Studying Sequent Systems via Non-deterministic Multiple-Valued Matrices

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We consider a family of sequent systems with "well-behaved" logical rules in which the cut rule and/or the identity-axiom are not present. We provide a semantic characterization of the logics induced by these systems in the form of non-deterministic three-valued or four-valued matrices. The semantics is used to study some important proof-theoretic properties of these systems. These results shed light on the dual semantic roles of the cut rule and the identity-axiom, showing that they are both crucial for having deterministic finite-valued semantics.*

Key words: proof theory, sequent systems, cut-elimination, semantic proofs, non-deterministic semantics, multiple-valued logics

1 INTRODUCTION

The family of *canonical sequent systems* was introduced and studied in [3]. The authors intended to formalize an important tradition in the philosophy of logic, according to which logical connectives are defined by well-behaved syntactic derivation rules. For this matter, the notion of a *canonical rule* was introduced (roughly speaking, that is a derivation rule in which exactly one occurrence of a connective is introduced and no other connective is mentioned). Canonical systems were in turn defined as sequent systems in which:

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(i) all logical rules are canonical; (ii) the two *identity rules* (the cut rule and the identity-axiom) and all usual structural rules (contraction, weakening, etc.) are included. The most prominent example of a canonical system is of course (the propositional fragment of) Gentzen's $\mathbf{LK}$ (the usual sequent system for classical logic). However, (infinitely) many more sequent systems belong to this family (a particularly useful one was recently suggested in [7]).

The study of canonical systems in [3] (see also [4]) mainly concerned a general soundness and completeness theorem, providing semantics for these systems in the form of two-valued non-deterministic matrices (Nmatrices) – a simple generalization of ordinary two-valued matrices. Unlike ordinary matrices, valuation functions in Nmatrices are allowed to non-deterministically choose the truth-value of some compound formulas out of some set of options. Non-determinism was proven to be inevitable to handle cases involving non truth-functional logics, where the meanings of compound formulas might not be uniquely determined by the meanings of their subformulas.

Given canonical system, the construction of an Nmatrix (in [3]) is modular, allowing to separate the semantic effect of each logical rule. Indeed, an addition of a logical (canonical) rule for some connective $\diamond$ results in a simple refinement of the Nmatrix, reducing the level of non-determinism in the truth-table of $\diamond$. However, while the semantic role of the logical rules is well-understood, the role of the two identity rules in canonical systems has not yet been studied. This is the subject of the current paper. We first define *semi-canonical* systems, which are obtained from canonical systems by omitting the cut rule and/or the identity-axiom. Then we study the semantics of logics induced by semi-canonical systems. As in canonical systems, we show that it cannot be captured using ordinary finite-valued matrices. But, unlike in the case of canonical systems, this is not a result of having non truth-functional connectives (in fact, even the semantics of $\mathbf{LK}$ without cut and/or identity-axiom cannot be given by ordinary finite-valued matrices). On the other hand, semantics for these logics can be provided using finite-valued (actually, three or four valued) Nmatrices. We show how to algorithmically construct these Nmatrices from the derivation rules of a given semi-canonical system. Finally, we prove some proof-theoretic properties of canonical and semi-canonical systems, demonstrating that the Nmatrix semantics is easily applicable for this matter. This includes a general strong cut-admissibility theorem for a large family of semi-canonical systems without identity-axioms.

Related Work Semantics for sequent systems without cut or identity-axiom was studied before. Following Schütte (see [10]), Girard (in [6]) studied the cut-free fragment of $\mathbf{LK}$, and provided semantics for this fragment using (non-deterministic) three-valued valuations.[†] Together with better understanding of the semantic role of the cut rule, this three-valued semantics was applied for proving several generalizations of the cut-elimination theorem (such as Takeuti's conjecture, see [6]). Later, the axiom-free fragment of $\mathbf{LK}$ was studied by Hösli and Jäger in [8]. As noted in [8], axiom-free systems play an important role in the proof-theoretic analysis of logic programming and in connection with the so called negation as failure. Hösli and Jäger provided a dual (non-deterministic) three-valued valuation semantics for axiom-free derivability in $\mathbf{LK}$. With respect to [6] and [8], the current work contributes in four main aspects:

1. Our results apply to the broad family of semi-canonical systems, of which the cut-free and the axiom-free fragments of $\mathbf{LK}$ are just particular examples. For these systems, we obtain practically the same semantics that was suggested in [6] and [8].

2. While the focus of [6] and [8] was on derivability between sequents, we also study the consequence relation between *formulas* induced by cut-free and axiom-free systems.

3. We formulate the two kinds of three-valued valuation semantics inside the well-studied framework of (three-valued) Nmatrices, exploiting some known general properties of Nmatrices.

4. In [8], it seems that the two dual kinds of three-valued valuation semantics cannot be combined. However, in this paper we show that a combination of them is obtained using *four-valued* Nmatrices. To the best of our knowledge, systems with neither cut nor identity-axiom were not studied before.

It should be mentioned, however, that [6] and [8] concerned also the usual quantifiers of $\mathbf{LK}$, while we only investigate *propositional* logics, leaving the more complicated first-order case (and beyond) to a future work.

[†] Note that cut-elimination implies that provability and cut-free provability coincide for $\mathbf{LK}$. However, it is well-known that cut-elimination fails in the presence of extra "non-logical" axioms (assumptions), and so the derivability relation induced by $\mathbf{LK}$ (which allows non-empty set of assumptions) is different from the one induced by its cut-free fragment.

2 LOGICS AND SEQUENT SYSTEMS

In what follows, $\mathcal{L}$ denotes a propositional language, and $Frm_{\mathcal{L}}$ denotes its set of wffs. We assume that $p_1, p_2, \ldots$ are the atomic formulas of any propositional language. An *$\mathcal{L}$-substitution* is a function $\sigma : Frm_{\mathcal{L}} \to Frm_{\mathcal{L}}$, such that $\sigma(\diamond(\psi_1, \ldots, \psi_n)) = \diamond(\sigma(\psi_1), \ldots, \sigma(\psi_n))$ for every n-ary connective $\diamond$ of $\mathcal{L}$. Substitutions are extended to sets of formulas in the obvious way.

Definition 1. *A relation $\vdash$ between sets of $\mathcal{L}$-formulas and $\mathcal{L}$-formulas is:*

Reflexive*:*	*if $\mathcal{T} \vdash \psi$ whenever $\psi \in \mathcal{T}$.*
Monotone*:*	*if $\mathcal{T}' \vdash \psi$ whenever $\mathcal{T} \vdash \psi$ and $\mathcal{T} \subseteq \mathcal{T}'$.*
Transitive*:*	*if $\mathcal{T}, \mathcal{T}' \vdash \varphi$ whenever $\mathcal{T} \vdash \psi$ and $\mathcal{T}', \psi \vdash \varphi$.*
Structural*:*	*if $\sigma(\mathcal{T}) \vdash \sigma(\psi)$ for every $\mathcal{L}$-substitution σ whenever $\mathcal{T} \vdash \psi$.*
Finitary*:*	*if $\Gamma \vdash \psi$ for some finite $\Gamma \subseteq \mathcal{T}$ whenever $\mathcal{T} \vdash \psi$.*
Consistent*:*	*if $\mathcal{T} \not\vdash \psi$ for some non-empty $\mathcal{T}$ and ψ.*

Definition 2. *A relation between sets of $\mathcal{L}$-formulas and $\mathcal{L}$-formulas which is reflexive, monotone and transitive is called a* Tarskian consequence relation (tcr) *for $\mathcal{L}$. A* (Tarskian propositional) logic *is a pair $\langle \mathcal{L}, \vdash \rangle$, where $\mathcal{L}$ is a propositional language, and $\vdash$ is a structural, finitary, consistent tcr for $\mathcal{L}$.*

The proof-theoretical way to define logics is based on a notion of a proof in some formal deduction system. In this paper we study *sequent systems*, these are axiomatic systems that manipulate higher-level constructs, called *sequents*, rather than the formulas themselves. There are several variants of what exactly constitutes a sequent. Here it is convenient to define sequents as expressions of the form $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite *sets* of formulas, and $\Rightarrow$ is a new symbol, not occurring in $\mathcal{L}$.[‡] Each sequent system induces a derivability relation between sets of sequents and sequents. We shall write $\mathcal{S} \vdash^{seq}_{\mathbf{G}} s$ if the sequent s is derivable in a sequent system $\mathbf{G}$ from a set $\mathcal{S}$ of sequents. The sequents of $\mathcal{S}$ are called *assumptions*. If $\vdash^{seq}_{\mathbf{G}} s$ (or formally, $\emptyset \vdash^{seq}_{\mathbf{G}} s$), we say that s is *provable* in $\mathbf{G}$.

All systems studied in this paper include the *weakening rule* (Weak) – the derivation rule allowing to infer a sequent of the form $\Gamma', \Gamma \Rightarrow \Delta, \Delta'$ from $\Gamma \Rightarrow \Delta$. In addition, usual systems include the following *identity rules*:

[‡] The sequent systems considered in this paper are all fully-structural, as they include the usual exchange, contraction and expansion rules (by expansion, we mean here the sequent rules allowing to "duplicate" formulas on both sides). For this reason, it is most convenient to define sequents using *sets*, so these structural rules are built-in and not included explicitly.

Definition 3. *The* cut rule (Cut) *is a derivation rule allowing to infer a sequent of the form* $\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2$ *from the sequents* $\Gamma_1 \Rightarrow \varphi, \Delta_1$ *and* $\Gamma_2, \varphi \Rightarrow \Delta_2$ *(*φ *is called the* cut-formula*). The* identity-axiom (Id) *is a rule allowing to infer sequents of the form* $\varphi \Rightarrow \varphi$ *(without premises).*

We shall refer to sequent systems that include (Cut) as (+C)-systems. Similarly, (+A)-systems are sequent systems that include (Id). We also use (−C), (−A), and their combinations. For example, (−C−A)-systems are sequent systems that include neither (Cut) nor (Id). In particular, (the propositional fragment of) Gentzen's system **LK** for classical logic is a (+C+A)-system.

Notation 1. *Given a (+C)-system* $\mathbf{G}$*,* $\mathbf{G}$*−C is the system obtained from* $\mathbf{G}$ *by omitting (Cut). Similarly,* $\mathbf{G}$*−A stands for omitting (Id), and* $\mathbf{G}$*−C−A stands for omitting both (Cut) and (Id).* $\mathbf{G}$*+C,* $\mathbf{G}$*+A,* $\mathbf{G}$*+C+A are defined similarly.*

Recall that sequent systems are a tool to handle consequence relations. The consequence relation induced by a given system is defined as follows:

Definition 4. *Given a sequent system* $\mathbf{G}$*, a set* $\mathcal{T}$ *of formulas and a formula* φ*:* $\mathcal{T} \vdash_{\mathbf{G}} \varphi$ *if* $\{\Rightarrow \psi \mid \psi \in \mathcal{T}\} \vdash^{seq}_{\mathbf{G}} \Rightarrow \varphi$*.*

It is clear that $\vdash_{\mathbf{G}}$ is a finitary tcr for *every* sequent system $\mathbf{G}$. Its structurality and consistency depend on $\mathbf{G}$.

Remark 1. *There is another natural way to obtain a relation between sets of formulas and formulas from a given sequent system (see [1]), that is to define that* $\mathcal{T} \vdash_{\mathbf{G}} \varphi$ *if* $\vdash^{seq}_{\mathbf{G}} \Gamma \Rightarrow \varphi$ *for some finite* $\Gamma \subseteq \mathcal{T}$*. It is easy to see that in many natural (+C+A)-systems the two definitions are equivalent (this holds for monotone and pure (+C+A)-systems, see [1]). The current paper, however, is about (−C)-systems and (−A)-systems, where the situation is different. Obviously, for every sequent system the definition above also leads to a monotone and finitary relation. Its reflexivity and transitivity, however, are not guaranteed. Note that the two definitions are also inequivalent in usual sequent systems for first-order logics, where they provide two natural consequence relations.*

3 CANONICAL AND SEMI-CANONICAL SEQUENT SYSTEMS

The framework of sequent systems described above is perhaps too broad to obtain any interesting general results. We put our focus on narrower families of sequent systems with "well-behaved" derivation rules. First we review the family of *canonical systems*, defined in [3].

Definition 5. *An n-*clause *is a sequent consisting only of atomic formulas from $\{p_1, \dots, p_n\}$.*

Definition 6. *A* canonical rule *for an n-ary connective $\diamond$ of $\mathcal{L}$ is an expression of the form $\mathcal{S}/C$, where $\mathcal{S}$ is a finite set of n-clauses (called* premises*), and C (the* conclusion*) is either $\Rightarrow \diamond(p_1, \dots, p_n)$ (in canonical* right rules*) or $\diamond(p_1, \dots, p_n) \Rightarrow$ (in canonical* left rules*). An* application *of a canonical right rule $\{\Pi_1 \Rightarrow \Sigma_1, \dots, \Pi_m \Rightarrow \Sigma_m\}/ \Rightarrow \diamond(p_1, \dots, p_n)$ is any inference step of the following form:*

$$\frac{\Gamma_1, \sigma(\Pi_1) \Rightarrow \sigma(\Sigma_1), \Delta_1 \quad \dots \quad \Gamma_m, \sigma(\Pi_m) \Rightarrow \sigma(\Sigma_m), \Delta_m}{\Gamma_1, \dots, \Gamma_m \Rightarrow \sigma(\diamond(p_1, \dots, p_n)), \Delta_1, \dots, \Delta_m}$$

where σ is an $\mathcal{L}$-substitution, and $\Gamma_1, \dots, \Gamma_m, \Delta_1, \dots, \Delta_m$ are arbitrary finite sets of formulas. Applications of left rules are defined similarly.

Example 1. *The usual derivation rules for the classical connectives can all be presented as canonical rules. For $\supset$, $\wedge$ and $\neg$ we have the following rules:*

$(\supset\Rightarrow)\ \{\Rightarrow p_1\ ,\ p_2 \Rightarrow\}/p_1 \supset p_2 \Rightarrow$ $\quad$ $(\Rightarrow\supset)\ \{p_1 \Rightarrow p_2\}/ \Rightarrow p_1 \supset p_2$
$(\wedge \Rightarrow)\{p_1, p_2 \Rightarrow\}/p_1 \wedge p_2 \Rightarrow$ $\quad$ $(\Rightarrow \wedge)\{\Rightarrow p_1\ ,\ \Rightarrow p_2\}/ \Rightarrow p_1 \wedge p_2$
$(\neg \Rightarrow)\{\Rightarrow p_1\}/\neg p_1 \Rightarrow$ $\quad$ $(\Rightarrow \neg)\{p_1 \Rightarrow\}/ \Rightarrow \neg p_1$

Applications of these rules have the form (respectively):

$$\frac{\Gamma_1 \Rightarrow \psi, \Delta_1 \quad \Gamma_2, \varphi \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2, \psi \supset \varphi \Rightarrow \Delta_1, \Delta_2} \qquad \frac{\Gamma, \psi \Rightarrow \varphi, \Delta}{\Gamma \Rightarrow \psi \supset \varphi, \Delta}$$

$$\frac{\Gamma, \psi, \varphi \Rightarrow \Delta}{\Gamma, \psi \wedge \varphi \Rightarrow \Delta} \qquad \frac{\Gamma_1 \Rightarrow \psi, \Delta_1 \quad \Gamma_2 \Rightarrow \varphi, \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \psi \wedge \varphi, \Delta_1, \Delta_2}$$

$$\frac{\Gamma \Rightarrow \psi, \Delta}{\Gamma, \neg\psi \Rightarrow \Delta} \qquad \frac{\Gamma, \psi \Rightarrow \Delta}{\Gamma \Rightarrow \neg\psi, \Delta}$$

Example 2. *An alternative implication, called primal-implication and denoted here by $\rightsquigarrow$, was studied in [7], and showed to be useful in certain applications. It is defined by the following two canonical rules:*

$(\rightsquigarrow\Rightarrow)\{\Rightarrow p_1\ ,\ p_2 \Rightarrow\}/p_1 \rightsquigarrow p_2 \Rightarrow$ $\quad$ $(\Rightarrow\rightsquigarrow)\{\Rightarrow p_2\}/ \Rightarrow p_1 \rightsquigarrow p_2$

Applications of these rules have the form (respectively):

$$\frac{\Gamma_1 \Rightarrow \psi, \Delta_1 \quad \Gamma_2, \varphi \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2, \psi \rightsquigarrow \varphi \Rightarrow \Delta_1, \Delta_2} \qquad \frac{\Gamma \Rightarrow \varphi, \Delta}{\Gamma \Rightarrow \psi \rightsquigarrow \varphi, \Delta}$$

Definition 7. *A sequent system is called* canonical *if it includes (Weak), (Cut), (Id), and each of its other rules is a canonical rule.*

The structure of the rules of canonical systems ensures that $\vdash_{\mathbf{G}}$ is structural for every canonical system $\mathbf{G}$. Clearly, the propositional fragment of $\mathbf{LK}$ can be presented as a canonical system. Many more canonical systems can be introduced with various new connectives. Not all combinations of canonical rules, however, are meaningful. A natural demand is that the premises of a right and a left rule for the same connective are contradictory. This is captured by the following coherence criterion ([3]).

Definition 8. *A canonical system* $\mathbf{G}$ *is called* coherent *if* $\mathcal{S}_1 \cup \mathcal{S}_2$ *is classically unsatisfiable whenever* $\mathbf{G}$ *includes two canonical rules of the form* $\mathcal{S}_1 / \Rightarrow \diamond(p_1, \dots, p_n)$ *and* $\mathcal{S}_2 / \diamond(p_1, \dots, p_n) \Rightarrow$ *for some connective* $\diamond$*.*

Coherence is a minimal demand from a canonical system. Indeed, it is proved in [3], that the tcr $\vdash_{\mathbf{G}}$ induced by a canonical system $\mathbf{G}$ is consistent *iff* $\mathbf{G}$ is coherent. It follows that all coherent canonical systems induce logics.

Remark 2. *It is also shown in [3] that all coherent canonical systems enjoy cut-admissibility. A new proof of this fact is given in the sequel (Corollary 3).*

Next, we define the family of semi-canonical systems, which are the (−C)-systems and (−A)-systems that are obtained from canonical systems.

Definition 9. *A* semi-canonical *system is a system obtained from a canonical system by omitting (Cut) and/or (Id).*

Remark 3. *Semi-canonical (−A)-systems may look strange at first sight. Indeed, (−A)-systems, in which each canonical rule has at least one premise, have no provable sequents (namely,* $\nvdash_{\mathbf{G}}^{seq} s$ *for every* s*). The interest in them arises when we consider derivations from non-empty sets of assumptions.*

Evidently, $\vdash_{\mathbf{G}}$ is a structural tcr for every semi-canonical system $\mathbf{G}$ (see Definitions 1 and 2). In semi-canonical (−C)-systems we have that $p_1 \nvdash_{\mathbf{G}} p_2$. It follows that for every semi-canonical (−C)-system $\mathbf{G}$ for $\mathcal{L}$, $\langle \mathcal{L}, \vdash_{\mathbf{G}} \rangle$ is a logic. On the other hand, in semi-canonical (+C−A)-systems the consistency of $\vdash_{\mathbf{G}}$ is not guaranteed. The coherence criterion is required in this case also (coherence of a semi-canonical system is defined exactly like coherence of canonical systems (Definition 8). Note, however, that there are non-coherent semi-canonical (+C−A)-systems that do induce a consistent tcr:

Example 3. *Consider a semi-canonical (+C−A)-system* $\mathbf{G}$*, whose only logical rules are* $\{p_1 \Rightarrow\} / \Rightarrow \circ p_1$ *and* $\{p_1 \Rightarrow\} / \circ p_1 \Rightarrow$ *(its language consists* *of one unary connective* $\circ$*).* $\mathbf{G}$ *is not coherent. But,* $\Rightarrow p_1 \not\vdash^{seq}_{\mathbf{G}} \Rightarrow p_2$*, and so* $p_1 \not\vdash_{\mathbf{G}} p_2$*. Therefore,* $\vdash_{\mathbf{G}}$ *is consistent, and* $\langle \mathcal{L}, \vdash_{\mathbf{G}} \rangle$ *is a logic.*

Next we provide an exact characterization of the semi-canonical (+C−A)-systems that induce logics:

Theorem 1. *Let* $\mathbf{G}$ *be a semi-canonical (+C−A)-system.*

1. *If* $\mathbf{G}$ *is coherent, then* $\vdash_{\mathbf{G}}$ *is consistent.*
2. *If each canonical left rule of* $\mathbf{G}$ *has at least one premise of the form* $\Pi \Rightarrow$ *, then* $\vdash_{\mathbf{G}}$ *is consistent.*
3. *Otherwise,* $\vdash_{\mathbf{G}}$ *is not consistent.*

Proof. If $\mathbf{G}$ is coherent, then $\mathbf{G}$+A is coherent. Obviously $\vdash_{\mathbf{G}} \subseteq \vdash_{\mathbf{G}+\mathtt{A}}$. The consistency of $\vdash_{\mathbf{G}}$ then follows from the consistency of $\vdash_{\mathbf{G}+\mathtt{A}}$ (recall that coherent canonical systems induce consistent tcrs, see [3]).

If 2 holds, then whenever $\Rightarrow p_1 \vdash^{seq}_{\mathbf{G}} \Gamma \Rightarrow \Delta$, we have that either $p_1 \in \Delta$, or Δ contains some compound formula (this can by shown using induction on the length of derivations). Thus $\Rightarrow p_1 \not\vdash^{seq}_{\mathbf{G}} \Rightarrow p_2$, and so $\vdash_{\mathbf{G}}$ is consistent.

Now, suppose that $\mathbf{G}$ is not coherent and that $\mathbf{G}$ includes a left rule $r = \mathcal{S}/\circ(p_1, \ldots, p_m) \Rightarrow$, where each premise $\Pi \Rightarrow \Sigma$ in $\mathcal{S}$ has non-empty Σ. We prove that $p_1 \vdash_{\mathbf{G}} p_2$. Since $\mathbf{G}$ is not coherent, $\mathbf{G}$ includes two canonical rules $r_1 = \mathcal{S}_1/\Rightarrow \diamond(p_1, \ldots, p_n)$ and $r_2 = \mathcal{S}_2/\diamond(p_1, \ldots, p_n) \Rightarrow$, such that $\mathcal{S}_1 \cup \mathcal{S}_2$ is classically satisfiable. Now, by possibly applying (Weak) on $\Rightarrow p_1$ and the rule r, we obtain a derivation of $\circ(p_1, \ldots, p_1) \Rightarrow$ from $\Rightarrow p_1$. Let u be the classical valuation satisfying $\mathcal{S}_1 \cup \mathcal{S}_2$. Let σ be a substitution, defined by $\sigma(p) = p_1$ if $u(p) = t$, and $\sigma(p) = \circ(p_1, \ldots, p_1)$ if $u(p) = f$. For every premise $\Pi \Rightarrow \Sigma \in \mathcal{S}_1 \cup \mathcal{S}_2$, either $\circ(p_1, \ldots, p_1) \in \sigma(\Pi)$ or $p_1 \in \sigma(\Sigma)$. By applying (Weak) either on $\circ(p_1, \ldots, p_1) \Rightarrow$ or on $\Rightarrow p_1$, we obtain that $\Rightarrow p_1 \vdash^{seq}_{\mathbf{G}} \sigma(s)$ for every $s \in \mathcal{S}_1 \cup \mathcal{S}_2$. By applying the rules r_1 and r_2, we obtain derivations of $\Rightarrow \sigma(\diamond(p_1, \ldots, p_n))$ and of $\sigma(\diamond(p_1, \ldots, p_n)) \Rightarrow$. A cut yields a derivation of the empty sequent, on which we can finally apply (Weak) to obtain $\Rightarrow p_2$. □

Note that we can consider only coherent (+C−A)-systems as (+C−A)- systems that induce logics, and exclude the systems that admit 2 above. This is justified by the following proposition:

Proposition 1. *Let* $\mathbf{G}$ *be a semi-canonical (−A)-system, in which every left rule has at least one premise of the form* $\Pi \Rightarrow$ *. Let* $\mathbf{G}'$ *be the system obtained from* $\mathbf{G}$ *by omitting all left rules. Then,* $\mathbf{G}'$ *is coherent and* $\vdash_{\mathbf{G}'} = \vdash_{\mathbf{G}}$*.*

Proof Outline. Obviously, $\mathbf{G}'$ is coherent and $\vdash_{\mathbf{G}'} \subseteq \vdash_{\mathbf{G}}$. For the converse, one shows that for every set $\mathcal{S}$ of sequents of the form $\Rightarrow \psi$, if $\mathcal{S} \vdash^{seq}_{\mathbf{G}} \Gamma \Rightarrow \Delta$ then $\mathcal{S} \vdash^{seq}_{\mathbf{G}'} \Rightarrow \Delta$. This is proved by induction of the length of the derivation in $\mathbf{G}$ of $\Gamma \Rightarrow \Delta$. The fact that $\vdash_{\mathbf{G}} \subseteq \vdash_{\mathbf{G}'}$ easily follows. $\square$

4 SEMANTICS FOR SEMI-CANONICAL SYSTEMS

In this section we provide semantics for logics induced by semi-canonical systems. This semantics is given in the form of non-deterministic matrices, which are a natural generalization of usual multiple-valued matrices.

Definition 10 ([3]). *A* non-deterministic matrix (Nmatrix) $\mathbf{M}$ *for $\mathcal{L}$ consists of: a non-empty set $\mathcal{V}_{\mathbf{M}}$ of* truth-values*, a non-empty proper subset $\mathcal{D}_{\mathbf{M}} \subset \mathcal{V}_{\mathbf{M}}$ of* designated truth-values*, and a function (called:* truth-table*) $\diamond_{\mathbf{M}}$ from ${\mathcal{V}_{\mathbf{M}}}^n$ to $P(\mathcal{V}_{\mathbf{M}}) \setminus \{\emptyset\}$ for every n-ary connective $\diamond$ of $\mathcal{L}$.*

Ordinary (deterministic) *matrices* correspond to the case when each $\diamond_{\mathbf{M}}$ is a function taking singleton values only (then it can be treated as a function to $\mathcal{V}_{\mathbf{M}}$). An Nmatrix $\mathbf{M}$ is *finite* if so is $\mathcal{V}_{\mathbf{M}}$. By an nNmatrix ($n \in \mathbb{N}$) we shall mean an Nmatrix for which $|\mathcal{V}_{\mathbf{M}}| = n$.

Definition 11. *A* valuation *in an Nmatrix $\mathbf{M}$ (for $\mathcal{L}$) is a function v from $Frm_{\mathcal{L}}$ to $\mathcal{V}_{\mathbf{M}}$, such that $v(\diamond(\psi_1, \ldots, \psi_n)) \in \diamond_{\mathbf{M}}(v(\psi_1), \ldots, v(\psi_n))$ for every compound formula $\diamond(\psi_1, \ldots, \psi_n) \in Frm_{\mathcal{L}}$. v is a* model *of a* formula *ψ if $v(\psi) \in \mathcal{D}_{\mathbf{M}}$. v is a* model *of a* set *$\mathcal{T}$ of formulas if it is a model of every $\psi \in \mathcal{T}$. In addition,* $\vdash_{\mathbf{M}}$, the tcr induced by $\mathbf{M}$*, is defined by: $\mathcal{T} \vdash_{\mathbf{M}} \psi$, if every valuation v in $\mathbf{M}$ which is a model of $\mathcal{T}$ is also a model of ψ.*

It is easy to verify that for every Nmatrix $\mathbf{M}$, $\vdash_{\mathbf{M}}$ is a structural consistent tcr (for consistency note that $p_1 \not\vdash_{\mathbf{M}} p_2$). In addition, it is proved in [3] that tcrs induced by finite Nmatrices are always finitary. It follows that $\langle \mathcal{L}, \vdash_{\mathbf{M}} \rangle$ is a logic for every finite Nmatrix $\mathbf{M}$ for $\mathcal{L}$.

In general, in order for a denotational semantics of a propositional logic to be useful and effective, it should be *analytic*. This means that to determine whether a formula φ follows from a theory $\mathcal{T}$, it suffices to consider *partial* valuations, defined on the set of all subformulas of the formulas in $\mathcal{T} \cup \{\varphi\}$. The semantics of Nmatrices is *analytic* in this sense:

Definition 12. *A* partial valuation *in an Nmatrix $\mathbf{M}$ (for $\mathcal{L}$) is a function v from some subset $\mathcal{E}$ of $Frm_{\mathcal{L}}$, which is closed under subformulas, to $\mathcal{V}_{\mathbf{M}}$, such that $v(\diamond(\psi_1, \ldots, \psi_n)) \in \diamond_{\mathbf{M}}(v(\psi_1), \ldots, v(\psi_n))$ for every compound* *formula* $\diamond(\psi_1, \ldots, \psi_n) \in \mathcal{E}$*. The notion of a* model *is defined for partial valuations exactly like it is defined for valuations.*

Proposition 2 ([4])**.** *Every partial valuation in some Nmatrix* $\mathbf{M}$ *can be extended to a (full) valuation in* $\mathbf{M}$*.*

Corollary 1. *Let* $\mathbf{M}$ *be an Nmatrix for* $\mathcal{L}$*.* $\mathcal{T} \vdash_{\mathbf{M}} \psi$ *iff for every partial valuation* v *in* $\mathbf{M}$ *defined on the set of subformulas of* $\mathcal{T}$ *and* ψ*:* v *is a model of* ψ *whenever it is a model of* $\mathcal{T}$*.*

As a result of the last corollary, we obtain that logics characterized by finite Nmatrices are decidable (see Theorem 28 in [4]).

The Nmatrices we construct for semi-canonical systems are based on the four truth-values $t, f, \top$, and $\bot$. We also use a partial order on $\{t, f, \top, \bot\}$, denoted by $\leq$, according to which $\bot$ is the minimal element, $\top$ – the maximal one, and t, f are intermediate incomparable values ($\leq$ is the transitive reflexive closure of $\{\langle \bot, t\rangle, \langle \bot, f\rangle, \langle t, \top\rangle, \langle f, \top\rangle\}$).[¶] In addition, the following definitions and propositions are used in the sequel:

Definition 13. *Let* $\mathbf{M}$ *be an Nmatrix with* $\mathcal{V}_{\mathbf{M}} \subseteq \{t, f, \top, \bot\}$*. A valuation* v *in* $\mathbf{M}$ *is a* model of a sequent $\Gamma \Rightarrow \Delta$ *if either* $v(\psi) \geq f$ *for some* $\psi \in \Gamma$*, or* $v(\psi) \geq t$ *for some* $\psi \in \Delta$*.* v *is a model of a set* $\mathcal{S}$ *of sequents if it is a model of every* $s \in \mathcal{S}$*. In addition,* $\vdash^{seq}_{\mathbf{M}}$*,* the relation induced by $\mathbf{M}$ between sets of sequents and sequents*, is defined as follows:* $\mathcal{S} \vdash^{seq}_{\mathbf{M}} s$*, if every valuation* v *in* $\mathbf{M}$ *which is a model of* $\mathcal{S}$ *is also a model of* s*.*

Proposition 3. *Let* $\mathbf{M}$ *be an Nmatrix satisfying* $\mathcal{V}_{\mathbf{M}} \subseteq \{t, f, \top, \bot\}$ *and* $\mathcal{D}_{\mathbf{M}} = \mathcal{V}_{\mathbf{M}} \cap \{t, \top\}$*.* $\mathcal{T} \vdash_{\mathbf{M}} \varphi$ *iff* $\{\Rightarrow \psi \mid \psi \in \mathcal{T}\} \vdash^{seq}_{\mathbf{M}} \Rightarrow \varphi$*.*

Proof. Follows from the fact that a valuation in $\mathbf{M}$ is a model of a sequent $\Rightarrow \psi$ iff it is a model of (the formula) ψ (since $\mathcal{D}_{\mathbf{M}} = \mathcal{V}_{\mathbf{M}} \cap \{t, \top\}$). $\square$

Definition 14. *Let* $x_1, \ldots, x_n \in \{t, f, \top, \bot\}$*. The tuple* $\langle x_1, \ldots, x_n\rangle$ satisfies *an* n*-clause* $\Pi \Rightarrow \Sigma$ *if there exists either some* $p_i \in \Pi$ *such that* $x_i \geq f$*, or some* $p_i \in \Sigma$ *such that* $x_i \geq t$*.* $\langle x_1, \ldots, x_n\rangle$ fulfils *a canonical rule* r *for an* n*-ary connective, if it satisfies every premise of* r*.*

Note that checking whether $\langle x_1, \ldots, x_n\rangle$ fulfils some rule is easy.

Proposition 4. *Let* $x_1, \ldots, x_n \in \{t, f, \bot\}$*, and let* $\mathbf{G}$ *be a semi-canonical system. If* $\mathbf{G}$ *is coherent then* $\langle x_1, \ldots, x_n\rangle$ *cannot fulfil* both *a right rule and a left rule of* $\mathbf{G}$ *for some* n*-ary connective* $\diamond$*.*

[¶] Note that $\leq$ is the "knowledge" partial order used in Dunn-Belnap matrix (see [5]).

The structure of the rest of this section is as follows. We begin with a (known) construction of 2Nmatrices for coherent canonical systems. Next, we separately deal with semi-canonical ($\mathtt{-C+A}$)-systems, coherent ($\mathtt{+C-A}$)-systems, and ($\mathtt{-C-A}$)-systems.

4.1 Canonical Systems

A general construction of 2Nmatrices to characterize logics induced by coherent canonical systems was given in [3]. It is also shown there that logics which are characterized by properly non-deterministic 2Nmatrices (i.e. $\diamond_{\mathbf{M}}(x_1,\ldots,x_n) = \{t,f\}$ for some connective $\diamond$ and $x_1,\ldots,x_n \in \{t,f\}$) cannot be characterized by finite (ordinary) matrices. Below we reproduce the construction of 2Nmatrices for coherent canonical systems.

Definition 15. *Let $\mathbf{G}$ be a coherent canonical system for $\mathcal{L}$. The Nmatrix $\mathbf{M_G}$ is defined by $\mathcal{V}_{\mathbf{M_G}} = \{t,f\}$, $\mathcal{D}_{\mathbf{M_G}} = \{t\}$, and for every n-ary connective $\diamond$ of $\mathcal{L}$: (i) $t \in \diamond_{\mathbf{M_G}}(x_1,\ldots,x_n)$ iff $\langle x_1,\ldots,x_n\rangle$ does not fulfil any left rule of $\mathbf{G}$ for $\diamond$; and (ii) $f \in \diamond_{\mathbf{M_G}}(x_1,\ldots,x_n)$ iff $\langle x_1,\ldots,x_n\rangle$ does not fulfil any right rule of $\mathbf{G}$ for $\diamond$.*

While this is not the formulation in [3], the same Nmatrix is obtained. Note that the coherence of $\mathbf{G}$ ensures that $\mathbf{M_G}$ is well-defined (see Proposition 4).

Example 4. *Suppose that the rules for $\supset$ and $\leadsto$ in a coherent canonical system $\mathbf{G}$ are those given in Examples 1-2. $\supset_{\mathbf{M_G}}$ and $\leadsto_{\mathbf{M_G}}$ are given below.*

$\supset_{\mathbf{M_G}}$	t	f
t	$\{t\}$	$\{f\}$
f	$\{t\}$	$\{t\}$

$\leadsto_{\mathbf{M_G}}$	t	f
t	$\{t\}$	$\{f\}$
f	$\{t\}$	$\{t,f\}$

Theorem 2 ([3]). *For every coherent canonical system $\mathbf{G}$, $\vdash^{seq}_{\mathbf{G}} = \vdash^{seq}_{\mathbf{M_G}}$.*

Following Proposition 3, we obtain the following corollary:

Corollary 2. *For every coherent canonical system $\mathbf{G}$, $\vdash_{\mathbf{G}} = \vdash_{\mathbf{M_G}}$.*

4.2 ($\mathtt{-C+A}$)-Systems

Semantics for semi-canonical ($\mathtt{-C+A}$)-systems is given below in the form of 3Nmatrices. First, we show that non-determinism is essential for this purpose.

Theorem 3. *Suppose that the language $\mathcal{L}$ contains an unary connective, denoted by $\neg$, and let $\mathbf{G}$ be a semi-canonical ($\mathtt{-C}$)-system, whose rules for $\neg$ are the usual rules. There is no finite (ordinary) matrix $\mathbf{M}$ such that $\vdash_{\mathbf{M}} \subseteq \vdash_{\mathbf{G}}$.*

Proof. Let $\mathbf{M}$ be an (ordinary) matrix, such that $\vdash_{\mathbf{M}} \subseteq \vdash_{\mathbf{G}}$. Note that for every $n \geq 0$, $\{\neg^i p_1 \mid i > n\} \not\vdash_{\mathbf{G}} \neg^n p_1$ (one can verify this using Theorem 4 below). Consequently, $\{\neg^i p_1 \mid i > n\} \not\vdash_{\mathbf{M}} \neg^n p_1$ for every $n \geq 0$. For every $n \geq 0$, let v_n be a valuation in $\mathbf{M}$, which is a model of $\{\neg^i p_1 \mid i > n\}$, and not of $\neg^n p_1$. Thus, for every $n > m \geq 0$, $v_m(\neg^n p_1)$ is designated, while $v_n(\neg^n p_1)$ is not designated. Since $\mathbf{M}$ is deterministic, this entails that for every $n > m \geq 0$, $\neg_{\mathbf{M}}{}^n(v_m(p_1))$ is designated, while $\neg_{\mathbf{M}}{}^n(v_n(p_1))$ is not designated. Hence $v_n(p_1) \neq v_m(p_1)$ for every $n > m \geq 0$. It follows that $\mathbf{M}$ is infinite. $\square$

Next we turn to the construction of 3Nmatrices (with the truth-values t, f and $\top$) for semi-canonical (`-C+A`)-systems.

Definition 16. *Let $\mathbf{G}$ be a semi-canonical (`-C+A`)-system for $\mathcal{L}$. $\mathbf{M_G}$ is defined by $\mathcal{V}_{\mathbf{M_G}} = \{t, f, \top\}$, $\mathcal{D}_{\mathbf{M_G}} = \{t, \top\}$, and for every n-ary connective $\diamond$ of $\mathcal{L}$: (i) $t \in \diamond_{\mathbf{M_G}}(x_1, \ldots, x_n)$ iff $\langle x_1, \ldots, x_n\rangle$ does not fulfil any left rule of $\mathbf{G}$ for $\diamond$; (ii) $f \in \diamond_{\mathbf{M_G}}(x_1, \ldots, x_n)$ iff $\langle x_1, \ldots, x_n\rangle$ does not fulfil any right rule of $\mathbf{G}$ for $\diamond$; and (iii) $\top \in \diamond_{\mathbf{M_G}}(x_1, \ldots, x_n)$ for every $x_1, \ldots, x_n$.*

Example 5. *Suppose that the rules for $\supset$ and $\wedge$ in some semi-canonical (`-C+A`)-system $\mathbf{G}$ are the usual rules. $\supset_{\mathbf{M_G}}$ and $\wedge_{\mathbf{M_G}}$ are given below.*

$\supset_{\mathbf{M_G}}$	t	f	$\top$
t	$\{t, \top\}$	$\{f, \top\}$	$\{\top\}$
f	$\{t, \top\}$	$\{t, \top\}$	$\{t, \top\}$
$\top$	$\{t, \top\}$	$\{\top\}$	$\{\top\}$

$\wedge_{\mathbf{M_G}}$	t	f	$\top$
t	$\{t, \top\}$	$\{f, \top\}$	$\{\top\}$
f	$\{f, \top\}$	$\{f, \top\}$	$\{f, \top\}$
$\top$	$\{\top\}$	$\{f, \top\}$	$\{\top\}$

The following propositions will be useful in the sequel:

Proposition 5. *Let $\mathbf{G}$ be a semi-canonical (`-C+A`)-system for $\mathcal{L}$. For every n-ary connective $\diamond$ of $\mathcal{L}$, and every $x_1, \ldots, x_n, y_1, \ldots, y_n \in \{t, f, \top\}$, if $x_i \leq y_i$ for every $1 \leq i \leq n$, then $\diamond_{\mathbf{M_G}}(y_1, \ldots, y_n) \subseteq \diamond_{\mathbf{M_G}}(x_1, \ldots, x_n)$.*

Proof. Let $x_1, \ldots, x_n, y_1, \ldots, y_n \in \{t, f, \top\}$. Suppose that $x_i \leq y_i$ for every $1 \leq i \leq n$. Assume that $x \notin \diamond_{\mathbf{M_G}}(x_1, \ldots, x_n)$. We prove that $x \notin \diamond_{\mathbf{M_G}}(y_1, \ldots, y_n)$. Obviously $x \in \{t, f\}$. Assume w.l.o.g. that $x = t$. Then the construction of $\mathbf{M_G}$ implies that $\langle x_1, \ldots, x_n\rangle$ fulfils some left rule r of $\mathbf{G}$ for $\diamond$. Since $x_i \leq y_i$ for every $1 \leq i \leq n$, we have that $\langle y_1, \ldots, y_n\rangle$ fulfils r. It follows that $t \notin \diamond_{\mathbf{M_G}}(y_1, \ldots, y_n)$. $\square$

Proposition 6. *Let $\mathbf{G}$ be a coherent canonical system for $\mathcal{L}$. For every n-ary connective $\diamond$ of $\mathcal{L}$ and $x_1, \ldots, x_n \in \{t, f\}$, $\diamond_{\mathbf{M_{G\text{-}C}}}(x_1, \ldots, x_n) = \diamond_{\mathbf{M_G}}(x_1, \ldots, x_n) \cup \{\top\}$.*

Remark 4. *Proposition 6 entails that for every coherent canonical system* $\mathbf{G}$ $\mathbf{M_G}$ *is a* simple refinement *of* $\mathbf{M}_{\mathbf{G}\text{-}\mathcal{C}}$ *(see [4]). Therefore,* $\vdash_{\mathbf{M}_{\mathbf{G}\text{-}\mathcal{C}}} \subseteq \vdash_{\mathbf{M_G}}$.

Next, we prove the correctness of this construction.

Theorem 4. *For every semi-canonical (*$-\mathcal{C}+\mathcal{A}$*)-system* $\mathbf{G}$*:* (i) $\vdash^{seq}_{\mathbf{G}} = \vdash^{seq}_{\mathbf{M_G}}$*; and* (ii) $\vdash_{\mathbf{G}} = \vdash_{\mathbf{M_G}}$.

Proof. (ii) follows from (i) using Proposition 3. We prove (i).

Soundness. Suppose that $\mathcal{S}_0 \vdash^{seq}_{\mathbf{G}} s_0$. Let v be a valuation in $\mathbf{M_G}$, which is a model of $\mathcal{S}_0$. We prove that v is a model of every sequent appearing in the derivation of s_0 from $\mathcal{S}_0$. It follows that v is a model of s_0. This obviously holds for all sequents of $\mathcal{S}_0$, and preserved by (Weak). To see that it holds in applications of (Id), note that every valuation over $\{t, f, \top\}$ is a model of every sequent of the from $\psi \Rightarrow \psi$. It remains to show that it is preserved in applications of the canonical rules of $\mathbf{G}$. Suppose that $\Gamma_1, \ldots, \Gamma_m \Rightarrow \sigma(\diamond(p_1, \ldots, p_n)), \Delta_1, \ldots, \Delta_m$ is derived from the sequents $\Gamma_i, \sigma(\Pi_i) \Rightarrow \sigma(\Sigma_i), \Delta_i$ for every $1 \leq i \leq m$, using the right canonical rule $r = \{\Pi_1 \Rightarrow \Sigma_1, \ldots, \Pi_m \Rightarrow \Sigma_m\}/ \Rightarrow \diamond(p_1, \ldots, p_n)$ (the proof is similar for left rules). If $v(\psi) \geq f$ for some ψ in some Γ_i, or $v(\psi) \geq t$ for some ψ in some Δ_i then obviously we are done. Assume otherwise. Then, our assumption about the premises of the application entails that for every $1 \leq i \leq m$, either $v(\sigma(p)) \geq f$ for some $p \in \Pi_i$, or $v(\sigma(p)) \geq t$ for some $p \in \Sigma_i$. Thus $\langle v(\sigma(p_1)), \ldots, v(\sigma(p_n))\rangle$ fulfils r. The definition of $\mathbf{M_G}$ ensures that $f \notin \diamond_{\mathbf{M_G}}(v(\sigma(p_1)), \ldots, v(\sigma(p_n)))$. Hence v is a model of $\Gamma_1, \ldots, \Gamma_m \Rightarrow \sigma(\diamond(p_1, \ldots, p_n)), \Delta_1, \ldots, \Delta_m$.

Completeness. Suppose that $\mathcal{S}_0 \nvdash^{seq}_{\mathbf{G}} \Gamma_0 \Rightarrow \Delta_0$. It is a routine matter to obtain *maximal* sets of formulas $\mathcal{T}, \mathcal{U}$ such that $\Gamma_0 \subseteq \mathcal{T}$ and $\Delta_0 \subseteq \mathcal{U}$, and $\mathcal{S}_0 \nvdash^{seq}_{\mathbf{G}} \Gamma \Rightarrow \Delta$ for every finite $\Gamma \subseteq \mathcal{T}$ and $\Delta \subseteq \mathcal{U}$. Define a function $v : Frm_{\mathcal{L}} \to \{t, f, \top\}$ as follows: $v(\psi) = t$ iff $\psi \in \mathcal{T}$, $v(\psi) = f$ iff $\psi \in \mathcal{U}$, and $v(\psi) = \top$ otherwise. The availability of (Id) ensures that $\psi \notin \mathcal{T} \cap \mathcal{U}$ for every ψ, and so v is well-defined. We prove that v is a valuation in $\mathbf{M_G}$. Suppose for contradiction that $v(\diamond(\psi_1, \ldots, \psi_n)) = t$ and $t \notin \diamond_{\mathbf{M_G}}(v(\psi_1), \ldots, v(\psi_n))$ for some formula $\diamond(\psi_1, \ldots, \psi_n)$ (the case that $v(\diamond(\psi_1, \ldots, \psi_n)) = f$ is similar). Then, there exists some left rule r of $\mathbf{G}$ for $\diamond$ which is fulfilled by $\langle v(\psi_1), \ldots, v(\psi_n)\rangle$. Let σ be a substitution such that $\sigma(p_i) = \psi_i$ for all $1 \leq i \leq n$, and let $\Pi \Rightarrow \Sigma$ be a premise of r. We show that there exist finite sets $\Gamma \subseteq \mathcal{T}$ and $\Delta \subseteq \mathcal{U}$ such that $\mathcal{S}_0 \vdash^{seq}_{\mathbf{G}} \Gamma, \sigma(\Pi) \Rightarrow \sigma(\Sigma), \Delta$. Since $\langle v(\psi_1), \ldots, v(\psi_n)\rangle$ fulfils r, there exists some $p_i \in \Pi$ such that $v(\psi_i) \geq f$, or some $p_i \in \Sigma$ such that $v(\psi_i) \geq t$.

Suppose the former holds (the latter is similar). Then $v(\psi_i) \neq t$, and hence $\psi_i \notin \mathcal{T}$. The maximality of $\mathcal{T}$ then entails that there exist finite sets $\Gamma \subseteq \mathcal{T}$ and $\Delta \subseteq \mathcal{U}$, such that $\mathcal{S}_0 \vdash^{seq}_{\mathbf{G}} \Gamma, \psi_i \Rightarrow \Delta$. Using (Weak) we obtain $\mathcal{S}_0 \vdash^{seq}_{\mathbf{G}} \Gamma, \sigma(\Pi) \Rightarrow \sigma(\Sigma), \Delta$. Now, by applying r on $\Gamma, \sigma(\Pi) \Rightarrow \sigma(\Sigma), \Delta$ for every premise $\Pi \Rightarrow \Sigma$ of r, we obtain that $\mathcal{S}_0 \vdash^{seq}_{\mathbf{G}} \Gamma, \diamond(\psi_1, \ldots, \psi_n) \Rightarrow \Delta$, for some finite sets $\Gamma \subseteq \mathcal{T}$ and $\Delta \subseteq \mathcal{U}$. This implies that $\diamond(\psi_1, \ldots, \psi_n) \notin \mathcal{T}$. But, this contradicts the fact that $v(\diamond(\psi_1, \ldots, \psi_n)) = t$.

Now, note that for every $\Gamma \Rightarrow \Delta \in \mathcal{S}_0$, we have $\mathcal{S}_0 \vdash^{seq}_{\mathbf{G}} \Gamma \Rightarrow \Delta$, and so there exists either some $\psi \in \Gamma$ such that $\psi \notin \mathcal{T}$, or some $\psi \in \Delta$ such that $\psi \notin \mathcal{U}$. Hence, there exists either some $\psi \in \Gamma$ such that $v(\psi) \geq f$, or some $\psi \in \Delta$ such that $v(\psi) \geq t$. Thus v is a model of $\Gamma \Rightarrow \Delta$. Finally, since $\Gamma_0 \subseteq \mathcal{T}$ and $\Delta_0 \subseteq \mathcal{U}$, $v(\psi) = t$ for every $\psi \in \Gamma_0$, and $v(\psi) = f$ for every $\psi \in \Delta_0$. Thus v is not a model of $\Gamma_0 \Rightarrow \Delta_0$. □

Remark 5. *The use of at least* three *truth-values is necessary in order to provide Nmatrices for every semi-canonical* ($-$C+A)*-system. To see this, consider for example a semi-canonical* ($-$C+A)*-system* $\mathbf{G}$*, that includes only the usual rules for* $\neg$*. Suppose for contradiction that there is a* 2*Nmatrix* $\mathbf{M}$*, such that* $\vdash_{\mathbf{G}} = \vdash_{\mathbf{M}}$*. Assume that* $\mathcal{V}_{\mathbf{M}} = \{t, f\}$ *and* $\mathcal{D}_{\mathbf{M}} = \{t\}$ *(obviously, any* 2*Nmatrix is isomorphic to such an Nmatrix). Then, since* $p_1, \neg p_1 \not\vdash_{\mathbf{G}} p_2$ *(this can be verified using Theorem 4),* $t \in \diamond_{\mathbf{M}}(t)$*. Since* $p_1 \not\vdash_{\mathbf{G}} \neg p_1$*,* $f \in \neg_{\mathbf{M}}(t)$*, and so* $\neg_{\mathbf{M}}(t) = \{t, f\}$*. Consider a partial valuation defined by* $v(p_1) = t$*,* $v(\neg p_1) = t$*,* $v(\neg\neg p_1) = f$ *and* $v(\psi)$ *is not defined for every other* $\psi \in Frm_{\mathcal{L}}$*, Then* v *is a partial valuation in* $\mathbf{M}$ *which is a model of* p_1 *but not of* $\neg\neg p_1$*. By Corollary 1,* $p_1 \not\vdash_{\mathbf{M}} \neg\neg p_1$*. This contradicts the fact that* $p_1 \vdash_{\mathbf{G}} \neg\neg p_1$*.*

Example 6. *Using the Nmatrix semantics of semi-canonical* ($-$C+A)*-systems, one can easily verify that using (Cut) is unavoidable in a given derivation. For example, we obviously have* $p_1 \supset p_2 \vdash_{\mathbf{LK}} p_1 \supset (p_3 \supset p_2)$*. Consider a partial valuation* v *in* $\mathbf{M}_{\mathbf{LK}_{-C}}$ *such that* $v(p_1) = v(p_3) = t$*,* $v(p_2) = v(p_3 \supset p_2) = f$*,* $v(p_1 \supset (p_3 \supset p_2)) = f$*,* $v(p_1 \supset p_2) = \top$*, and* $v(\psi)$ *is not defined for every other formula* ψ *(it is straightforward to verify that this is indeed a partial valuation in* $\mathbf{M}_{\mathbf{LK}_{-C}}$*).* v *is a model of* $p_1 \supset p_2$ *but not of* $p_1 \supset (p_3 \supset p_2)$*. By Corollary 1,* $p_1 \supset p_2 \not\vdash_{\mathbf{M}_{\mathbf{LK}_{-C}}} p_1 \supset (p_3 \supset p_2)$*. Theorem 4 entails that* $p_1 \supset p_2 \not\vdash_{\mathbf{LK}_{-C}} p_1 \supset (p_3 \supset p_2)$*. In other words,* $\Rightarrow p_1 \supset (p_3 \supset p_2)$ *has no cut-free derivation in* $\mathbf{LK}$ *from* $\Rightarrow p_1 \supset p_2$*.*

4.3 (+C−A)-Systems

Semantics for coherent semi-canonical (+C−A)-systems is given below in the form of 3Nmatrices. Again, we show that non-determinism is essential.

Theorem 5. *Suppose that $\mathcal{L}$ contains a unary connective, denoted by $\neg$, and let $\mathbf{G}$ be a semi-canonical ($\texttt{-A}$)-system, whose rules for $\neg$ are the usual rules. There is no finite (ordinary) matrix $\mathbf{M}$ such that $\vdash_{\mathbf{G}}=\vdash_{\mathbf{M}}$.*

Proof. Note that (i) $\neg^n p_1 \vdash_{\mathbf{G}} \neg^m p_1$ for every $n, m \in \mathbb{N}_{even}$ provided that $n \leq m$; and (ii) $\neg^{n+2} p_1 \not\vdash_{\mathbf{G}} \neg^n p_1$ for every $n \in \mathbb{N}$ (one can verify this using Theorem 6 below). Now, assume that $\vdash_{\mathbf{G}}=\vdash_{\mathbf{M}}$ for some (ordinary) matrix. For every $n \geq 0$, let v_n be a valuation in $\mathbf{M}$, which is a model of $\neg^{n+2} p_1$, and not a model of $\neg^n p_1$. We show that $v_n(p_1) \neq v_m(p_1)$ for every $n, m \in \mathbb{N}_{even}$ such that $n < m$ (and so, $\mathbf{M}$ is infinite). Let $n, m \in \mathbb{N}_{even}$ such that $n < m$. Then, since v_n is a model of $\neg^{n+2} p_1$, (i) implies that v_n is a model of $\neg^m p_1$. On the other hand, v_m is not a model of $\neg^m p_1$. This implies (using the fact that $\mathbf{M}$ is deterministic) that $v_n(p_1) \neq v_m(p_1)$. $\square$

Next we turn to the construction of 3Nmatrices (with the truth-values t, f and $\perp$) for coherent semi-canonical ($\texttt{+C-A}$)-systems. Note that the coherence of $\mathbf{G}$ ensures that $\mathbf{M_G}$ constructed below is well-defined (see Proposition 4).

Definition 17. *Let $\mathbf{G}$ be a coherent semi-canonical ($\texttt{+C-A}$)-system for $\mathcal{L}$. $\mathbf{M_G}$ is defined by $\mathcal{V}_{\mathbf{M_G}} = \{t, f, \perp\}$, $\mathcal{D}_{\mathbf{M_G}} = \{t\}$, and for every n-ary connective $\diamond$ of $\mathcal{L}$: (i) $t \in \diamond_{\mathbf{M_G}}(x_1, \ldots, x_n)$ iff $\langle x_1, \ldots, x_n\rangle$ does not fulfil any left rule of $\mathbf{G}$ for $\diamond$; (ii) $f \in \diamond_{\mathbf{M_G}}(x_1, \ldots, x_n)$ iff $\langle x_1, \ldots, x_n\rangle$ does not fulfil any right rule of $\mathbf{G}$ for $\diamond$; and (iii) $\perp \in \diamond_{\mathbf{M_G}}(x_1, \ldots, x_n)$ iff $\langle x_1, \ldots, x_n\rangle$ does not fulfil any rule of $\mathbf{G}$ for $\diamond$.*

Example 7. *Suppose that the rules for $\supset$ and $\wedge$ in a coherent semi-canonical ($\texttt{+C-A}$)-system $\mathbf{G}$ are the usual rules. $\supset_{\mathbf{M_G}}$ and $\wedge_{\mathbf{M_G}}$ are given below.*

$\supset_{\mathbf{M_G}}$	t	f	$\perp$
t	$\{t\}$	$\{f\}$	$\{t, f, \perp\}$
f	$\{t\}$	$\{t\}$	$\{t\}$
$\perp$	$\{t\}$	$\{t, f, \perp\}$	$\{t, f, \perp\}$

$\wedge_{\mathbf{M_G}}$	t	f	$\perp$
t	$\{t\}$	$\{f\}$	$\{t, f, \perp\}$
f	$\{f\}$	$\{f\}$	$\{f\}$
$\perp$	$\{t, f, \perp\}$	$\{f\}$	$\{t, f, \perp\}$

Example 8. *Consider a unary connective $\triangledown$ defined by the canonical rule: $\{p_1 \Rightarrow p_1\}$ / $\Rightarrow \triangledown p_1$. Its applications allow to infer $\Gamma \Rightarrow \triangledown\varphi, \Delta$ from $\Gamma, \varphi \Rightarrow \varphi, \Delta$. According to the construction above, the semantics of $\triangledown$ is given by: $\triangledown_{\mathbf{M_G}}(t) = \triangledown_{\mathbf{M_G}}(f) = \{t\}$, and $\triangledown_{\mathbf{M_G}}(\perp) = \{t, f, \perp\}$.*

Theorem 6. *For every coherent semi-canonical ($\texttt{+C-A}$)-system $\mathbf{G}$: (i) $\vdash^{seq}_{\mathbf{G}} = \vdash^{seq}_{\mathbf{M_G}}$; and (ii) $\vdash_{\mathbf{G}} = \vdash_{\mathbf{M_G}}$.*

Proof. (ii) follows from (i) using Proposition 3. We prove (i).

Soundness. Suppose that $\mathcal{S}_0 \vdash^{seq}_{\mathbf{G}} s_0$. Let v be a valuation in $\mathbf{M_G}$, which is a model of $\mathcal{S}_0$. We prove that v is a model of every sequent appearing in the derivation of s_0 from $\mathcal{S}_0$. This obviously holds for all sequents of $\mathcal{S}_0$. As in the proof of Theorem 4, one shows that this property preserved by (Weak) and by applications of the canonical rules of $\mathbf{G}$. It remains to consider applications of (Cut). Suppose that $\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2$ is derived from the sequents $\Gamma_1 \Rightarrow \psi, \Delta_1$ and $\Gamma_2, \psi \Rightarrow \Delta_2$. If $v(\psi) \neq t$ (i.e. $v(\psi) \in \{f, \bot\}$) then our assumption about $\Gamma_1 \Rightarrow \psi, \Delta_1$ entails that $v(\varphi) = f$ for some $\varphi \in \Gamma_1$, or $v(\varphi) = t$ for some $\varphi \in \Delta_1$. Otherwise, $v(\psi) = t$, and our assumption about $\Gamma_2, \psi \Rightarrow \Delta_2$ entails that $v(\varphi) = f$ for some $\varphi \in \Gamma_2$, or $v(\varphi) = t$ for some $\varphi \in \Delta_2$. In any case, v is a model of $\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2$.

Completeness. Suppose that $\mathcal{S}_0 \nvdash^{seq}_{\mathbf{G}} \Gamma_0 \Rightarrow \Delta_0$. It is a routine matter to obtain *maximal* sets of formulas $\mathcal{T}$,$\mathcal{U}$ such that $\Gamma_0 \subseteq \mathcal{T}$ and $\Delta_0 \subseteq \mathcal{U}$, and $\mathcal{S}_0 \nvdash^{seq}_{\mathbf{G}} \Gamma \Rightarrow \Delta$ for every finite $\Gamma \subseteq \mathcal{T}$ and $\Delta \subseteq \mathcal{U}$. Next, we recursively construct a valuation v in $\mathbf{M_G}$. For atomic formulas $v(p) = \bot$ iff $p \in \mathcal{T} \cap \mathcal{U}$, $v(p) = t$ iff $p \in \mathcal{T}$ and $p \notin \mathcal{U}$, and otherwise $v(p) = f$. Now let $\diamond$ be an n-ary connective of $\mathcal{L}$, and suppose that $v(\psi_1), \ldots, v(\psi_n)$ were defined. We define $v(\diamond(\psi_1, \ldots, \psi_n)) = x$ if $\diamond_{\mathbf{M_G}}(v(\psi_1), \ldots, v(\psi_n)) = \{x\}$ (for $x \in \{t, f\}$), and otherwise $v(\diamond(\psi_1, \ldots, \psi_n))$ is defined in the same way as $v(p)$ is defined above. v is obviously a valuation in $\mathbf{M_G}$.

Next, we simultaneously prove by induction on the build-up of formulas that: (a) if $\varphi \in \mathcal{T}$ then $v(\varphi) \neq f$; and (b) if $\varphi \in \mathcal{U}$ then $v(\varphi) \neq t$. For atomic formulas (a) and (b) directly follow from the definition above. Let $\diamond$ be an n-ary connective of $\mathcal{L}$. Suppose (a) and (b) hold for $\psi_1, \ldots, \psi_n$, and let $\theta = \diamond(\psi_1, \ldots, \psi_n)$. We show that (a) holds for θ. (b) is proved similarly. Suppose that $\theta \in \mathcal{T}$. We prove that $\diamond_{\mathbf{M_G}}(v(\psi_1), \ldots, v(\psi_n)) \neq \{f\}$. The definition of v then ensures that $v(\theta) \neq f$. Assume (for contradiction) that $\diamond_{\mathbf{M_G}}(v(\psi_1), \ldots, v(\psi_n)) = \{f\}$. According to the construction of $\mathbf{M_G}$, there exists some left rule r of $\mathbf{G}$ for $\diamond$, which is fulfilled by $\langle v(\psi_1), \ldots, v(\psi_n)\rangle$. Let σ be a substitution such that $\sigma(p_i) = \psi_i$ for all $1 \leq i \leq n$, and let $\Pi \Rightarrow \Sigma$ be a premise of r. We show that there exist finite sets $\Gamma \subseteq \mathcal{T}$ and $\Delta \subseteq \mathcal{U}$ such that $\mathcal{S}_0 \vdash^{seq}_{\mathbf{G}} \Gamma, \sigma(\Pi) \Rightarrow \sigma(\Sigma), \Delta$. Since $\langle v(\psi_1), \ldots, v(\psi_n)\rangle$ fulfils r, there is some $p_i \in \Pi$ such that $v(\psi_i) = f$ or some $p_i \in \Sigma$ such that $v(\psi_i) = t$. Suppose the former holds (the latter is similar). Then by the induction hypothesis $\psi_i \notin \mathcal{T}$. The maximality of $\mathcal{T}$ then entails that there exist finite sets $\Gamma \subseteq \mathcal{T}$ and $\Delta \subseteq \mathcal{U}$, such that $\mathcal{S}_0 \vdash^{seq}_{\mathbf{G}} \Gamma, \psi_i \Rightarrow \Delta$. Using (Weak) we obtain $\mathcal{S}_0 \vdash^{seq}_{\mathbf{G}} \Gamma, \sigma(\Pi) \Rightarrow \sigma(\Sigma), \Delta$. Now, by applying r on

$\Gamma, \sigma(\Pi) \Rightarrow \sigma(\Sigma), \Delta$ for every premise $\Pi \Rightarrow \Sigma$ of r, we obtain a derivation from $\mathcal{S}_0$ of $\Gamma, \theta \Rightarrow \Delta$ in $\mathbf{G}$, for some finite sets $\Gamma \subseteq \mathcal{T}$ and $\Delta \subseteq \mathcal{U}$. But since $\theta \in \mathcal{T}$, this contradicts the properties of $\mathcal{T}$ and $\mathcal{U}$.

It remains to show that v is a model of $\mathcal{S}_0$, but not of $\Gamma_0 \Rightarrow \Delta_0$. For the latter, note that since $\Gamma_0 \subseteq \mathcal{T}$ and $\Delta_0 \subseteq \mathcal{U}$, (a) and (b) imply that v is not a model in $\Gamma_0 \Rightarrow \Delta_0$. Now let $\Gamma \Rightarrow \Delta$ be a sequent of $\mathcal{S}_0$. Suppose first that $\Gamma \Rightarrow \Delta$ is a sequent of the form $\varphi \Rightarrow \varphi$. Since $\varphi \Rightarrow \varphi$ has a derivation from $\mathcal{S}_0$, $\varphi \notin \mathcal{T} \cap \mathcal{U}$. Hence, $v(\varphi) \neq \bot$, and so v is a model of $\Gamma \Rightarrow \Delta$. Next, suppose that $\Gamma \Rightarrow \Delta$ is not a sequent of the form $\varphi \Rightarrow \varphi$. Note that for every $\psi \in \Gamma \cup \Delta$ either $\psi \in \mathcal{T}$ or $\psi \in \mathcal{U}$. (Otherwise, there exist sets $\Gamma_1, \Gamma_2 \subseteq \mathcal{T}$ and $\Delta_1, \Delta_2 \subseteq \mathcal{U}$, such that $\mathcal{S}_0 \vdash^{seq}_{\mathbf{G}} \Gamma_1 \Rightarrow \psi, \Delta_1$ and $\mathcal{S}_0 \vdash^{seq}_{\mathbf{G}} \Gamma_2, \psi \Rightarrow \Delta_2$. By applying (Cut) on ψ, $\mathcal{S}_0 \vdash^{seq}_{\mathbf{G}} \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2$, in contradiction to the properties of $\mathcal{T}$ and $\mathcal{U}$.) Since $\mathcal{S}_0 \nvdash^{seq}_{\mathbf{G}} \Gamma \Rightarrow \Delta$, there either exists some $\psi \in \Gamma$ such that $\psi \notin \mathcal{T}$, or some $\psi \in \Delta$ such that $\psi \notin \mathcal{U}$. It follows that there either exists some $\psi \in \Gamma$ such that $\psi \notin \mathcal{T}$ and $\psi \in \mathcal{U}$, or some $\psi \in \Delta$ such that $\psi \in \mathcal{T}$ and $\psi \notin \mathcal{U}$. Using (a) and (b) (and the fact that $v(\psi) = \bot$ only if $\psi \in \mathcal{T} \cap \mathcal{U}$), it follows that $v(\psi) = f$ for some $\psi \in \Gamma$, or $v(\psi) = t$ for some $\psi \in \Delta$. Hence v is a model of $\Gamma \Rightarrow \Delta$. □

Example 9. *Using the Nmatrix semantics of semi-canonical (*`+C−A`*)-systems, one can easily verify that using (Id) is unavoidable in a given derivation. For example, clearly* $\neg p_1 \vdash_{\mathbf{LK}} \neg(p_1 \wedge p_2)$. *It is easy to see that a function* v *defined by* $v(p_1) = \bot$, $v(\neg p_1) = v(p_2) = v(p_1 \wedge p_2) = t$, $v(\neg(p_1 \wedge p_2)) = f$ *(*v *is not defined for other formulas) is a partial valuation in* $\mathbf{M}_{\mathbf{LK}\text{-}\mathtt{A}}$*, which is a model of* $\neg p_1$ *but not of* $\neg(p_1 \wedge p_2)$*. By Corollary 1,* $\neg p_1 \nvdash_{\mathbf{M}_{\mathbf{LK}\text{-}\mathtt{A}}} \neg(p_1 \wedge p_2)$*. Theorem 6 entails that* $\neg p_1 \nvdash_{\mathbf{LK}\text{-}\mathtt{A}} \neg(p_1 \wedge p_2)$*. In other words,* $\Rightarrow \neg(p_1 \wedge p_2)$ *has no identity-axiom-free derivation in* $\mathbf{LK}$ *from* $\Rightarrow \neg p_1$*.*

4.4 (−C−A)-Systems

Finally, we provide Nmatrix semantics for semi-canonical (`−C−A`)-systems, in the form of 4Nmatrices (using the truth-values $t, f, \top$ and $\bot$).

Definition 18. *Let* $\mathbf{G}$ *be a semi-canonical (*`−C−A`*)-system for* $\mathcal{L}$*.* $\mathbf{M}_{\mathbf{G}}$ *is defined by* $\mathcal{V}_{\mathbf{M}_{\mathbf{G}}} = \{t, f, \top, \bot\}$, $\mathcal{D}_{\mathbf{M}_{\mathbf{G}}} = \{t\}$*, and for every* n*-ary connective* $\diamond$ *of* $\mathcal{L}$*:* (i) $t \in \diamond_{\mathbf{M}_{\mathbf{G}}}(x_1, \ldots, x_n)$ *iff* $\langle x_1, \ldots, x_n \rangle$ *does not fulfil any left rule of* $\mathbf{G}$ *for* $\diamond$*;* (ii) $f \in \diamond_{\mathbf{M}_{\mathbf{G}}}(x_1, \ldots, x_n)$ *iff* $\langle x_1, \ldots, x_n \rangle$ *does not fulfil any right rule of* $\mathbf{G}$ *for* $\diamond$*;* (iii) $\bot \in \diamond_{\mathbf{M}_{\mathbf{G}}}(x_1, \ldots, x_n)$ *iff* $\langle x_1, \ldots, x_n \rangle$ *does not fulfil any rule of* $\mathbf{G}$ *for* $\diamond$*; and* (iv) $\top \in \diamond_{\mathbf{M}_{\mathbf{G}}}(x_1, \ldots, x_n)$ *for every* $x_1, \ldots, x_n$*.*

Example 10. *Suppose that the rules for* $\supset$ *in some semi-canonical (*`−C−A`*)-*

system $\mathbf{G}$ *are the usual rules.* $\supset_{\mathbf{M_G}}$ *is given below.*

$\supset_{\mathbf{M_G}}$	t	f	$\top$	$\bot$
t	$\{t, \top\}$	$\{f, \top\}$	$\{\top\}$	$\{t, f, \top, \bot\}$
f	$\{t, \top\}$	$\{t, \top\}$	$\{t, \top\}$	$\{t, \top\}$
$\top$	$\{t, \top\}$	$\{\top\}$	$\{\top\}$	$\{t, \top\}$
$\bot$	$\{t, \top\}$	$\{t, f, \top, \bot\}$	$\{t, \top\}$	$\{t, f, \top, \bot\}$

Theorem 7. *For every semi-canonical (*$-$`C`$-$`A`*)-system* $\mathbf{G}$*:* $(i) \vdash^{seq}_{\mathbf{G}} = \vdash^{seq}_{\mathbf{M_G}}$*; and* $(ii) \vdash_{\mathbf{G}} = \vdash_{\mathbf{M_G}}$.

Proof Outline. (ii) follows from (i) using Proposition 3. Soundness in (i) is proved similarly to the proof of Theorem 4. Completeness is also very similar, where the refuting valuation v in $\mathbf{M_G}$ is defined as follows: $v(\psi) = t$ iff $\psi \in \mathcal{T}$ and $\psi \notin \mathcal{U}$, $v(\psi) = f$ iff $\psi \in \mathcal{U}$ and $\psi \notin \mathcal{T}$, $v(\psi) = \bot$ iff $\psi \in \mathcal{T}$ and $\psi \in \mathcal{U}$, and $v(\psi) = \top$ otherwise. We leave the details for the reader. $\square$

5 PROOF-THEORETIC APPLICATIONS

The semantics for semi-canonical system can be easily exploited to obtain some proof-theoretic properties of semi-canonical and canonical systems. First, we provide a new proof of the fact that all coherent canonical systems enjoy cut-admissibility.

Definition 19. *Let* $\mathbf{G}$ *be a (*$+$`C`*)-system.* $\mathbf{G}$ *enjoys* cut-admissibility *if* $\vdash^{seq}_{\mathbf{G}} s$ *implies* $\vdash^{seq}_{\mathbf{G}-\mathtt{C}} s$*.* $\mathbf{G}$ *enjoys* strong cut-admissibility *if* $\mathcal{S} \vdash^{seq}_{\mathbf{G}} s$ *implies that there exists a derivation of* s *from* $\mathcal{S}$ *in* $\mathbf{G}$ *in which only formulas occurring in* $\mathcal{S}$ *serve as cut-formulas. (A formula* φ *occurs in* $\mathcal{S}$*, if there exists some sequent* $\Gamma \Rightarrow \Delta \in \mathcal{S}$ *such that* $\varphi \in \Gamma \cup \Delta$*.)*

Strong cut-admissibility is clearly stronger than cut-admissibility. In semi-canonical ($+$`C`$+$`A`)-systems, they are actually equivalent (proved in [1] for $\mathbf{LK}$, but the proof works the same for any semi-canonical ($+$`C`$+$`A`)-system). The following semantic property is the key for cut-admissibility.

Theorem 8. *Let* $\mathbf{G}$ *be a coherent canonical system. If every valuation in* $\mathbf{M_G}$ *is a model of a sequent* s*, then so does every valuation in* $\mathbf{M}_{\mathbf{G}-\mathtt{C}}$*.*

Proof. Suppose that there exists a valuation v' in $\mathbf{M}_{\mathbf{G}-\mathtt{C}}$ which is not a model of s. We construct a function $v : \mathit{Frm}_{\mathcal{L}} \to \{t, f\}$, such that v is a valuation in $\mathbf{M_G}$, and $v(\psi) \leq v'(\psi)$ for every $\psi \in \mathit{Frm}_{\mathcal{L}}$. In particular, v will not be a model of s. The construction of v is done by recursion on the build-up of formulas. First, for atomic formulas, we (arbitrarily) choose $v(p)$ to be either t or f, so that $v(p) \leq v'(p)$ would hold. Now, let $\diamond$ be an n-ary connective of $\mathcal{L}$, and suppose $v(\psi_i)$ was defined for every $1 \leq i \leq n$. We choose $v(\diamond(\psi_1, \ldots, \psi_n))$ to be equal to $v'(\diamond(\psi_1, \ldots, \psi_n))$, if the latter is either t or f. Otherwise, we choose $v(\diamond(\psi_1, \ldots, \psi_n))$ to be some element of $\diamond_{\mathbf{M_G}}(v(\psi_1), \ldots, v(\psi_n))$. Obviously, $v(\psi) \leq v'(\psi)$ for every $\psi \in \mathit{Frm}_{\mathcal{L}}$. To see that v is a valuation in $\mathbf{M_G}$, suppose w.l.o.g. that $\diamond_{\mathbf{M_G}}(v(\psi_1), \ldots, v(\psi_n)) = \{t\}$. We show that $v(\diamond(\psi_1, \ldots, \psi_n)) = t$. By Proposition 6, $f \notin \diamond_{\mathbf{M}_{\mathbf{G}\text{-}\mathtt{C}}}(v(\psi_1), \ldots, v(\psi_n))$. Now, since $v(\psi) \leq v'(\psi)$ for every $\psi \in \mathit{Frm}_{\mathcal{L}}$, Proposition 5 entails that $f \notin \diamond_{\mathbf{M}_{\mathbf{G}\text{-}\mathtt{C}}}(v'(\psi_1), \ldots, v'(\psi_n))$. Hence, since v' is a valuation in $\mathbf{M}_{\mathbf{G}\text{-}\mathtt{C}}$, $v'(\diamond(\psi_1, \ldots, \psi_n)) \neq f$. Now, if $v'(\diamond(\psi_1, \ldots, \psi_n)) = \top$, then $v(\diamond(\psi_1, \ldots, \psi_n)) \in \diamond_{\mathbf{M_G}}(v(\psi_1), \ldots, v(\psi_n))$ by definition. Otherwise, $v(\diamond(\psi_1, \ldots, \psi_n)) = v'(\diamond(\psi_1, \ldots, \psi_n)) = t$. □

Corollary 3. *Every coherent canonical system enjoys cut-admissibility.*

Proof. Suppose that $\vdash^{seq}_{\mathbf{G}} s$. Then, by Theorem 2, every valuation in $\mathbf{M_G}$ is a model of s. By Theorem 8, every valuation in $\mathbf{M}_{\mathbf{G}\text{-}\mathtt{C}}$ is a model of s. Theorem 4 implies that $\vdash^{seq}_{\mathbf{G}\text{-}\mathtt{C}} s$. □

Next, we show that Theorem 6 and its proof entail that coherent semi-canonical (+C−A)-systems also enjoy strong cut-admissibility. In fact we prove something stronger:

Corollary 4. *Let* $\mathbf{G}$ *be a coherent semi-canonical (+C−A)-system. Assume that* $\mathcal{S} \vdash^{seq}_{\mathbf{G}} s$*. Then there exists a derivation of s from* $\mathcal{S}$ *in* $\mathbf{G}$*, in which only formulas occurring in* $\mathcal{S} \setminus \{\psi \Rightarrow \psi \mid \psi \in \mathit{Frm}_{\mathcal{L}}\}$ *serve as cut-formulas.*

Proof. Suppose that there does not exist such a derivation. We prove that $\mathcal{S} \not\vdash^{seq}_{\mathbf{G}} s$. By Theorem 6, it suffices to show that there exists a valuation in $\mathbf{M_G}$, which is a model of $\mathcal{S}$, but not of s. Such a valuation is constructed exactly like in the proof of Theorem 6. Note that only cuts on formulas occurring in $\mathcal{S}_0 \setminus \{\psi \Rightarrow \psi \mid \psi \in \mathit{Frm}_{\mathcal{L}}\}$ are needed in this proof. □

Finally, the semantics of coherent semi-canonical (+C−A)-systems can be used to provide a new proof of the equivalence between axiom-expansion and determinism in coherent canonical systems (shown first in [2]).

Definition 20. *Let* $\diamond$ *be an* n*-ary connective.*

- $\diamond$ *admits* axiom-expansion *in a canonical system* $\mathbf{G}$ *if* $\{p_i \Rightarrow p_i \mid 1 \leq i \leq n\} \vdash^{seq}_{\mathbf{G}\text{-}C\text{-}A} \diamond(p_1, \ldots, p_n) \Rightarrow \diamond(p_1, \ldots, p_n)$.

- $\diamond$ *is* deterministic *in an Nmatrix* $\mathbf{M}$ *if* $\diamond_{\mathbf{M}}$ *takes only singleton values.*

Theorem 9. *A connective* $\diamond$ *admits axiom-expansion in a coherent canonical system* $\mathbf{G}$ *iff* $\diamond$ *is deterministic in* $\mathbf{M}_{\mathbf{G}}$.

Proof. Let $\diamond$ be an n-ary connective. Denote the sequent $\diamond(p_1,\ldots,p_n) \Rightarrow \diamond(p_1,\ldots,p_n)$ by s_0, and the set $\{p_i \Rightarrow p_i \mid 1 \leq i \leq n\}$ by $\mathcal{S}_0$. Suppose that $\mathcal{S}_0 \not\vdash^{seq}_{\mathbf{G}\text{-}\mathtt{C}\text{-}\mathtt{A}} s_0$. By Corollary 4, $\mathcal{S}_0 \not\vdash_{\mathbf{G}\text{-}\mathtt{A}} s_0$. By Theorem 6, there exists some valuation v in $\mathbf{M}_{\mathbf{G}\text{-}\mathtt{A}}$, which is a model of $\mathcal{S}_0$, but not of s_0. Therefore, $v(p_i) \in \{t,f\}$ for every $1 \leq i \leq n$, and $v(\diamond(p_1,\ldots,p_n)) = \perp$. Thus $\perp \in \diamond_{\mathbf{M}_{\mathbf{G}\text{-}\mathtt{A}}}(v(p_1),\ldots,v(p_n))$. The construction of $\mathbf{M}_{\mathbf{G}\text{-}\mathtt{A}}$ then entails that $\langle v(p_1),\ldots,v(p_n)\rangle$ does not fulfil any rule for $\diamond$ in $\mathbf{G}$. This implies that $\diamond_{\mathbf{M}_{\mathbf{G}}}(v(p_1),\ldots,v(p_n)) = \{t,f\}$.

Now, assume that $\diamond_{\mathbf{M}_{\mathbf{G}}}(x_1,\ldots,x_n) = \{t,f\}$ for $x_1,\ldots,x_n \in \{t,f\}$. Hence, $\langle x_1,\ldots,x_n\rangle$ does not fulfil any rule for $\diamond$ of $\mathbf{G}$. In this case, we have $\diamond_{\mathbf{M}_{\mathbf{G}\text{-}\mathtt{A}}}(x_1,\ldots,x_n) = \{t,f,\perp\}$. Consider a partial valuation v in $\mathbf{M}_{\mathbf{G}\text{-}\mathtt{A}}$, that assigns x_i to p_i for $1 \leq i \leq n$, and $\perp$ to $\diamond(p_1,\ldots,p_n)$. By Proposition 2, there exists a (full) valuation v' in $\mathbf{M}_{\mathbf{G}\text{-}\mathtt{A}}$ that extends v. v' is a model of $\mathcal{S}_0$, but not of s_0. By Theorem 6, $\mathcal{S}_0 \not\vdash^{seq}_{\mathbf{G}\text{-}\mathtt{A}} s_0$. It follows that $\mathcal{S}_0 \not\vdash^{seq}_{\mathbf{G}\text{-}\mathtt{C}\text{-}\mathtt{A}} s_0$. $\square$

6 CONCLUSIONS AND FURTHER RESEARCH

The results of this paper shed light on the semantic roles of (Cut) and (Id) in sequent systems. The availability of each of these two components rules out one truth-value ($\top$ or $\perp$), and reduces the level of non-determinism in the induced Nmatrix. The presence of both of them (as happens in $\mathbf{LK}$) is crucial for having deterministic semantics.

It would be interesting and useful to extend the results of this paper to broader families of sequent (and higher-level objects) systems without (Cut) and/or (Id). This includes first-order canonical systems, many-sided sequent systems, and systems allowing larger variety of logical rules. Finally, a natural question, arising from the definition of semi-canonical systems, concerns the necessity of the weakening rule. It is interesting to understand the semantic effect of (Weak) in canonical systems, and develop semantics for systems lacking this rule. Finite Nmatrices do not suffice for this purpose:

Proposition 7. *Suppose that* $\mathcal{L}$ *contains two binary connectives denoted by* $\vee$ *and* $\sqcup$. *Let* $\mathbf{G}$ *be a sequent system consisting only on (Cut), (Id) and the canonical rules:* $\{\Rightarrow p_1, p_2\}/\Rightarrow p_1 \vee p_2$, $\{\Rightarrow p_1\}/\Rightarrow p_1 \sqcup p_2$, *and* $\{\Rightarrow p_2\}/\Rightarrow p_1 \sqcup p_2$. *There is no finite Nmatrix* $\mathbf{M}$ *such that* $\vdash_{\mathbf{G}} = \vdash_{\mathbf{M}}$.

Proof Outline. Suppose otherwise, and let $\mathbf{M}$ be a finite Nmatrix for which $\vdash_{\mathbf{G}}=\vdash_{\mathbf{M}}$. Let n denote the size of $\mathcal{V}_{\mathbf{M}}$ and let $m = \binom{n+1}{2}$. Let $\varphi_1, \ldots, \varphi_m$ be an enumeration of all formulas in the set $\{p_i \vee p_j \mid 1 \leq i < j \leq n+1\}$. Let $\psi_1 = \varphi_1$, and for every $2 \leq i \leq m$ let $\psi_i = \psi_{i-1} \sqcup \varphi_i$. It is possible to show that $p_1, \ldots, p_m \not\vdash_{\mathbf{G}} \psi_m$. We show that every valuation in $\mathbf{M}$ which is a model of $p_1, \ldots, p_m$ is also a model of ψ_m. Let v be a valuation in $\mathbf{M}$, and suppose that $v(p_i) \in \mathcal{D}_{\mathbf{M}}$ for every $1 \leq i \leq m$. Since $|\mathcal{V}_{\mathbf{M}}| = n$, there exist $1 \leq i < j \leq n+1$ such that $v(p_i) = v(p_j)$. Denote by k the index for which $\varphi_k = p_i \vee p_j$. Note that $p_1 \vdash_{\mathbf{G}} p_1 \vee p_1$, and so for every $x \in \mathcal{D}_{\mathbf{M}}$, $\vee_{\mathbf{M}}(x, x) \subseteq \mathcal{D}_{\mathbf{M}}$. It follows that $\vee_{\mathbf{M}}(v(p_i), v(p_j)) \subseteq \mathcal{D}_{\mathbf{M}}$. Hence, $v(\varphi_k) \in \mathcal{D}_{\mathbf{M}}$. This entails that $v(\psi_m) \in \mathcal{D}_{\mathbf{M}}$ (since $p_1 \vdash_{\mathbf{G}} p_1 \sqcup p_2$ and $p_2 \vdash_{\mathbf{G}} p_1 \sqcup p_2$, we have $\sqcup_{\mathbf{M}}(x, y) \subseteq \mathcal{D}_{\mathbf{M}}$ if $x \in \mathcal{D}_{\mathbf{M}}$ or $y \in \mathcal{D}_{\mathbf{M}}$). $\square$

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