 21. *For all Px86$_{sim}$-consistent executions G, G', if $G \equiv_{\mathsf{p}} G'$, then $G_{\mathsf{p}} \equiv_{\mathsf{pp}} G'_{\mathsf{p}}$.*

Proof. Pick arbitrary Px86$_{\text{sim}}$-consistent executions G, G'. From the definitions of G_{p} and G'_{p}, and since:

1) $G.\mathsf{mo}$ and $G'.\mathsf{mo}$ are total on WU and $G.\mathsf{mo} \subseteq G.\mathsf{tso}$ and $G'.\mathsf{mo} \subseteq G'.\mathsf{tso}$,
2) $G.\mathsf{mo}$=$G_{\mathsf{p}}.\mathsf{mo}$, $G'.\mathsf{mo}$=$G'_{\mathsf{p}}.\mathsf{mo}$, $G_{\mathsf{p}}.\mathsf{mo} \subseteq G_{\mathsf{p}}.\mathsf{tso}_{\mathsf{p}}$, $G'_{\mathsf{p}}.\mathsf{mo} \subseteq G'_{\mathsf{p}}.\mathsf{tso}_{\mathsf{p}}$, and thus
3) $G.\mathsf{tso}|_{WU_x} = G_{\mathsf{p}}.\mathsf{tso}_{\mathsf{p}}|_{WU_x}$ and $G'.\mathsf{tso}|_{WU_x} = G'_{\mathsf{p}}.\mathsf{tso}_{\mathsf{p}}|_{WU_x}$ for each location x,
4) $G.\mathsf{tso}|_{WU_x} \subseteq G.\mathsf{nvo}$ and $G'.\mathsf{tso}|_{WU_x} \subseteq G'.\mathsf{nvo}$ for each location x,
5) $G_{\mathsf{p}}.\mathsf{tso}_{\mathsf{p}}|_{WU_x} \subseteq G_{\mathsf{p}}.\mathsf{nvo}_{\mathsf{p}}$ and $G'_{\mathsf{p}}.\mathsf{tso}_{\mathsf{p}}|_{WU_x} \subseteq G'_{\mathsf{p}}.\mathsf{nvo}_{\mathsf{p}}$ for each location x,
6) $G.P = G_{\mathsf{p}}.P$ and $G'.P = G'_{\mathsf{p}}.P$

we then know $G.\mathsf{nvo}|_{G.P \cap WU_x} = G_{\mathsf{p}}.\mathsf{nvo}_{\mathsf{p}}|_{G_{\mathsf{p}}.P \cap WU_x}$ and $G'.\mathsf{nvo}|_{G'.P \cap WU_x} = G'_{\mathsf{p}}.\mathsf{nvo}_{\mathsf{p}}|_{G'_{\mathsf{p}}.P \cap WU_x}$. Consequently, since $G \equiv_{\mathsf{p}} G'$, from the definition of $\equiv_{\mathsf{pp}}$ we have $G_{\mathsf{p}} \equiv_{\mathsf{pp}} G'_{\mathsf{p}}$, as required. □

Lemma 22. *For all Px86$_{sim}^{'}$-consistent tuples G, G', if $G \equiv_{\mathsf{pp}} G'$, then: $\forall G_{\mathsf{t}} \in G_{\mathsf{T}}, G'_{\mathsf{t}} \in G'_{\mathsf{T}}.\ G_{\mathsf{t}} \equiv_{\mathsf{pp}} G'_{\mathsf{t}}$.*

The proof of this lemma is analogous to that of previous lemma and is thus omitted here.

Lemma 23. *For all C, C': $\mathsf{C} \equiv_{\mathsf{p}} \mathsf{C}' \overset{def}{\Leftrightarrow} \mathsf{C} \equiv_{\mathsf{pp}} \mathsf{C}'$*

Proof. Follows from Lemma 20, Lemma 21, Lemma 22 and the definitions of $\equiv_{\mathsf{p}}$ and $\equiv_{\mathsf{pp}}$. □

Proposition 3. *For all Px86$_{sim}^{'}$-consistent tuples G: $G.(\mathsf{tso}_{\mathsf{p}})_e \subseteq (G.\mathsf{po} \cap G.\mathsf{tso}_{\mathsf{p}})^?; ((G.\mathsf{mo}_e \cup \mathsf{rf}_e); (G.\mathsf{po} \cap G.\mathsf{tso}_{\mathsf{p}})^?)^+$ and $G.(\mathsf{tso}_{\mathsf{p}})_i \subseteq G.\mathsf{po}$.*

In what follows we write RU for $R \cup U$.

D.1 Soundness of Transformation Step 1 ($\mathsf{flush}_{\mathsf{opt}}$ Reordering)

Lemma 24. *For all $x, y \in Loc$, $X \in CL$, $G{=}(E, I, P, \mathrm{po}, \mathrm{rf}, \mathrm{mo})$, $fo \in E \cap FO_x$ and $e \in FO \cup FL \cup W_y$, if G is $Px86^{'}_{sim}$-consistent, $x \in X$, $y \notin X$ and $(fo, e) \in \mathrm{po}|_{imm}$, then there exists G' such that G' is $Px86^{'}_{sim}$-consistent, $G \equiv_{\mathsf{pp}} G'$ and $G'.\mathrm{po} = \mathrm{po}'$ with $\mathrm{po}' = (G.\mathrm{po} \setminus \{(fo, e)\}) \cup \{(e, fo)\}$.*

Proof. Pick arbitrary $x, y \in \mathrm{Loc}$, $X \in \mathrm{CL}$, $G = (E, I, P, \mathrm{po}, \mathrm{rf}, \mathrm{mo})$, $fo \in E \cap FO_x$ and $e \in R \cup FO \cup FL \cup W_y$ such that G is $\mathrm{Px86}^{'}_{\mathrm{sim}}$-consistent, $x \in X$, $y \notin X$ and $(fo, e) \in \mathrm{po}|_{\mathrm{imm}}$. There are now three cases to consider:

Case $e \in FO$

Let $G' = (E, I, P, \mathrm{po}', \mathrm{rf}, \mathrm{mo})$. Note that since $fo, e \in FO$ and $(fo, e) \in \mathrm{po}|_{\mathrm{imm}}$, given the definitions of G, G' and $\mathrm{tso}_{\mathsf{p}}$ we know that $G.\mathrm{tso}_{\mathsf{p}}{=}G'.\mathrm{tso}_{\mathsf{p}}$ and that $(fo, e), (e, fo) \notin G.\mathrm{tso}_{\mathsf{p}}$ and $(fo, e), (e, fo) \notin G'.\mathrm{tso}_{\mathsf{p}}$. Consequently since G is $\mathrm{Px86}^{'}_{\mathrm{sim}}$-consistent, by definition we know that G' is also $\mathrm{Px86}^{'}_{\mathrm{sim}}$-consistent. Moreover, from the definitions of G, G' and $\equiv_{\mathsf{pp}}$ we know $G \equiv_{\mathsf{pp}} G'$, as required.

Case $e \in FL$

There are now to cases consider: 1) $\mathtt{loc}(e) \notin X$; or 2) $\mathtt{loc}(e) \in X$. The proof of case (1) is analogous to that of previous case (when $e \in FO$) and is omitted here.

In case (2), let $G' = (E, I, P', \mathrm{po}', \mathrm{rf}, \mathrm{mo})$, where

$$P' \triangleq \begin{cases} P & \text{if } (fo \in P \wedge e \in P) \vee (fo \notin P \wedge e \notin P) \\ P \setminus \{fo\} & \text{if } fo \in P \wedge e \notin P \end{cases}$$

Note that as $(fo, e) \in G.\mathrm{po}$ and, $fo \in FO_X$, $e \in FL_X$ and G is $\mathrm{Px86}^{'}_{\mathrm{sim}}$-consistent, we know that $(fo, e) \in G.\mathrm{tso}_{\mathsf{p}}$ and thus $(fo, e) \in G.\mathrm{nvo}_{\mathsf{p}}$. As such, the case where $fo \notin P \wedge e \in P$ does not arise (since $dom(G.\mathrm{nvo}_{\mathsf{p}}; [P]) \subseteq G.\mathrm{nvo}_{\mathsf{p}}$).

First we establish that $(fo, e) \in (G.\mathrm{tso}_{\mathsf{p}})|_{\mathrm{imm}}$. We proceed by contradiction. Let us assume that $(fo, e) \notin (G.\mathrm{tso}_{\mathsf{p}})|_{\mathrm{imm}}$. Since form the definition of $G.\mathrm{tso}_{\mathsf{p}}$ we know $(fo, e) \in G.\mathrm{tso}_{\mathsf{p}})$, from Prop. 3 we then know that there exists $w_1, w_2 \in WU$, $r \in R \cup WU$ and a such that $fo \xrightarrow{\mathrm{po} \cap G.\mathrm{tso}_{\mathsf{p}}} w_1 \xrightarrow{\mathrm{mo}_e \cup \mathrm{rf}_e} a \xrightarrow{G.\mathrm{tso}_{\mathsf{p}}^{?}} w_2 \xrightarrow{\mathrm{mo}_e \cup \mathrm{rf}_e} r \xrightarrow{\mathrm{po} \cap G.\mathrm{tso}_{\mathsf{p}}} e$. However, as $(fo, e) \in \mathrm{po}|_{\mathrm{imm}}$, we also have $r \xrightarrow{\mathrm{po}} fo$. Moreover, since $r \in R \cup WU$, we then have $r \xrightarrow{G.\mathrm{tso}_{\mathsf{p}}} w_1$. Consequently we have $r \xrightarrow{G.\mathrm{tso}_{\mathsf{p}}} w_1 \xrightarrow{\mathrm{mo}_e \cup \mathrm{rf}_e} a \xrightarrow{G.\mathrm{tso}_{\mathsf{p}}^{?}} w_2 \xrightarrow{\mathrm{mo}_e \cup \mathrm{rf}_e} r$. That is, we have $(r, r) \in \mathrm{tso}_{\mathsf{p}}$, leading to a contradiction.

As $(fo, e) \in (G.\mathrm{tso}_{\mathsf{p}})|_{\mathrm{imm}}$, from the definition of $G.\mathrm{nvo}_{\mathsf{p}}$ we also have $(fo, e) \in (G.\mathrm{nvo}_{\mathsf{p}})|_{\mathrm{imm}}$.

Next note that it is straightforward to demonstrate that $G'.\mathrm{tso}_{\mathsf{p}} = \big((G.\mathrm{tso}_{\mathsf{p}} \setminus \{(fo, e)\}) \cup \{(e, fo)\}\big)^{+}$. Consequently, from the definition of $G'.\mathrm{nvo}_{\mathsf{p}}$ and $G.\mathrm{nvo}_{\mathsf{p}}$ we also have $G'.\mathrm{nvo}_{\mathsf{p}} = \big((G.\mathrm{nvo}_{\mathsf{p}} \setminus \{(fo, e)\}) \cup \{(e, fo)\}\big)^{+}$.

We next demonstrate that G' is $\mathrm{Px86}^{'}_{\mathrm{sim}}$-consistent. (P-EXEC), (??) and (P-TSO-RF1) follow immediately from the definition of G'. For (P-TSO-ORDER) we proceed by contradiction. Let us assume that there exists a such that $(a, a) \in G'.\mathrm{tso}_{\mathsf{p}}$. As $G.\mathrm{tso}_{\mathsf{p}}$ is acyclic, we then know that $a \xrightarrow{G.\mathrm{tso}_{\mathsf{p}}} e \xrightarrow{\mathrm{po}' \cap G'.\mathrm{tso}_{\mathsf{p}}} fo \xrightarrow{G.\mathrm{tso}_{\mathsf{p}}} a$. From Prop. 3 we then know that either 1) $\mathtt{tid}(e) = \mathtt{tid}(a)$ and $a \xrightarrow{\mathrm{po}} e \xrightarrow{\mathrm{po}' \cap G'.\mathrm{tso}_{\mathsf{p}}} fo \xrightarrow{G.\mathrm{po}} a$; or 2) $\mathtt{tid}(a) \neq \mathtt{tid}(e)$ and $a \xrightarrow{(G'.\mathrm{tso}_{\mathsf{p}})_e} e \xrightarrow{\mathrm{po}' \cap G'.\mathrm{tso}_{\mathsf{p}}} fo \xrightarrow{(G'.\mathrm{tso}_{\mathsf{p}})_e} a$. In case (1) as $(fo, e) \in \mathrm{po}|_{\mathrm{imm}}$, we also have $(a, fo) \in \mathrm{po}$ and thus we have $a \xrightarrow{\mathrm{po}} fo \xrightarrow{G.\mathrm{po}} a$, i.e. $(a, a) \in \mathrm{po}$, leading to a contradiction as G is $\mathrm{Px86}^{'}_{\mathrm{sim}}$-consistent. In case (2) from Prop. 3 we know

there exist $w_1, w_2 \in WU$, $r \in R \cup WU$ and b such that: $a \xrightarrow{G.\mathsf{tso}_\mathsf{p}^?} w_1 \xrightarrow{G.\mathsf{mo}_e \cup \mathsf{rf}_e} r \xrightarrow{(G.\mathsf{tso}_\mathsf{p} \cap G.\mathsf{po})} e \xrightarrow{G'.\mathsf{po}} fo \xrightarrow{G.\mathsf{po} \cap G.\mathsf{tso}_\mathsf{p}} w_2 \xrightarrow{G.\mathsf{mo}_e \cup \mathsf{rf}_e} b \xrightarrow{G.\mathsf{tso}_\mathsf{p}} a$. As $(fo, e) \in \mathsf{po}|_{\mathsf{imm}}$ and $(r, e) \in \mathsf{po}$, we also have $(r, fo) \in \mathsf{po}$. Consequently, given the definition of $G.\mathsf{tso}_\mathsf{p}$ and since $r \in R \cup WU$ and $w_2 \in WU$, we have $(r, w_2) \in G.\mathsf{tso}_\mathsf{p}$. That is, we have $a \xrightarrow{G.\mathsf{tso}_\mathsf{p}} r \xrightarrow{G.\mathsf{tso}_\mathsf{p}} w_2 \xrightarrow{G.\mathsf{tso}_\mathsf{p}} a$, i.e. $(a, a) \in G.\mathsf{tso}_\mathsf{p}$, leading to a contradiction.

For (P-TSO-RF2) we proceed by contradiction. Let us assume there exist w, w', r such that $(w, w') \in G'.\mathsf{tso}_\mathsf{p}$, $(w, r) \in G'.\mathsf{rf}$ and $(w', r) \in G'.\mathsf{tso}_\mathsf{p} \cup G.\mathsf{po}$. From the definition of G' we then have $(w, r) \in G.\mathsf{rf}$ and that $(w, w') \in G'.\mathsf{mo}{=}G.\mathsf{mo} \subseteq G.\mathsf{tso}_\mathsf{p}$ and thus $(w, w') \in G.\mathsf{tso}_\mathsf{p}$. As G is $\mathrm{Px86}^{'}_{\mathrm{sim}}$-consistent, we then know that $(w', r) \notin G.\mathsf{po} \cup G.\mathsf{tso}_\mathsf{p}$. Consequently, since $(w', r) \in G'.\mathsf{po} \cup G'.\mathsf{tso}_\mathsf{p}$, given the definition of $G'.\mathsf{tso}_\mathsf{p}$ and $G'.\mathsf{po}$ we know that $w' \xrightarrow{G.\mathsf{tso}_\mathsf{p}} e \xrightarrow{G'.\mathsf{tso}_\mathsf{p} \cap G'.\mathsf{po}} fo \xrightarrow{G.\mathsf{tso}_\mathsf{p}} r$. It is then straightforward to show that since $(fo, e) \in G.\mathsf{po}|_{\mathsf{imm}}$, $fo \xrightarrow{G.\mathsf{tso}_\mathsf{p}} r$, $fo \in FO$ and $e \in FL$, from the definition of $G.\mathsf{tso}_\mathsf{p}$ we also have $(e, r) \in G.\mathsf{tso}_\mathsf{p}$. As such we have $w' \xrightarrow{G.\mathsf{tso}_\mathsf{p}} e \xrightarrow{G.\mathsf{tso}_\mathsf{p}} r$. That is, $(w', r) \in G.\mathsf{tso}_\mathsf{p}$. This however leads to a contradiction as we established above that $(w', r) \notin G.\mathsf{po} \cup G.\mathsf{tso}_\mathsf{p}$.

For (P-NVO-ORDER) note that by definition $G'.\mathsf{nvo}_\mathsf{p} \subseteq G'.\mathsf{tso}_\mathsf{p}$. As such, since we established that $G'.\mathsf{tso}_\mathsf{p}$ is a strict order, and since $G'.\mathsf{nvo}_\mathsf{p}$ is by definition transitive, we also know that $G'.\mathsf{nvo}_\mathsf{p}$ is a strict order as required.

For (P-NVO-P) we proceed by contradiction. Let us assume there exist a, b such that $a \notin P'$, $b \in P'$ and $(a, b) \in G'.\mathsf{nvo}_\mathsf{p}$. There are now three cases to consider: 1) $fo \in P \wedge e \in P$; 2) $fo \notin P \wedge e \notin P$ 3) $fo \in P \wedge e \notin P$.

In case (1), we then have $P' = P$ and thus since $a \notin P'$ and $b \in P'$, we also have $a \notin P$ and $b \in P$. Moreover, as G is consistent and $G'.\mathsf{nvo}_\mathsf{p} = \left((G.\mathsf{nvo}_\mathsf{p} \setminus \{(fo, e)\}) \cup \{(e, fo)\}\right)^+$, we then know that $a \xrightarrow{G.\mathsf{nvo}_\mathsf{p}} e \xrightarrow{G'.\mathsf{nvo}_\mathsf{p}} fo \xrightarrow{G.\mathsf{nvo}_\mathsf{p}^?} b$. On the other hand, as G is consistent and thus $dom(G.\mathsf{nvo}_\mathsf{p}; [P]) \subseteq G.\mathsf{nvo}_\mathsf{p}$, and $e \in P$, we also have $a \in P$, leading to a contradiction as we also established that $a \notin P$.

Similarly in case (2), we have $P' = P$ and thus since $a \notin P'$ and $b \in P'$, we also have $a \notin P$ and $b \in P$. Moreover, as G is consistent and $G'.\mathsf{nvo}_\mathsf{p} = \left((G.\mathsf{nvo}_\mathsf{p} \setminus \{(fo, e)\}) \cup \{(e, fo)\}\right)^+$, we then know that $a \xrightarrow{G.\mathsf{nvo}_\mathsf{p}^?} e \xrightarrow{G'.\mathsf{nvo}_\mathsf{p}} fo \xrightarrow{G.\mathsf{nvo}_\mathsf{p}} b$. On the other hand, as G is consistent and thus $dom(G.\mathsf{nvo}_\mathsf{p}; [P]) \subseteq G.\mathsf{nvo}_\mathsf{p}$, and $b \in P$, we also have $fo \in P$, contradicting the assumption of case (2).

In case (3), we have $P' = P \setminus \{fo\}$ and thus since $b \in P'$, we also have $b \in P$. Moreover, as G is consistent and $G'.\mathsf{nvo}_\mathsf{p} = \left((G.\mathsf{nvo}_\mathsf{p} \setminus \{(fo, e)\}) \cup \{(e, fo)\}\right)^+$, we then know that $a \xrightarrow{G.\mathsf{nvo}_\mathsf{p}^?} e \xrightarrow{G'.\mathsf{nvo}_\mathsf{p}} fo \xrightarrow{G.\mathsf{nvo}_\mathsf{p}^?} b$. From the definition of $G.\mathsf{nvo}_\mathsf{p}$ we know that $G.\mathsf{nvo}_\mathsf{p}|_{E \setminus I} \subseteq G.\mathsf{tso}_\mathsf{p}$ and thus $fo \xrightarrow{G.\mathsf{tso}_\mathsf{p}^?} b$. Moreover, as $(fo, e) \in \mathsf{po}|_{\mathsf{imm}}$ and $fo \xrightarrow{G.\mathsf{tso}_\mathsf{p}^?} b$, from the definition of $G.\mathsf{tso}_\mathsf{p}$ it is straightforward to demonstrate that we also have $e \xrightarrow{G.\mathsf{tso}_\mathsf{p}} b$, and thus since $e \in FL$, we also have $e \xrightarrow{G.\mathsf{nvo}_\mathsf{p}} b$. Consequently, since $b \in P$ and from the $\mathrm{Px86}^{'}_{\mathrm{sim}}$-consistency of G we have $dom(G.\mathsf{nvo}_\mathsf{p}; [P]) \subseteq G.\mathsf{nvo}_\mathsf{p}$, we also have $e \in P$, contradicting the assumption of case (3).

Finally, we demonstrate that $G \equiv_{\mathsf{pp}} G'$. Pick an arbitrary location x. We are then required to show that $\mathsf{max}_{\mathsf{pp}}(G, x) = \mathsf{max}_{\mathsf{pp}}(G', x)$. Let us proceed by contradiction and assume that $\mathsf{max}_{\mathsf{pp}}(G, x) \neq \mathsf{max}_{\mathsf{pp}}(G', x)$. That is, there exist $w, w' \in E \cap WU_x$ such that $w \neq w'$, $w \in P$,

$w' \in P'$ (and thus from the definition of P') $w' \in P$, and $w = \max\left(G.\mathsf{nvo_p}|_{P\cap WU_x}\right)$ and $w' = \max\left(G'.\mathsf{nvo_p}|_{P\cap WU_x}\right)$. Moreover, as established above after the definition of $\equiv_{\mathsf{pp}}$, $\mathsf{nvo_p}$ is total on the writes of each location and thus we have $w' \xrightarrow{G.\mathsf{nvo_p}} w$ and $w \xrightarrow{G'.\mathsf{nvo_p}} w'$. On the other hand, since $G'.\mathsf{nvo_p} = \left((G.\mathsf{nvo_p} \setminus \{(fo,e)\}) \cup \{(e,fo)\}\right)^+$, we then have $w \xrightarrow{G.\mathsf{nvo_p}} e \xrightarrow{G'.\mathsf{nvo_p}} fo \xrightarrow{G.\mathsf{nvo_p}} w'$. From the definition of $G.\mathsf{nvo_p}$ we know that $G.\mathsf{nvo_p} \subseteq G.\mathsf{tso_p}$ and thus $fo \xrightarrow{G.\mathsf{tso_p}} w'$. Furthermore, as $(fo,e) \in \mathrm{po}|_{\mathrm{imm}}$ and $fo \xrightarrow{G.\mathsf{tso_p}} w'$, from the definition of $G.\mathsf{tso_p}$ it is straightforward to demonstrate that we also have $e \xrightarrow{G.\mathsf{tso_p}} w'$, and thus since $e \in FL$, we also have $e \xrightarrow{G.\mathsf{nvo_p}} w'$. That is, we have $w \xrightarrow{G.\mathsf{nvo_p}} e \xrightarrow{G.\mathsf{nvo_p}} w'$, i.e. $(w,w') \in G.\mathsf{nvo_p}$. Consequently, as $w' \in P$, we have $w \neq \max\left(G.\mathsf{nvo_p}|_{P\cap WU_x}\right)$, leading to a contradiction.

Case $e \in W_y$
The proof of this case is analogous to that of the first case (when $e \in FO$) and is omitted here. □

D.2 Soundness of Transformation Step 2 (Replacing $\mathsf{flush_{opt}}$ with flush)

Definition 17 (Blind persists)**.** Given G=$(E, I, P, \mathrm{po}, \mathsf{rf}, \mathsf{mo})$ and $e \in E \cap (FO \cup FL)$, e is a *blind persist* in G iff $[D_X]; G.\mathsf{tso_p}; \mathsf{mo}; \mathrm{po}; [\{e\}] \not\subseteq G.\mathsf{tso_p}$; otherwise e is *non-blind*.

Lemma 25. *For all* G=$(E, I, P, \mathrm{po}, \mathsf{rf}, \mathsf{mo})$, $X \in CL$, $x \in X$ *and* $fo \in E \cap FO_x$, *if* G *is* $Px86^{'}_{sim}$*-consistent, and* fo *is non-blind in* G $(rng([\{fo\}]; \mathrm{po}|_{imm}) \subseteq MF \cup SF \cup U)$, *then there exists* $fl \in FL_x$ *such that:* $\mathtt{tid}(fl) = \mathtt{tid}(fo)$, $\mathtt{id}(fl) = \mathtt{id}(fo)$, $G' = (E', I, P', \mathrm{po}', \mathsf{rf}, \mathsf{mo})$ *is* $Px86^{'}_{sim}$*-consistent and* $G \equiv_{\mathsf{pp}} G'$, *where* $E' = (E \setminus \{fo\}) \cup \{fl\}$, $\mathrm{po}' = \mathrm{po}|_{E\setminus\{fo\}} \cup \left\{(a,fl),(fl,b) \;\middle|\; (a,fo),(fo,b) \in \mathrm{po}\right\}$ *and*

$$P' = \begin{cases} (P \setminus \{fo\}) \cup \{fl\} & \text{if } fo \in P \\ P & \text{otherwise} \end{cases}$$

Proof. Pick arbitrary G=$(E, I, P, \mathrm{po}, \mathsf{rf}, \mathsf{mo})$, $X \in \mathrm{CL}$, $x \in X$ and $fo \in E \cap FO_x$ such that G is $\mathrm{Px86}^{'}_{\mathrm{sim}}$-consistent, $rng([\{fo\}]; \mathrm{po}|_{\mathrm{imm}}) \subseteq MF \cup SF \cup U$ and $[D_X]; G.\mathsf{tso_p}; \mathsf{mo}; \mathrm{po}; [\{fo\}] \subseteq G.\mathsf{tso_p}$. Note that it is straightforward to demonstrate that:

$$G'.\mathsf{tso_p} = \left(G.\mathsf{tso_p}|_{E\setminus\{fo\}} \cup \left\{(a,fl),(fl,b) \;\middle|\; (a,fo),(fo,b) \in G.\mathsf{tso_p}\right\} \cup S\right)^+$$

where $S = \left\{(e,fl),(f,fl) \;\middle|\; (e,fl),(f,fl) \in \mathrm{po} \wedge e \in FL \cup W \wedge \mathtt{loc}(e) \notin X \wedge f \in FO_X\right\}$.

Note that since $rng([\{fo\}]; \mathrm{po}|_{\mathrm{imm}}) \subseteq MF \cup SF \cup U$, for all a, b such that $a \xrightarrow{S\subseteq \mathrm{po}} fo \xrightarrow{G.\mathsf{tso_p}} b$, we also have $a \xrightarrow{G.\mathsf{tso_p}} b$. As such, from the definition of $G'.\mathsf{nvo_p}$ and $G.\mathsf{nvo_p}$ we also have $G'.\mathsf{nvo_p} = (G.\mathsf{nvo_p}|_{E\setminus\{fo\}} \cup A_1 \cup A_2)^+$ where:

$$A_1 = \left\{(a,fl) \;\middle|\; a \in D \wedge a \xrightarrow{G.\mathsf{tso_p}} \xrightarrow{S} fo \wedge \mathtt{loc}(a) \in X\right\} \qquad A_2 = \left\{(fl,b) \;\middle|\; b \in D \wedge fo \xrightarrow{G.\mathsf{tso_p}} b\right\}$$

We next demonstrate that $A_1 \subseteq \left\{(a,fl) \;\middle|\; (a,fo) \in G.\mathsf{tso_p}\right\}$. Let us proceed by contradiction and assume there exists $a \in D$ such that $(a,fl) \in A_1$ and $(a,fo) \notin G.\mathsf{tso_p}$. That is, there exists e such that $a \xrightarrow{G.\mathsf{tso_p}} e \xrightarrow{S\subseteq \mathrm{po}} fo$. From Prop. 3 there are then three cases to consider: 1) $(a,e) \in \mathrm{po}$; or 2) there exist $w_1, w_2 \in WU$ such that $a \xrightarrow{\mathrm{po}\cap G.\mathsf{tso_p}} w_1 \xrightarrow{\mathsf{mo}_e} w_2 \xrightarrow{\mathrm{po}} e$; or 3) there exist $w \in WU, r \in RU$ such that $a \xrightarrow{\mathrm{po}\cap G.\mathsf{tso_p}} w \xrightarrow{\mathsf{rf}_e} r \xrightarrow{\mathrm{po}} e$. In case (1) we then have $a \xrightarrow{\mathrm{po}} e \xrightarrow{S\subseteq \mathrm{po}} fo$ and thus $(a,fo) \in \mathrm{po}$. Consequently, since $\mathtt{loc}(a), \mathtt{loc}(fo) \in X$, from the definition of $G.\mathsf{tso_p}$ we have $(a,fo) \in G.\mathsf{tso_p}$, leading to a contradiction. In case (2) since $[D_X]; G.\mathsf{tso_p}; \mathsf{mo}_e; \mathrm{po}; [\{fo\}] \subseteq G.\mathsf{tso_p}$ and $a \xrightarrow{\mathrm{po}\cap G.\mathsf{tso_p}}$

$w_1 \xrightarrow{\mathsf{mo}_e} w_2 \xrightarrow{\mathsf{po}} e \xrightarrow{S \subseteq \mathsf{po}} fo$, we also have $(a, fo) \in G.\mathsf{tso}_\mathsf{p}$, leading to a contradiction. In case (3) since $r \in RU$ and $r \xrightarrow{\mathsf{po}} e \xrightarrow{S \subseteq \mathsf{po}} fo$ and thus $(r, fo) \in \mathsf{po}$, from the definition of $G.\mathsf{tso}_\mathsf{p}$ we also have $(r, fo) \in G.\mathsf{tso}_\mathsf{p}$. We then have $a \xrightarrow{\mathsf{po} \cap G.\mathsf{tso}_\mathsf{p}} w \xrightarrow{\mathsf{rf}_e} r \xrightarrow{G.\mathsf{tso}_\mathsf{p}} fo$, i.e. $(a, fo) \in G.\mathsf{tso}_\mathsf{p}$, leading to a contradiction.

We next show that G' is $\mathrm{Px86}^{'}_{\mathrm{sim}}$-consistent. (P-EXEC) and (P-TSO-RF1) follow immediately from the definition of G'. For (P-TSO-ORDER) we proceed by contradiction. Let us assume that there exists a such that $(a, a) \in G'.\mathsf{tso}_\mathsf{p}$. As $G.\mathsf{tso}_\mathsf{p}$ is acyclic, we then know that $a \xrightarrow{G.\mathsf{tso}_\mathsf{p}} e \xrightarrow{S} fl \xrightarrow{G.\mathsf{tso}_\mathsf{p}} a$. From Prop. 3 we then know there exists e such that either 1) $\mathtt{tid}(fl) = \mathtt{tid}(a)$ and $a \xrightarrow{\mathsf{po}} e \xrightarrow{S} fl \xrightarrow{G.\mathsf{po}} a$; or 2) $\mathtt{tid}(fl) \neq \mathtt{tid}(a)$ and $a \xrightarrow{(G'.\mathsf{tso}_\mathsf{p})_e} e \xrightarrow{S} fl \xrightarrow{(G'.\mathsf{tso}_\mathsf{p})_e} a$. In case (1) we know $S \subseteq \mathsf{po}'$, and since from the definition of po' we know $(e, fl) \in \mathsf{po}'$ implies $(e, fo) \in \mathsf{po}$, we then have $a \xrightarrow{\mathsf{po}} e \xrightarrow{\mathsf{po}} fo \xrightarrow{G.\mathsf{po}} a$, i.e. $(a, a) \in \mathsf{po}$, leading to a contradiction as G is $\mathrm{Px86}^{'}_{\mathrm{sim}}$-consistent.

In case (2) from Prop. 3 and the definition of $G'.\mathsf{tso}_\mathsf{p}$ we know there exist $w_1, w_2 \in WU$, $r \in R \cup WU$ and b such that: $fo \xrightarrow{\mathsf{po} \cap G.\mathsf{tso}_\mathsf{p}} w_2$ and $a \xrightarrow{G.\mathsf{tso}_\mathsf{p}^?} w_1 \xrightarrow{G.\mathsf{mo}_e \cup \mathsf{rf}_e} r \xrightarrow{(G.\mathsf{tso}_\mathsf{p} \cap \mathsf{po})^?} e \xrightarrow{S} fl \xrightarrow{\mathsf{po}' \cap G'.\mathsf{tso}_\mathsf{p}} w_2 \xrightarrow{G.\mathsf{tso}_\mathsf{p}} a$. Since $r \xrightarrow{(G.\mathsf{tso}_\mathsf{p} \cap G.\mathsf{po})^?} e$, and as $(e, fl) \in S \subseteq \mathsf{po}'$ implies $(e, fo) \in \mathsf{po}$, we then have $(r, fo) \in \mathsf{po}$. That is, we have $r \xrightarrow{(G.\mathsf{tso}_\mathsf{p} \cap \mathsf{po})^?} e \xrightarrow{\mathsf{po}} fo \xrightarrow{\mathsf{po}} w_2$; i.e. $(r, w_2) \in \mathsf{po}$. Consequently, from the definition of $G.\mathsf{tso}_\mathsf{p}$ and since $r \in R \cup WU$ and $w_2 \in WU$, we also have $(r, w_2) \in G.\mathsf{tso}_\mathsf{p}$. We thus have $a \xrightarrow{G.\mathsf{tso}_\mathsf{p}^?} w_1 \xrightarrow{G.\mathsf{mo}_e \cup \mathsf{rf}_e} r \xrightarrow{G.\mathsf{tso}_\mathsf{p}} w_2 \xrightarrow{G.\mathsf{tso}_\mathsf{p}} a$. That is, we have $(a, a) \in G.\mathsf{tso}_\mathsf{p}$, leading to a contradiction as G is $\mathrm{Px86}^{'}_{\mathrm{sim}}$-consistent.

For (P-TSO-RF2) we proceed by contradiction. Let us assume that $G'.\mathsf{tso}_\mathsf{p} \cup G'.\mathsf{rb}$ is acyclic. From the definitions of $G'.\mathsf{tso}_\mathsf{p}$ and $G'.\mathsf{rb}$ we then know that there exist $w \in WU, r \in RU$ such that $(w, r) \in G'.\mathsf{tso}_\mathsf{p}$ and $(r, w) \in G'.\mathsf{rb}$. However since G is $\mathrm{Px86}^{'}_{\mathrm{sim}}$-consistent and thus (P-TSO-RF2) holds for G, from the definition of G' we know there exists e such that $w \xrightarrow{G.\mathsf{tso}_\mathsf{p}} e \xrightarrow{S \cap G.\mathsf{po}} fo \xrightarrow{G.\mathsf{tso}_\mathsf{p}} r$. Consequently, from Prop. 3 and the definition of $G'.\mathsf{tso}_\mathsf{p}$ we know that either 1) there exists $e' \in MF \cup U$ such that $fo \xrightarrow{G.\mathsf{po}} e' \xrightarrow{G.\mathsf{po}} r$; or 2) there exists $w' \in WU$ and e' such that $fo \xrightarrow{G.\mathsf{po}} w' \xrightarrow{G.\mathsf{rf}_e \cup \mathsf{mo}_e} e' \xrightarrow{G.\mathsf{tso}_\mathsf{p} \cap G.\mathsf{po}} r$; or In case (1), since we have $w \xrightarrow{G.\mathsf{tso}_\mathsf{p}} e \xrightarrow{S \cap G.\mathsf{po}} fo \xrightarrow{G.\mathsf{po}} e' \xrightarrow{G.\mathsf{po}} r$ and $e' \in MF \cup U$, from the definition of $G.\mathsf{tso}_\mathsf{p}$ we also have $w \xrightarrow{G.\mathsf{tso}_\mathsf{p}} r$. We then have $w \xrightarrow{G.\mathsf{tso}_\mathsf{p}} r \xrightarrow{G.\mathsf{rb}}$, contradicting the (P-TSO-RF2) property of G since it is $\mathrm{Px86}^{'}_{\mathrm{sim}}$-consistent.

In case (2) from Prop. 3 there are again two cases to consider: i) $w \xrightarrow{G.\mathsf{tso}_\mathsf{p} \cap \mathsf{po}} e$ or ii) there exists $e'' \in WU \cup R$ such that $w \xrightarrow{G.(\mathsf{po} \cap \mathsf{tso}_\mathsf{p}^?) G.\mathsf{rf}_e \cup \mathsf{mo}_e} e'' \xrightarrow{G.\mathsf{po}^?} e$. In case (2.i) we then have $w \xrightarrow{G.\mathsf{tso}_\mathsf{p} \cap \mathsf{po}} e \xrightarrow{G.\mathsf{po}} fo \xrightarrow{G.\mathsf{po}} w'$ and thus $(w, w') \in G.\mathsf{po}$. As such, since $w, w' \in WU$ we also have $(w, w') \in G.\mathsf{tso}_\mathsf{p}$. That is, we have $w \xrightarrow{G.\mathsf{tso}_\mathsf{p}} w' \xrightarrow{G.\mathsf{rf}_e \cup \mathsf{mo}_e} e' \xrightarrow{G.\mathsf{tso}_\mathsf{p} \cap G.\mathsf{po}} r$, i.e. $(w, r) \in G.\mathsf{tso}_\mathsf{p}$. We then have $w \xrightarrow{G.\mathsf{tso}_\mathsf{p}} r \xrightarrow{G.\mathsf{rb}}$, contradicting the (P-TSO-RF2) property of G since it is $\mathrm{Px86}^{'}_{\mathrm{sim}}$-consistent. Similarly, in case (2.ii) we have $e'' \xrightarrow{G.\mathsf{po}^?} e \xrightarrow{S \cap G.\mathsf{po}} fo \xrightarrow{G.\mathsf{po}} w'$ and thus $(e'', w') \in G.\mathsf{po}$. as such, since $e'' \in WU \cup R$ and $w' \in WU$, from the definition of $G.\mathsf{tso}_\mathsf{p}$ we also have $(e'', w') \in G.\mathsf{tso}_\mathsf{p}$. That is, we have $w \xrightarrow{G.(\mathsf{po} \cap \mathsf{tso}_\mathsf{p}^?) G.\mathsf{rf}_e \cup \mathsf{mo}_e} e'' \xrightarrow{G.\mathsf{tso}_\mathsf{p}} w' \xrightarrow{G.\mathsf{rf}_e \cup \mathsf{mo}_e} e' \xrightarrow{G.\mathsf{tso}_\mathsf{p} \cap G.\mathsf{po}} r$, and thus $(w, r) \in G.\mathsf{tso}_\mathsf{p}$. We then have $w \xrightarrow{G.\mathsf{tso}_\mathsf{p}} r \xrightarrow{G.\mathsf{rb}}$, contradicting the (P-TSO-RF2) property of G since it is $\mathrm{Px86}^{'}_{\mathrm{sim}}$-consistent.

For (P-NVO-ORDER) note that by definition $G'.\mathsf{nvo}_\mathsf{p} \subseteq G'.\mathsf{tso}_\mathsf{p}$. As such, since we established that $G'.\mathsf{tso}_\mathsf{p}$ is a strict order, and since $G'.\mathsf{nvo}_\mathsf{p}$ is by definition transitive, we also know that $G'.\mathsf{nvo}_\mathsf{p}$ is a strict order as required.

For (P-NVO-P) we proceed by contradiction. Let us assume there exist a, b such that $a \notin P'$, $b \in P'$ and $(a, b) \in G'.\mathsf{nvo}_\mathsf{p}$. Given the definition of P' and since $G'.\mathsf{nvo}_\mathsf{p} = (G.\mathsf{nvo}_\mathsf{p}|_{E\setminus\{fo\}} \cup A_1 \cup A_2)^+$ and G is $\text{Px86}^{'}_{\text{sim}}$-consistent, we then know that $a \xrightarrow{A_1} fl \xrightarrow{A_2} b, \mathtt{loc}(a) \in X, b \in P$ and $a \notin P$. Consequently, from the definition of A_2 and since $A_1 \subseteq \left\{(a, fl) \,\middle|\, (a, fo) \in G.\mathsf{tso}_\mathsf{p}\right\}$ we have $a \xrightarrow{G.\mathsf{tso}_\mathbf{p}} fo \xrightarrow{G.\mathsf{tso}_\mathbf{p}} b$. As such, since $\mathtt{loc}(a), \mathtt{loc}(fo) \in X$, from the definition of $G.\mathsf{nvo}_\mathsf{p}$ we have $a \xrightarrow{G.\mathsf{nvo}_\mathbf{p}} fo \xrightarrow{G.\mathsf{nvo}_\mathbf{p}} b$. This, however contradicts the (P-NVO-P) property of G as G is $\text{Px86}^{'}_{\text{sim}}$-consistent, $b \in P$ and $a \notin P$.

Finally, we demonstrate that $G \equiv_{\mathsf{pp}} G'$. Pick an arbitrary location x. We are then required to show that $\mathsf{max}_{\mathsf{pp}}(G, x) = \mathsf{max}_{\mathsf{pp}}(G', x)$. Let us proceed by contradiction and assume that $\mathsf{max}_{\mathsf{pp}}(G, x) \neq \mathsf{max}_{\mathsf{pp}}(G', x)$. That is, there exist $w, w' \in E \cap WU_x$ such that $w \neq w'$, $w \in P$, $w' \in P'$, $w = \max\left(G.\mathsf{nvo}_\mathsf{p}|_{P \cap WU_x}\right)$ and $w' = \max\left(G'.\mathsf{nvo}_\mathsf{p}|_{P' \cap WU_x}\right)$. As $w' \in P'$, from the definition of P' we then know $w' \in P$. Similarly, as $w \in P$, from the definition of P' we then know $w \in P'$. There are now two cases to consider: 1) $w \xrightarrow{\mathsf{mo}} w'$; or 2) $w' \xrightarrow{\mathsf{mo}} w$.

In case (1) from the definition of $G.\mathsf{tso}_\mathsf{p}$ we have $w \xrightarrow{G.\mathsf{tso}_\mathbf{p}} w'$ and thus since from the definition of $G.\mathsf{tso}_\mathsf{p}$ we have $G.\mathsf{tso}_\mathsf{p}|_{D_x} \subseteq G.\mathsf{nvo}_\mathsf{p}$, we also have $w \xrightarrow{G.\mathsf{nvo}_\mathbf{p}} w'$. This however leads to a contradiction as $w' \in P$ and thus $\max\left(G.\mathsf{nvo}_\mathsf{p}|_{P \cap WU_x}\right) \neq w$. Similarly in case (2) from the definition of $G'.\mathsf{tso}_\mathsf{p}$ we have $w' \xrightarrow{G.\mathsf{tso}_\mathbf{p}} w$ and thus since from the definition of $G'.\mathsf{tso}_\mathsf{p}$ we have $G'.\mathsf{tso}_\mathsf{p}|_{D_x} \subseteq G'.\mathsf{nvo}_\mathsf{p}$, we also have $w' \xrightarrow{G'.\mathsf{nvo}_\mathbf{p}} w$. This however leads to a contradiction as $w \in P'$ and thus $\max\left(G'.\mathsf{nvo}_\mathsf{p}|_{P' \cap WU_x}\right) \neq w'$. □