Automated Support for the Investigation of Paraconsistent and Other Logics*

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Abstract. We automate the construction of analytic sequent calculi and effective semantics for a large class of logics formulated as Hilbert calculi. Our method applies to infinitely many logics, which include the family of paraconsistent C-systems, as well as to other logics for which neither analytic calculi nor suitable semantics have so far been available.

1 Introduction

Non-classical logics are often introduced using Hilbert systems. Intuitionistic, modal and paraconsistent logics are just a few cases in point. The usefulness of such logics, however, strongly depends on two essential components. The first is an intuitive *semantics*, which can provide insights into the logic. A desirable property of such semantics is *effectiveness*, in the sense that it naturally induces a decision procedure for the logic. Examples of such semantics include finite-valued matrices, and their generalizations: non-deterministic finite-valued matrices (*Nmatrices*) and partial Nmatrices (*PNmatrices*) (see [5, 6]). The second component is a corresponding *analytic calculus*, i.e. a calculus whose proofs only consist of concepts already contained in the result. Analytic calculi are useful for establishing various properties of the corresponding logics, and are also the key for developing automated reasoning methods for them.

In this paper we provide both methodologies and practical tools for an *automatic generation* of analytic sequent calculi and effective semantics for a large class **H** of Hilbert systems. This is a concrete step towards a systematization of the vast variety of existing non-classical logics and the developement of tools for designing new application-oriented logics, see e.g. [11].

The calculi in **H** are obtained (i) by extending the language of CL^+, the positive fragment of classical logic, to a language $\mathcal{L}_\mathcal{U}$ which includes also a finite set $\mathcal{U}$ of unary connectives, and (ii) by adding to a Hilbert axiomatization HCL^+ of CL^+ axioms over $\mathcal{L}_\mathcal{U}$ of a certain general form. **H** contains infinitely many systems, which include well-known Hilbert calculi, the simplest and best known of which is the standard calculus for classical logic, obtained by adding to HCL^+ the usual axioms for negation. Another example of calculi in **H** is the family of *paraconsistent logics* known as C-systems [8, 10].

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Given a system $H \in \mathbf{H}$, our algorithm proceeds in two steps. First we introduce a sequent calculus G equivalent to H. This is done by suitably adapting the procedure in [9], where certain Hilbert axioms are transformed into equivalent (sequent and hypersequent) structural rules. In contrast to [9], however, here the rules extracted from the axioms of H are logical rules in Gentzen's terminology, that is they introduce logical connectives. The analyticity of the resulting calculus depends on the interaction between these rules. This is not anymore a local check and needs instead a *"global view"* on the obtained calculus, which is provided by the semantics constructed in the second step. This semantics is given in the framework of PNmatrices – a generalization of usual many-valued matrices in which each entry in the truth-tables of the logical connectives consists of a *possibly empty set* of options (see [6]). This framework allows non-deterministic semantics, and also, using empty sets of options makes it possible to forbid some combinations of truth values. However, it is still effective, as it guarantees the decidability of the corresponding sequent calculus. As a corollary it follows that each system $H \in \mathbf{H}$ is decidable. Furthermore, we show that the PNmatrix constructed for H is an Nmatrix (i.e., it has no empty sets in the truth-tables) iff G enjoys a certain generalized analyticity property.

Related Work: A semi-automated procedure to define semantics and analytic calculi for the family of C-systems was introduced in [4]. A corresponding Nmatrix was constructed there for each system in the family, and was then used for introducing a corresponding analytic sequent calculus. However, the construction of Nmatrices out of the Hilbert calculi is done manually, and it requires some ingenuity. In this paper we provide a full automation of the generation of effective semantics and analytic calculi for all the systems considered in [4], which have finite-valued semantics. Our method also applies to infinitely many other extensions of CL^+, which had so far no available semantics or adequate calculi. These include some logics defined in [1], finding semantics for which was left as an open problem. It should be noted that our algorithm reverses the steps taken in [4]: it first extracts suitable sequent rules from the axioms of $\mathbf{H}$, and uses them to "read off" the semantics.

Implementation: Our method is implemented in the Prolog system *Paralyzer*, available at `www.logic.at/people/lara/paralyzer.html`. For any set of axioms over $\mathcal{L}_{\mathcal{U}}$ of a certain general form *Paralyzer* (PARAconsistent (and other) logics anaLYZER) outputs: (a) a set of corresponding sequent rules, and (b) the associated PNmatrix. The user can choose whether to start as basic system with HCL^+ or with the system BK from [4], obtained by augmenting HCL^+ with the axioms $(\mathbf{n_1})$, $(\mathbf{b})$ and $(\mathbf{k})$ (cf. Fig. 1). In the latter case, by exploiting the invertibility of the sequent rules for $\circ$, (a) and (b) for the C-systems having finite-valued semantics coincide with the results in [4].

2 Step 1: From Hilbert Systems to Sequent Calculi

The first step of our method consists of a mapping from a family $\mathbf{H}$ of Hilbert systems into a family $\mathbf{G}$ of "well-behaved" sequent calculi.

2.1 The Family H

In what follows, $\mathcal{L}$ denotes a propositional language, and $\mathit{wff}_{\mathcal{L}}$ is its set of formulas. We assume that the atomic formulas of $\mathcal{L}$ are $\{p_1, p_2, \ldots\}$. $\mathcal{L}_{cl}^{+}$ is the language of CL^{+}, the positive fragment of (propositional) classical logic, consisting of the binary connectives $\wedge, \vee$ and $\supset$. We consider languages that extend $\mathcal{L}_{cl}^{+}$ with finitely many new unary connectives (such as $\neg$ and $\circ$). Henceforth $\mathcal{U}$ denotes an arbitrary finite set of unary connectives, and $\mathcal{L}_{\mathcal{U}}$ denotes the extension of $\mathcal{L}_{cl}^{+}$ with the connectives of $\mathcal{U}$. For a Hilbert system H, we write $\Gamma \vdash_H \varphi$ if φ is provable in H from a finite set Γ of formulas. HCL^{+} denotes any Hilbert calculus for $\mathcal{L}_{cl}^{+}$, which is sound and complete for CL^{+}. **H** is a family of axiomatic extensions of HCL^{+}, each of which is in the language $\mathcal{L}_{\mathcal{U}}$ for some $\mathcal{U}$. These systems are obtained by augmenting HCL^{+} with axioms[3] of the form defined below.

Definition 1. *Let $\mathcal{U} = \{\star_1, \ldots, \star_n\}$. $\mathbf{Ax}_{\mathcal{U}}$ is the set of $\mathcal{L}_{\mathcal{U}}$-formulas generated by the following grammar (where S is the initial variable):*

$$
\begin{array}{ll}
S = R_p \mid R_1 \mid R_2 & P_1 = (P_1 \diamond P_1) \mid \star p_1 \mid p_1 \mid p_2 \mid \ldots \\
R_p = (R_p \diamond P_1) \mid (P_1 \diamond R_p) \mid \star p_1 & P_2 = (P_2 \diamond P_2) \mid \star p_1 \mid \star p_2 \mid p_1 \mid p_2 \mid p_3 \mid \ldots \\
R_1 = (R_1 \diamond P_1) \mid (P_1 \diamond R_1) \mid \star\star p_1 & \diamond = \wedge, \vee, \supset \\
R_2 = (R_2 \diamond P_2) \mid (P_2 \diamond R_2) \mid \star(p_1 \diamond p_2) & \star = \star_1 \mid \ldots \mid \star_n
\end{array}
$$

N :	$(\mathbf{n_1})$	$p_1 \vee \neg p_1$	$(\mathbf{n_2})$	$p_1 \supset (\neg p_1 \supset p_2)$
	$(\mathbf{c})$	$\neg\neg p_1 \supset p_1$	$(\mathbf{e})$	$p_1 \supset \neg\neg p_1$
	$(\mathbf{n^l_\wedge})$	$\neg(p_1 \wedge p_2) \supset (\neg p_1 \vee \neg p_2)$	$(\mathbf{n^r_\wedge})$	$(\neg p_1 \vee \neg p_2) \supset \neg(p_1 \wedge p_2)$
	$(\mathbf{n^l_\vee})$	$\neg(p_1 \vee p_2) \supset (\neg p_1 \wedge \neg p_2)$	$(\mathbf{n^r_\vee})$	$(\neg p_1 \wedge \neg p_2) \supset \neg(p_1 \vee p_2)$
	$(\mathbf{n^l_\supset})$	$\neg(p_1 \supset p_2) \supset (p_1 \wedge \neg p_2)$	$(\mathbf{n^r_\supset})$	$(p_1 \wedge \neg p_2) \supset \neg(p_1 \supset p_2)$
C :	$(\mathbf{b})$	$p_1 \supset (\neg p_1 \supset (\circ p_1 \supset p_2))$	$(\mathbf{r_\diamond})$	$\circ(p_1 \diamond p_2) \supset (\circ p_1 \vee \circ p_2)$
	$(\mathbf{k})$	$\circ p_1 \vee (p_1 \wedge \neg p_1)$	$(\mathbf{i})$	$\neg\circ p_1 \supset (p_1 \wedge \neg p_1)$
	$(\mathbf{o^1_\diamond})$	$\circ p_1 \supset \circ(p_1 \diamond p_2)$	$(\mathbf{o^2_\diamond})$	$\circ p_2 \supset \circ(p_1 \diamond p_2)$
	$(\mathbf{a_\diamond})$	$(\circ p_1 \wedge \circ p_2) \supset \circ(p_1 \diamond p_2)$	$(\mathbf{a_\neg})$	$\circ p_1 \supset \circ\neg p_1$

Fig. 1. Examples of formulas in $\mathbf{Ax}_{\{\neg,\circ\}}$ ($\diamond \in \{\vee, \wedge, \supset\}$)

Definition 2. *A Hilbert calculus H for a language $\mathcal{L}_{\mathcal{U}}$ is called a $\mathcal{U}$-*extension *of HCL^{+} if it is obtained by augmenting HCL^{+} with a finite set of axioms from $\mathbf{Ax}_{\mathcal{U}}$. We denote by **H** the family of all $\mathcal{U}$-extensions of HCL^{+} for some $\mathcal{U}$.*

The family **H** contains infinitely many systems, which include many well-known Hilbert calculi. The most important member of **H** is the standard calculus for (propositional) classical logic, obtained by adding $(\mathbf{n_1})$ and $(\mathbf{n_2})$ to HCL^{+} (cf. Fig. 1). Other important examples include various systems for paraconsistent logics [4, 7, 8, 10].

Remark 1. Paraconsistent logics are logics which are tolerant of inconsistent theories, i.e. there are some formulas ψ, φ, such that: $\psi, \neg\psi \nvdash \varphi$. One well-known

[3] By *axioms* we actually mean *axiom schemata.*

family of paraconsistent logics, formulated in terms of Hilbert calculi, is known as C-systems [4, 7, 8, 10]. In this family the notion of consistency is internalized into the object language by employing a unary consistency operator $\circ$, the intuitive meaning of $\circ\psi$ being "ψ is consistent". Clearly, a system which includes the standard axiom for negation ($\mathbf{n_2}$) (Fig. 1) cannot induce a paraconsistent logic. Many C-systems include instead the weaker axiom ($\mathbf{b}$), and in addition also the axiom ($\mathbf{n_1}$). Furthermore, different C-systems employ different subsets of the axioms from the set $\mathbf{C}$ (Fig. 1), which express various properties of the operator $\circ$. For instance, axiom ($\mathbf{a}_\vee$) says that the consistency of two formulas implies the consistency of their disjunction. The axiom ($\mathbf{o}^{\mathbf{1}}_\vee$) expresses another form of consistency propagation: the consistency of a formula implies the consistency of its disjunction with any other formula. By adding to HCL^+ various combinations of axioms from Fig. 1, we obtain a wider family of systems (not all of them paraconsistent), many of which are studied in [2, 4].

2.2 The Family G

The sequent calculi we will consider, formulated label-style, are as follows:

Definition 3. *1. A* labelled $\mathcal{L}$-formula *has the form $b : \psi$, where $b \in \{f, t\}$ and $\psi \in wff_\mathcal{L}$. An* $\mathcal{L}$-sequent *is a finite set of labelled $\mathcal{L}$-formulas. The usual sequent notation $\psi_1, \ldots, \psi_n \Rightarrow \varphi_1, \ldots, \varphi_m$ is interpreted as the set $\{f : \psi_1, \ldots, f : \psi_n, t : \varphi_1, \ldots, t : \varphi_m\}$.*

2. *An* $\mathcal{L}$-substitution *is a function $\sigma : wff_\mathcal{L} \to wff_\mathcal{L}$, such that $\sigma(\diamond(\psi_1, \ldots, \psi_n)) = \diamond(\sigma(\psi_1), \ldots, \sigma(\psi_n))$ for every n-ary connective $\diamond$ of $wff_\mathcal{L}$. $\mathcal{L}$-substitutions are naturally extended to labelled $\mathcal{L}$-formulas and $\mathcal{L}$-sequents.*
3. *An* $\mathcal{L}$-rule *is an expression of the form Q/s, where Q is a finite set of $\mathcal{L}$-sequents (called* premises*) and s is an $\mathcal{L}$-sequent (called* conclusion*). An* application *of an $\mathcal{L}$-rule Q/s is any inference step inferring the $\mathcal{L}$-sequent $\sigma(s) \cup c$ from the set of $\mathcal{L}$-sequents $\{\sigma(q) \cup c \mid q \in Q\}$, where σ is an $\mathcal{L}$-substitution, and c is an $\mathcal{L}$-sequent.*
4. *A sequent calculus G for $\mathcal{L}$ consists of a finite set of $\mathcal{L}$-rules. We write $\mathcal{S} \vdash_G s$ whenever the $\mathcal{L}$-sequent s is derivable from the set $\mathcal{S}$ of $\mathcal{L}$-sequents in G.*

Example 1. Formulated according to Def. 3, the standard sequent calculus LK^+ for CL^+ is the set of $\mathcal{L}^+_{cl}$-rules consisting of the following elements:

(id)	$\emptyset/\{f : p_1, t : p_1\}$	(cut)	$\{\{f : p_1\}, \{t : p_1\}\}/\emptyset$
$(W\Rightarrow)$	$\{\emptyset\}/\{f : p_1\}$	$(\Rightarrow W)$	$\{\emptyset\}/\{t : p_1\}$
$(\wedge\Rightarrow)$	$\{\{f : p_1, f : p_2\}\}/\{f : p_1 \wedge p_2\}$	$(\Rightarrow\wedge)$	$\{\{t : p_1\}, \{t : p_2\}\}/\{t : p_1 \wedge p_2\}$
$(\vee\Rightarrow)$	$\{\{f : p_1\}, \{f : p_2\}\}/\{f : p_1 \vee p_2\}$	$(\Rightarrow\vee)$	$\{\{t : p_1, t : p_2\}\}/\{t : p_1 \vee p_2\}$
$(\supset\Rightarrow)$	$\{\{t : p_1\}, \{f : p_2\}\}/\{f : p_1 \supset p_2\}$	$(\Rightarrow\supset)$	$\{\{f : p_1, t : p_2\}\}/\{t : p_1 \supset p_2\}$

G is a family of sequent calculi, each of which is in the language $\mathcal{L}_\mathcal{U}$ for some $\mathcal{U}$. These calculi are obtained by augmenting LK^+ with *simple* rules:

$(\Rightarrow \neg)$	$\{\{f : p_1\}\}/\{t : \neg p_1\}$	$\dfrac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma \Rightarrow \neg\varphi, \Delta}$
$(\circ \Rightarrow)$	$\{\{t : p_1\}, \{t : \neg p_1\}\}/\{f : \circ p_1\}$	$\dfrac{\Gamma \Rightarrow \varphi, \Delta \quad \Gamma \Rightarrow \neg\varphi, \Delta}{\Gamma, \circ\varphi \Rightarrow \Delta}$
$(\neg\neg \Rightarrow)$	$\{\{f : p_1\}\}/\{f : \neg\neg p_1\}$	$\dfrac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma, \neg\neg\varphi \Rightarrow \Delta}$
$(\Rightarrow \neg\wedge)_1$	$\{\{t : \neg p_1\}\}/\{t : \neg(p_1 \wedge p_2)\}$	$\dfrac{\Gamma \Rightarrow \neg\varphi, \Delta}{\Gamma \Rightarrow \neg(\varphi \wedge \psi), \Delta}$

Fig. 2. Examples of $\mathcal{L}_{\{\neg,\circ\}}$-rules and their applications forms

Definition 4. *A $\mathcal{U}_n$-premise ($n = 1, 2$) is an $\mathcal{L}_\mathcal{U}$-sequent of the form $\{b : p_n\}$ or $\{b : \star p_n\}$, where $b \in \{f, t\}$ and $\star \in \mathcal{U}$. An $\mathcal{L}_\mathcal{U}$-rule Q/s is ($b \in \{f, t\}$, $\star, \rhd \in \mathcal{U}$ and $\diamond \in \{\wedge, \vee, \supset\}$):*

- primitive *if $s = \{b : \star p_1\}$ and Q consists only of $\mathcal{U}_1$-premises.*
- onevar *if $s = \{b : \star \rhd p_1\}$ and Q consists only of $\mathcal{U}_1$-premises.*
- twovar *if $s = \{b : \star(p_1 \diamond p_2)\}$ and Q consists only of $\mathcal{U}_1$-premises and $\mathcal{U}_2$-premises.*
- simple *if it is either a primitive, a onevar or a twovar rule.*

Example 2. $(\Rightarrow \neg)$ is primitive, $(\neg\neg \Rightarrow)$ onevar, and $(\Rightarrow \neg\wedge)_1$ twovar (cf. Fig. 2).

Distinguishing between the types of rules above will be crucial for the semantic definitions of Section 3.2. As we shall see, rules of different types will play different semantic roles: the primitive rules will determine the truth values in the PNmatrices, while the onevar and twovar rules will dictate the truth-tables of the unary and binary connectives respectively.

Definition 5. *A sequent calculus G for $\mathcal{L}_\mathcal{U}$ is called a* $\mathcal{U}$-extension *of LK^+ if it is obtained by augmenting LK^+ with a finite set of simple $\mathcal{L}_\mathcal{U}$-rules. We denote by $\mathbf{G}$ the family of all $\mathcal{U}$-extensions of LK^+ for some $\mathcal{U}$.*

2.3 Mapping from H to G

Given a Hilbert system $H \in \mathbf{H}$ we show how to construct a sequent calculus $G_H \in \mathbf{G}$ which is equivalent in the following sense:

Definition 6. *A sequent calculus G is* equivalent *to a Hilbert system H if for every finite set $\Gamma \cup \{\varphi\}$ of formulas: $\Gamma \vdash_H \varphi$ iff $\vdash_G \Gamma \Rightarrow \varphi$.*

Fact 1. *LK^+ is equivalent to HCL^+.*

We denote by $H \cup \{\varphi\}$ the Hilbert system obtained from H by adding the axiom φ, and by $G \cup R$ the sequent calculus extending G with the set R of rules.

Definition 7. *Let R and R' be two sets of $\mathcal{L}$-rules, and G be a sequent calculus for $\mathcal{L}$. R and R' are* equivalent *in G if $Q \vdash_{G\cup R'} s$ for every $Q/s \in R$, and $Q \vdash_{G\cup R} s$ for every $Q/s \in R'$.*

Definition 8. *An $\mathcal{L}_{\mathcal{U}}$-rule Q/s is* invertible *in G if $s \vdash_G q$ for every $q \in Q$.*

The key observations for our transformation procedure are: (i) the invertibility of the rules for $\wedge, \vee$ and $\supset$ in LK^+, (ii) Lemma 1, known as Ackermann's lemma and used, e.g. in [9] for substructural logics, and (iii) Lemma 2, which allows the generated rules to obey a (weaker form of) subformula property.

Lemma 1. *Let G be a sequent calculus for $\mathcal{L}$ extending LK^+. Let s be an $\mathcal{L}$-sequent, and γ be a labelled formula in s. The $\mathcal{L}$-rule $\emptyset/s$ is equivalent in G to the rule $r = \{\{\bar{b} : \varphi\} \mid b : \varphi \in s \setminus \{\gamma\}\}/\{\gamma\}$ (where $\overline{f} = t$ and $\overline{t} = f$).*

Proof. $\{\{\bar{b} : \varphi\} \mid b : \varphi \in s \setminus \{\gamma\}\} \vdash_{G\cup\{\emptyset/s\}} \gamma$ is obtained by applying the rule $\emptyset/s$ and then have multiple applications of (cut) (preceded by suitable applications of $(W\Rightarrow)$ and $(\Rightarrow W)$). To prove $\vdash_{G\cup\{r\}} s$ we first use (id) to obtain $\{f : \psi, t : \psi\}$ for every $\psi \in \{\varphi \mid b : \varphi \in s \setminus \{\gamma\}\}$ followed by suitable applications of $(W\Rightarrow)$ and $(\Rightarrow W)$. The claim then follows by applying r. $\square$

Lemma 2. *Let G be a sequent calculus for $\mathcal{L}$ extending LK^+. Let s be an $\mathcal{L}$-sequent, and let $s' = s \cup \{b : p\}$, where $b \in \{f, t\}$ and p is an atomic formula that does not occur in s. Then, $\vdash_{G\cup\{\emptyset/s'\}} \Gamma \Rightarrow \varphi$ iff $\vdash_{G\cup\{\emptyset/s\}} \Gamma \Rightarrow \varphi$, for every sequent $\Gamma \Rightarrow \varphi$.*

Proof. Clearly, $\vdash_{G\cup\{\emptyset/s'\}} \subseteq \vdash_{G\cup\{\emptyset/s\}}$ (applications of $\emptyset/s'$ can be simulated using $(W\Rightarrow)$ or $(\Rightarrow W)$, and $\emptyset/s$). For the converse direction, we distinguish two cases according to b. If $b = f$ then every application of $\emptyset/s$ deriving $\sigma(s)$ can be simulated in $G \cup \{\emptyset/s'\}$ by using (cut) on $\sigma(s) \cup \{f : p_1 \supset p_1\}$ (obtained by $\emptyset/s'$ in which p is substituted with $p_1 \supset p_1$) and $\sigma(s) \cup \{t : p_1 \supset p_1\}$, derivable in LK^+. If $b = t$ we need a proof transformation: every application of $\emptyset/s$ in a derivation of $\Gamma \Rightarrow \varphi$ is replaced with an application of $\emptyset/s'$, in which p is substituted with φ. $t : \varphi$ is then propagated till the end sequent. $\square$

Theorem 1. *Every $H \in \mathbf{H}$ has an equivalent sequent calculus $G_H \in \mathbf{G}$.*

Proof. Follows by repeatedly applying the following procedure (starting from HCL^+ and LK^+). Let $H \in \mathbf{H}$ and $G \in \mathbf{G}$ be an equivalent sequent calculus for $\mathcal{L}_{\mathcal{U}}$ and let $\psi \in \mathbf{Ax}_{\mathcal{U}}$. We show how to construct a finite (possibly empty) set R' of simple $\mathcal{L}_{\mathcal{U}}$-rules such that $H \cup \{\psi\}$ is equivalent to $G \cup R'$.

First, it is easy to see that $H \cup \{\psi\}$ is equivalent to $G \cup \{r_\psi\}$, where r_ψ is the rule $\emptyset/\{t : \psi\}$. For the right-to-left direction consider a proof of a sequent $\Gamma \Rightarrow \varphi$ in $G \cup \{r_\psi\}$, and transform it into a proof of $\Gamma, \psi \Rightarrow \varphi$ in G, by replacing every application of r_ψ with the identity axiom $\{f : \psi, t : \psi\}$, and propagating $f : \psi$ through the derivation till the end sequent. The equivalence of H and G entails that $\Gamma, \psi \vdash_H \varphi$, and it immediately follows that $\Gamma \vdash_{H\cup\{\psi\}} \varphi$.

Now, starting from r_ψ and using the invertibility of the rules for $\wedge, \vee$ and $\supset$, we obtain a finite set of rules R, such that (i) R is equivalent to $\{r_\psi\}$ in G, and (ii) each $r \in R$ has the form $\emptyset/s$, where s has one of the following forms, according to whether ψ is generated by R_p, R_1 or R_2 in the grammar of Def. 1:

1. s consists of at least one labelled formula of the form $b : \star p_1$ ($b \in \{f, t\}$, $\star \in \mathcal{U}$), and any number of labelled formulas $b : p_i$ ($b \in \{f, t\}$, $i \geq 1$).
2. s consists of exactly one labelled formula of the form $b : \star \triangleright p_1$ ($b \in \{f, t\}$, $\star, \triangleright \in \mathcal{U}$), and any number of labelled formulas of the form $b : p_i$ or $b : \star p_1$ ($b \in \{f, t\}$, $i \geq 1$, and $\star \in \mathcal{U}$).
3. s consists of exactly one labelled formula of the form $b : \star(p_1 \diamond p_2)$ ($b \in \{f, t\}$, $\star \in \mathcal{U}$, $\diamond \in \{\wedge, \vee, \supset\}$), and any number of labelled formulas of the form $b : p_i$, $b : \star p_1$, or $b : \star p_2$ ($b \in \{f, t\}$, $i \geq 1$, and $\star \in \mathcal{U}$).

Obviously, we can discard all rules $\emptyset/s$ of R for which $\{f : p_i, t : p_i\} \subseteq s$ for some $i \geq 1$. By Lemma 2, for each rule $\emptyset/s$ left in R: if s has the form 1 or 2 above, we can omit from s all labelled formulas of the form $b : p_i$ for $i > 1$, and similarly, if s has the form 3, all labelled formulas of the form $b : p_i$ for $i > 2$. By Lemma 1 the resulting rules can be transformed into equivalent simple $\mathcal{L}_{\mathcal{U}}$-rules. $\square$

The proof above is constructive, and induces an algorithm to extract simple $\mathcal{L}_{\mathcal{U}}$-rules out of axioms in $\mathbf{Ax}_{\mathcal{U}}$.

Example 3. Let (**b**) be the axiom $p_1 \supset (\neg p_1 \supset (\circ p_1 \supset p_2))$. Consider the rule $\emptyset/\{t : p_1 \supset (\neg p_1 \supset (\circ p_1 \supset p_2))\}$. Using the invertibility of $(\Rightarrow\supset)$ we obtain an equivalent rule $\emptyset/\{f : p_1, f : \neg p_1, f : \circ p_1, t : p_2\}$. By Lemma 2 we get $\emptyset/\{f : p_1, f : \neg p_1, f : \circ p_1\}$. The primitive rule $\{\{t : p_1\}, \{t : \neg p_1\}\}/\{f : \circ p_1\}$ (or $\{\{t : p_1\}, \{t : \circ p_1\}\}/\{f : \neg p_1\}$) then follows by Lemma 1.

3 Step 2: Extracting Semantics

We define finite-valued semantics, using partial non-deterministic matrices, for every calculus in $\mathbf{G}$.

3.1 Partial Non-deterministic Matrices

Partial non-deterministic matrices were introduced in [6] in the context of labelled sequent calculi. They generalize the notion of non-deterministic matrices by allowing *empty* sets of options in the truth-tables of the logical connectives. This feature makes it possible to semantically characterize every $G \in \mathbf{G}$. Below we shortly reproduce and adapt to our context the basic definitions from [6].

Definition 9. *A* partial non-deterministic matrix *(*PNmatrix*) $\mathcal{M}$ for $\mathcal{L}$ consists of: (i) a set $\mathcal{V}_{\mathcal{M}}$ of truth values, (ii) a subset $\mathcal{D}_{\mathcal{M}} \subseteq \mathcal{V}_M$ (designated truth values), and (iii) a truth-table $\diamond_{\mathcal{M}} : {\mathcal{V}_{\mathcal{M}}}^n \to P(\mathcal{V}_{\mathcal{M}})$ for every n-ary connective $\diamond$ of $\mathcal{L}$.*

Definition 10. *Let $\mathcal{M}$ be a PNmatrix for $\mathcal{L}$, and $\mathcal{W}$ be a set of $\mathcal{L}$-formulas closed under subformulas.*

1. *A* $\mathcal{W}$-valuation *is a function* v *from* $\mathcal{W}$ *to some set* $\mathcal{V}$ *(of* truth values*). A* $wff_{\mathcal{L}}$*-valuation is also called* an $\mathcal{L}$-valuation.
2. *A* $\mathcal{W}$-valuation v *is called* $\mathcal{M}$*-legal if its range is* $\mathcal{V}_{\mathcal{M}}$*, and it respects the truth-tables of* $\mathcal{M}$*, i.e.* $v(\diamond(\psi_1, \ldots, \psi_n)) \in \diamond_{\mathcal{M}}(v(\psi_1), \ldots, v(\psi_n))$ *for every compound formula* $\diamond(\psi_1, \ldots, \psi_n) \in \mathcal{W}$.
3. *A* $\mathcal{W}$*-valuation* v *satisfies an* $\mathcal{L}$*-sequent* s *for* $\mathcal{M}$ *(denoted by* $v \models_{\mathcal{M}} s$*) if either* $v(\varphi) \in \mathcal{D}_{\mathcal{M}}$ *for some* $t : \varphi \in s$*, or* $v(\varphi) \notin \mathcal{D}_{\mathcal{M}}$ *for some* $f : \varphi \in s$.
4. *Given an* $\mathcal{L}$*-sequent* s*,* $\vdash^{\mathcal{W}}_{\mathcal{M}} s$ *if* $v \models_{\mathcal{M}} s$ *for every* $\mathcal{M}$*-legal* $\mathcal{W}$*-valuation* v*. We write* $\vdash_{\mathcal{M}} s$ *instead of* $\vdash^{wff_{\mathcal{L}}}_{\mathcal{M}} s$.

Clearly, every (ordinary) matrix can be identified with a PNmatrix, in which all truth-tables take only singletons.

Example 4. The (positive fragment of the) standard classical matrix can be identified with the PNmatrix $\mathcal{M}_{LK^+}$ defined as:

1. $\mathcal{V}_{\mathcal{M}_{LK^+}} = \{f, t\}$, $\mathcal{D}_{\mathcal{M}_{LK^+}} = \{t\}$.
2. $\wedge_{\mathcal{M}_{LK^+}}$, $\vee_{\mathcal{M}_{LK^+}}$, and $\supset_{\mathcal{M}_{LK^+}}$ are defined according to the classical truth-tables (singletons are used instead of values, e.g. $\wedge_{\mathcal{M}_{LK^+}}(t, f) = \{f\}$).

Fact 2. $\mathcal{M}_{LK^+}$ *is sound and complete for* LK^+ *(i.e.* $\vdash_{LK^+} s$ *iff* $\vdash_{\mathcal{M}_{LK^+}} s$*).*

3.2 PNmatrices for $\mathcal{U}$-extensions of LK^+

Until the end of this section, let G be some $\mathcal{U}$-extension of LK^+. The main idea behind the construction of a PNmatrix $\mathcal{M}_G$ for G is to use truth values as "information carriers" (along the lines of [1]) in the following sense. In addition to determining whether φ is "true", the truth value of φ contains also information about the "truth/falsity" of all the formulas of the form $\star\varphi$ for $\star \in \mathcal{U}$. To this end, instead of using the truth values $\{f, t\}$, we use extended truth values, which are tuples over $\{f, t\}$ of size $|\mathcal{U}|+1$. The first element of each such a tuple u, denoted by u^0, is reserved for representing the "truth/falsity" of φ. Each connective $\star \in \mathcal{U}$ is then (arbitrarily) allocated one of the remaining elements. We shall denote by $u^\star$ the element of u allocated for $\star \in \mathcal{U}$. Thus whenever φ is assigned the truth value u, φ is "true" iff $u^0 = t$, and for each $\star \in \mathcal{U}$, $\star\varphi$ is "true" iff $u^\star = t$. However, in constructing $\mathcal{M}_G$ not all the possible tuples will be used as truth values: only those that "respect" the primitive rules of G (cf. Def. 4). This is formalized as follows:

Notation 1. *We denote by* $\mathsf{V}_{\mathcal{U}}$ *the set of all* $(|\mathcal{U}| + 1)$*-tuples over* $\{f, t\}$.

Definition 11. *A tuple* $u \in \mathsf{V}_{\mathcal{U}}$ satisfies *a* $\mathcal{U}_1$*-premise* q*, if either* $q = \{u^0 : p_1\}$*, or* $q = \{u^\star : \star p_1\}$ *for some* $\star \in \mathcal{U}$*.* u respects *a primitive rule* $Q/\{b : \star p_1\}$ *if* $u^\star = b$ *whenever* u *satisfies every* $q \in Q$.

Definition 12. $\mathcal{V}_{\mathcal{M}_G}$ *(the set of truth values of the PNmatrix* $\mathcal{M}_G$*) is the set of all tuples in* $\mathsf{V}_{\mathcal{U}}$ *which respect all primitive rules of* G*. In addition, the set of designated truth values* $\mathcal{D}_{\mathcal{M}_G}$ *is* $\{u \in \mathcal{V}_{\mathcal{M}_G} \mid u^0 = t\}$.

Example 5. Suppose that $\mathcal{U} = \{\neg\}$, and that the only primitive rule of G is $\{\{f : p_1\}\}/\{t : \neg p_1\}$. A pair $u \in \mathcal{V}_{\mathcal{M}_G}$ respects $(\Rightarrow \neg)$ iff $u^{\neg} = t$ whenever $u^0 = f$. Thus we obtain $\mathcal{V}_{\mathcal{M}_G} = \{\langle f, t\rangle, \langle t, f\rangle, \langle t, t\rangle\}$ (here $u^{\neg}$ is the second component of each pair). The designated values are: $\mathcal{D}_{\mathcal{M}_G} = \{\langle t, f\rangle, \langle t, t\rangle\}$.

Having defined the truth values of $\mathcal{M}_G$, we proceed to providing a truth-table $\triangleright_{\mathcal{M}_G}$ for each (unary) connective $\triangleright \in \mathcal{U}$. This is done according to the *onevar rules* of G of the form $Q/\{b : \star \triangleright p_1\}$.

Definition 13. *Let* $\triangleright \in \mathcal{U}$*. For every* $u_1 \in \mathcal{V}_{\mathcal{M}_G}$*,* $\triangleright_{\mathcal{M}_G}(u_1)$ *is the set of all tuples* $u \in \mathcal{V}_{\mathcal{M}_G}$ *such that: (i)* $u^0 = u_1^{\triangleright}$*; and (ii) for every onevar rule of* G *of the form* $Q/\{b : \star \triangleright p_1\}$*, if* u_1 *satisfies every* $q \in Q$ *then* $u^{\star} = b$*.*

Intuitively, condition (i) forces the information about the "truth/falsity" of $\triangleright\varphi$ carried in the truth value of $\triangleright\varphi$ (in the first bit of this tuple) to be equal to the one carried in the truth value of φ.

Example 6. Following Example 5, suppose that G's only onevar rule of the form $Q/\{b : \star\neg p_1\}$ is $\{\{f : p_1\}\}/\{f : \neg\neg p_1\}$. Let us explain, e.g., how $\neg_{\mathcal{M}_G}(\langle t, f\rangle)$ is obtained. The only tuple from $\mathcal{V}_{\mathcal{M}_G} = \{\langle f, t\rangle, \langle t, f\rangle, \langle t, t\rangle\}$ satisfying condition (i) (that is, whose first component is $\langle t, f\rangle^{\neg} = f$) is $u = \langle f, t\rangle$. Condition (ii) holds trivially for u, as $\langle t, f\rangle$ does not satisfy the premise $\{f : p_1\}$ of the above rule. Thus we obtain: $\neg_{\mathcal{M}_G}(\langle t, f\rangle) = \{\langle f, t\rangle\}$. Similarly, we get $\neg_{\mathcal{M}_G}(\langle f, t\rangle) = \{\langle t, f\rangle\}$, and $\neg_{\mathcal{M}_G}(\langle t, t\rangle) = \{\langle t, f\rangle, \langle t, t\rangle\}$.

To complete the construction of $\mathcal{M}_G$, we provide the truth-tables of the binary connectives, using the *twovar rules.*

Definition 14. *A pair of tuples* $\langle u_1, u_2\rangle \in \mathsf{V}_{\mathcal{U}}{}^2$ satisfies *a* $\mathcal{U}_1$*-premise* q*, if* u_1 *satisfies* q*.* $\langle u_1, u_2\rangle$ satisfies *a* $\mathcal{U}_2$*-premise* q*, if* u_2 *satisfies* q*.*

Definition 15. *Let* $\diamond \in \{\wedge, \vee, \supset\}$*. For every* $u_1, u_2 \in \mathcal{V}_{\mathcal{M}_G}$*,* $\diamond_{\mathcal{M}_G}(u_1, u_2)$ *is the set of all tuples* $u \in \mathcal{V}_{\mathcal{M}_G}$ *satisfying: (i)* $u^0 \in \diamond_{\mathcal{M}_{LK^+}}(u_1^0, u_2^0)$*; and (ii) for every twovar rule of* G *of the form* $Q/\{b : \star(p_1 \diamond p_2)\}$*, if* $\langle u_1, u_2\rangle$ *satisfies every* $q \in Q$ *then* $u^{\star} = b$*.*

Intuitively, condition (i) ensures that $\diamond$ behaves as the corresponding classical connective, and condition (ii) provides the correspondence between the truth-table of $\diamond$ and the twovar rules that involve $\diamond$.

Example 7. Following Example 5, suppose that G's only twovar rule of the form $Q/\{b : \star(p_1 \wedge p_2)\}$ is $(\Rightarrow \neg\wedge)_1$ (see Fig. 2). A pair of values $\langle u_1, u_2\rangle \in \mathcal{V}_{\mathcal{M}_G}{}^2$ satisfies the premise of $(\Rightarrow \neg\wedge)_1$ iff $u_1^{\neg} = t$. In this case we require that for every $u \in \wedge_{\mathcal{M}_G}(u_1, u_2)$ we have $u^{\neg} = t$. Thus we obtain the following table for $\wedge$:

$\widetilde{\wedge}$	$\langle f, t\rangle$	$\langle t, f\rangle$	$\langle t, t\rangle$
$\langle f, t\rangle$	$\{\langle f, t\rangle\}$	$\{\langle f, t\rangle\}$	$\{\langle f, t\rangle\}$
$\langle t, f\rangle$	$\{\langle f, t\rangle\}$	$\{\langle t, f\rangle, \langle t, t\rangle\}$	$\{\langle t, f\rangle, \langle t, t\rangle\}$
$\langle t, t\rangle$	$\{\langle f, t\rangle\}$	$\{\langle t, t\rangle\}$	$\{\langle t, t\rangle\}$

3.3 Soundness and Completeness

We turn to prove the correctness of the construction of $\mathcal{M}_G$. We establish strong forms of soundness and completeness, to be used later in the characterization of analyticity of G. The main idea is to maintain a correlation between the formulas used in the derivation, and the formulas from the domain of the valuations. In what follows $\mathcal{W}$ is an arbitrary set of $\mathcal{L}_{\mathcal{U}}$-formulas closed under subformulas. We use the following additional notations and definitions:

Notation 2. *Let s be an $\mathcal{L}_{\mathcal{U}}$-sequent.*

1. *$sub[s]$ denotes the set of subformulas of all $\mathcal{L}_{\mathcal{U}}$-formulas occurring in s.*
2. *s is called a* $\mathcal{W}$-sequent *if $sub[s] \subseteq \mathcal{W}$.*
3. *We write $\vdash_G^{\mathcal{W}} s$ if there exists a proof of s in G consisting only of $\mathcal{W}$-sequents.*

Definition 16. *The sets $\mathcal{U}^+(\mathcal{W})$ and $\mathcal{U}^-(\mathcal{W})$ are defined as follows:*
$\mathcal{U}^-(\mathcal{W}) = \mathcal{W} \backslash \{\star\psi \in \mathcal{W} \mid \star \in \mathcal{U}, \star\psi \text{ is not a proper subformula of a formula in } \mathcal{W}\}$
$\mathcal{U}^+(\mathcal{W}) = \mathcal{W} \cup \{\star\psi \mid \star \in \mathcal{U}, \psi \in \mathcal{U}^-(\mathcal{W})\}$

Example 8. For $\mathcal{U} = \{\neg\}$ and $\mathcal{W} = \{p_1, p_2, \neg p_1, \neg p_2, p_1 \vee p_2, \neg p_1 \vee p_2, \neg(p_1 \vee p_2)\}$, we have $\mathcal{U}^-(\mathcal{W}) = \{p_1, p_2, \neg p_1, p_1 \vee p_2, \neg p_1 \vee p_2\}$, and $\mathcal{U}^+(\mathcal{W}) = \mathcal{W} \cup \{\neg\neg p_1, \neg(\neg p_1 \vee p_2)\}$.

Remark 2. Note that $\psi \in \mathcal{U}^-(\mathcal{W})$ whenever $\star\psi \in \mathcal{U}^+(\mathcal{W})$ for some $\star \in \mathcal{U}$.

The weaker notion of satisfaction, introduced in the following definition, is needed later to characterize (a generalized form of) analyticity.

Definition 17. *A $\mathcal{U}^-(\mathcal{W})$-valuation $v : \mathcal{U}^-(\mathcal{W}) \to \mathsf{V}_{\mathcal{U}}$* w-satisfies *a $\mathcal{U}^+(\mathcal{W})$-sequent s if there exists some labelled formula $b : \psi \in s$, such that either (i) ψ does not have the form $\star\varphi$ and $v(\psi)^0 = b$; or (ii) $\psi = \star\varphi$ (for some $\star \in \mathcal{U}$ and $\varphi \in \mathcal{U}^-(\mathcal{W})$) and $v(\varphi)^\star = b$.*

Theorem 2 (Soundness). *Let s be a $\mathcal{W}$-sequent. If $\vdash_G^{\mathcal{U}^+(\mathcal{W})} s$, then every $\mathcal{M}_G$-legal $\mathcal{U}^-(\mathcal{W})$-valuation w-satisfies s.*

Proof. It suffices to show that whenever an $\mathcal{M}_G$-legal $\mathcal{U}^-(\mathcal{W})$-valuation w-satisfies the premises of some application of an $\mathcal{L}_{\mathcal{U}}$-rule $r = Q/s$ of G consisting solely of formulas from $\mathcal{U}^+(\mathcal{W})$, it also w-satisfies its conclusion. Consider such an application of r inferring $\sigma(s) \cup c$ from the set $\{\sigma(q) \cup c \mid q \in Q\}$, where c is an $\mathcal{L}_{\mathcal{U}}$-sequent, and σ is an $\mathcal{L}_{\mathcal{U}}$-substitution. Assume that $\sigma(p_1) = \psi_1$ and $\sigma(p_2) = \psi_2$. Let v be an $\mathcal{M}_G$-legal $\mathcal{U}^-(\mathcal{W})$-valuation, and suppose that v w-satisfies $\sigma(q) \cup c$ for every $q \in Q$. We prove that v w-satisfies $\sigma(s) \cup c$. Clearly, if v w-satisfies c, then we are done. Suppose otherwise. Then our assumption entails that it w-satisfies $\sigma(q)$ for every $q \in Q$. We show that in this case v w-satisfies $\sigma(s)$ (and so it w-satisfies $\sigma(s) \cup c$). For $r \in LK^+$ the claim is easy and left for the reader. Otherwise, r is a simple rule. Three cases can occur:

– Suppose that $r = Q/\{b : \triangleright p_1\}$ is a primitive rule. Note that since we only consider applications of r consisting solely of formulas from $\mathcal{U}^+(\mathcal{W})$, we have that $\triangleright\psi_1 \in \mathcal{U}^+(\mathcal{W})$ and so $\psi_1 \in \mathcal{U}^-(\mathcal{W})$. The fact v w-satisfies $\sigma(q)$ for every $q \in Q$ implies that $v(\psi_1)$ satisfies every $q \in Q$. To see this, consider the following cases:
  - Assume that $q = \{b : p_1\}$, and ψ_1 does not have the form $\star\varphi$. Since v w-satisfies $\sigma(q)$, $v(\psi_1)^0 = b$.
  - Assume that $q = \{b : p_1\}$, and ψ_1 has the form $\star\varphi$. Since v w-satisfies $\sigma(q)$, $v(\varphi)^\star = b$. Since v is $\mathcal{M}_G$-legal, $v(\star\varphi)^0 = b$.
  - Assume that $q = \{b : \star p_1\}$. Since v w-satisfies $\sigma(q)$, $v(\psi_1)^\star = b$.

  In all cases, we obtain that $v(\psi_1)$ satisfies q. Now, since $v(\psi_1) \in \mathcal{V}_{\mathcal{M}_G}$, $v(\psi_1)$ respects r, and so $v(\psi_1)^\triangleright = b$. Thus v w-satisfies $\{b : \triangleright\psi_1\}$.
– Suppose that $r = Q/\{b : \star \triangleright p_1\}$ is a onevar rule. As in the previous case, $v(\psi_1)$ satisfies every $q \in Q$. Thus, since $v(\triangleright\psi_1) \in \triangleright_{\mathcal{M}_G}(v(\psi_1))$, we have $v(\triangleright\psi_1)^\star = b$. It follows that v w-satisfies $\{b : \star \triangleright \psi_1\}$.
– Suppose that $r = Q/\{b : \star(p_1 \diamond p_2)\}$ is a twovar rule. Similarly to the previous cases, we have $\langle v(\psi_1), v(\psi_2)\rangle$ satisfies every $q \in Q$. Thus, since $v(\psi_1 \diamond \psi_2) \in \diamond_{\mathcal{M}_G}(v(\psi_1), v(\psi_2))$, we have that $v(\psi_1 \diamond \psi_2)^\star = b$. It follows that v w-satisfies $\{b : \star(\psi_1 \diamond \psi_2)\}$. □

Theorem 3 (Completeness). *If every $\mathcal{M}_G$-legal $\mathcal{U}^-(\mathcal{W})$-valuation w-satisfies a $\mathcal{W}$-sequent s_0, then $\vdash_G^{\mathcal{U}^+(\mathcal{W})} s_0$.*

Proof. Suppose that $\nvdash_G^{\mathcal{U}^+(\mathcal{W})} s_0$. We construct an $\mathcal{M}_G$-legal $\mathcal{U}^-(\mathcal{W})$-valuation v that does not w-satisfy s_0. Call a set Ω of labelled $\mathcal{L}_\mathcal{U}$-formulas *maximal* if it satisfies the following conditions: (i) Ω consists of labelled $\mathcal{L}_\mathcal{U}$-formulas of the form $b : \psi$ for $\psi \in \mathcal{U}^+(\mathcal{W})$; (ii) $\nvdash_G^{\mathcal{U}^+(\mathcal{W})} s$ for every $\mathcal{L}_\mathcal{U}$-sequent $s \subseteq \Omega$; and (iii) For every formula $\psi \in \mathcal{U}^+(\mathcal{W})$ and $b \in \{f, t\}$, if $b : \psi \notin \Omega$ then $\vdash_G^{\mathcal{U}^+(\mathcal{W})} s \cup \{b : \psi\}$ for some $\mathcal{L}_\mathcal{U}$-sequent $s \subseteq \Omega$. It is straightforward to construct a maximal set Ω that extends s_0.

Note that the availability of the cut rule implies that for every $\psi \in \mathcal{U}^+(\mathcal{W})$, either $f : \psi \in \Omega$ or $t : \psi \in \Omega$ (otherwise, we would have $\vdash_G^{\mathcal{U}^+(\mathcal{W})} s_1 \cup \{f : \psi\}$ and $\vdash_G^{\mathcal{U}^+(\mathcal{W})} s_2 \cup \{t : \psi\}$ for $s_1, s_2 \subseteq \Omega$, and by applying weakenings (the rules $(W \Rightarrow)$ and $(\Rightarrow W)$) and (cut) we could obtain $\vdash_G^{\mathcal{U}^+(\mathcal{W})} s_1 \cup s_2$). Similarly, the availability of the identity axiom implies that for every $\psi \in \mathcal{U}^+(\mathcal{W})$, either $f : \psi \notin \Omega$ or $t : \psi \notin \Omega$ (otherwise, the fact that $\vdash_G^{\mathcal{U}^+(\mathcal{W})} \{f : \psi, t : \psi\}$ would contradict Ω's properties).

Let $v : \mathcal{U}^-(\mathcal{W}) \to \mathsf{V}_\mathcal{U}$ be a $\mathcal{U}^-(\mathcal{W})$-valuation defined by: $v(\psi)^0 = t$ iff $f : \psi \in \Omega$, and for every $\star \in \mathcal{U}$: $v(\psi)^\star = t$ iff $f : \star\psi \in \Omega$. Thus we have that for every $\psi \in \mathcal{U}^-(\mathcal{W})$ and $b \in \{f, t\}$, $v(\psi)^0 = b$ iff $b : \psi \notin \Omega$, and for every $\star \in \mathcal{U}$ $v(\psi)^\star = b$ iff $b : \star\psi \notin \Omega$. We show that v does not w-satisfy s_0. Let $b : \psi \in s_0$ such that ψ does not have the form $\star\varphi$. Thus $\psi \in \mathcal{U}^-(\mathcal{W})$, and since $s_0 \subseteq \Omega$, $v(\psi)^0 \neq b$. Similarly, let $b : \psi \in s_0$ such that ψ does have the form $\psi = \star\varphi$ (for some $\star \in \mathcal{U}$ and $\mathcal{L}_\mathcal{U}$-formula φ). Thus $\varphi \in \mathcal{U}^-(\mathcal{W})$, and since $s_0 \subseteq \Omega$, $v(\varphi)^\star \neq b$.

To show that v is $\mathcal{M}_G$-legal, we use the following properties:

($*$) Let σ be an $\mathcal{L}_{\mathcal{U}}$-substitution, such that $\sigma(p_1) \in \mathcal{U}^-(\mathcal{W})$. If $v(\sigma(p_1))$ satisfies a $\mathcal{U}_1$-premise q then $\vdash_G^{\mathcal{U}^+(\mathcal{W})} s \cup \sigma(q)$ for some $\mathcal{L}_{\mathcal{U}}$-sequent $s \subseteq \Omega$.

To see this, note that if $v(\sigma(p_1))$ satisfies q then one of the following holds:

- $q = b : p_1$ and $v(\sigma(p_1))^0 = b$. Thus $b : \sigma(p_1) \notin \Omega$, and since $\sigma(p_1) \in \mathcal{U}^+(\mathcal{W})$, we obtain that $\vdash_G^{\mathcal{U}^+(\mathcal{W})} s \cup \{b : \sigma(p_1)\}$ for some $\mathcal{L}_{\mathcal{U}}$-sequent $s \subseteq \Omega$.
- $q = b : \star p_1$ and $v(\sigma(p_1))^\star = b$. Thus $b : \star\sigma(p_1) \notin \Omega$, and since $\star\sigma(p_1) \in \mathcal{U}^+(\mathcal{W})$, we obtain that $\vdash_G^{\mathcal{U}^+(\mathcal{W})} s \cup \{b : \star\sigma(p_1)\}$ for some $\mathcal{L}_{\mathcal{U}}$-sequent $s \subseteq \Omega$.

Similarly, we have the following:

($**$) Let q be a $\mathcal{U}_1$-premise or a $\mathcal{U}_2$-premise, and σ be an $\mathcal{L}_{\mathcal{U}}$-substitution, such that $\sigma(p_1), \sigma(p_2) \in \mathcal{U}^-(\mathcal{W})$. If $\langle v(\sigma(p_1)), v(\sigma(p_2))\rangle$ satisfies q, then $\vdash_G^{\mathcal{U}^+(\mathcal{W})} s \cup \sigma(q)$ for some $\mathcal{L}_{\mathcal{U}}$-sequent $s \subseteq \Omega$.

We show that $\mathcal{V}_{\mathcal{M}_G}$ is the range of v. Let $\psi \in \mathcal{U}^-(\mathcal{W})$. To prove that $v(\psi) \in \mathcal{V}_{\mathcal{M}_G}$, we show that $v(\psi)$ respects all primitive rules of G. Consider a primitive rule of G, $r = Q/\{b : \star p_1\}$. Suppose that $v(\psi)$ satisfies every $q \in Q$. We show that $v(\psi)^\star = b$. Let σ be any $\mathcal{L}_{\mathcal{U}}$-substitution, assigning ψ to p_1. By ($*$), for every $q \in Q$, there exists some $\mathcal{L}_{\mathcal{U}}$-sequent $s_q \subseteq \Omega$ such that $\vdash_G^{\mathcal{U}^+(\mathcal{W})} s_q \cup \sigma(q)$. By applying weakenings and the rule r, we obtain that $\vdash_G^{\mathcal{U}^+(\mathcal{W})} \bigcup_{q\in Q} s_q \cup \{b : \star\psi\}$ (here we use the fact that $\star\psi \in \mathcal{U}^+(\mathcal{W})$ since $\psi \in \mathcal{U}^-(\mathcal{W})$). This implies that $b : \star\psi \notin \Omega$, and so $v(\psi)^\star = b$.

Finally, we show that v respects the truth-tables of $\mathcal{M}_G$:

(1) Let $\rhd\psi \in \mathcal{U}^-(\mathcal{W})$ (where $\rhd \in \mathcal{U}$). We show that $v(\rhd\psi) \in \rhd_{\mathcal{M}_G}(v(\psi))$. By the construction of $\rhd_{\mathcal{M}_G}$, it suffices to show: (i) $v(\rhd\psi)^0 = v(\psi)^{\rhd}$; and (ii) $v(\rhd\psi)^\star = b$ for every onevar rule $r = Q/\{b : \star \rhd p_1\}$ of G for which $v(\psi)$ satisfies every $q \in Q$. (i) trivially holds using the definition of v. For (ii), let $r = Q/\{b : \star \rhd p_1\}$ be a onevar rule of G, and suppose that $v(\psi)$ satisfies every $q \in Q$. We prove that $v(\rhd\psi)^\star = b$. Let σ be any $\mathcal{L}_{\mathcal{U}}$-substitution, assigning ψ to p_1. By ($*$) (note that $\psi \in \mathcal{U}^-(\mathcal{W})$ since $\mathcal{U}^-(\mathcal{W})$ is closed under subformula), for every $q \in Q$, there exists some $\mathcal{L}_{\mathcal{U}}$-sequent $s_q \subseteq \Omega$ such that $\vdash_G^{\mathcal{U}^+(\mathcal{W})} s_q \cup \sigma(q)$. By applying weakenings and the rule r, we obtain that $\vdash_G^{\mathcal{U}^+(\mathcal{W})} \bigcup_{q\in Q} s_q \cup \{b : \star \rhd \psi\}$ (note that $\star \rhd \psi \in \mathcal{U}^+(\mathcal{W})$ since $\rhd\psi \in \mathcal{U}^-(\mathcal{W})$). This implies that $b : \star \rhd \psi \notin \Omega$, and so $v(\rhd\psi)^\star = b$.

(2) Let $\psi_1 \diamond \psi_2 \in \mathcal{U}^-(\mathcal{W})$ for $\diamond \in \{\wedge, \vee, \supset\}$. We show that $v(\psi_1 \diamond \psi_2) \in \diamond_{\mathcal{M}_G}(v(\psi_1), v(\psi_2))$. Here it suffices to show: (i) $v(\psi_1 \diamond \psi_2)^\star = b$ for every twovar rule $r = Q/\{b : \star(p_1 \diamond p_2)\}$ of G for which $\langle v(\psi_1), v(\psi_2)\rangle$ satisfies every $q \in Q$; and (ii) $v(\diamond(\psi_1, \psi_2))^0 \in \diamond_{\mathcal{M}_{LK+}}(v(\psi_1)^0, v(\psi_2)^0)$. We prove (i) and leave (ii) to the reader. Let $r = Q/\{b : \star(p_1 \diamond p_2)\}$ be a twovar rule of G, and suppose that $\langle v(\psi_1), v(\psi_2)\rangle$ satisfies every $q \in Q$. We prove that $v(\psi_1 \diamond \psi_2)^\star = b$. Let σ be any $\mathcal{L}_{\mathcal{U}}$-substitution, assigning ψ_1 to p_1, and ψ_2 to p_2. By ($**$), for every $q \in Q$, there exists some $\mathcal{L}_{\mathcal{U}}$-sequent $s_q \subseteq \Omega$ such that $\vdash_G^{\mathcal{U}^+(\mathcal{W})} s_q \cup \sigma(q)$. By applying weakenings and the rule r, we obtain that $\vdash_G^{\mathcal{U}^+(\mathcal{W})} \bigcup_{q\in Q} s_q \cup \{b : \star(\psi_1 \diamond \psi_2)\}$

(note that $\star(\psi_1 \diamond \psi_2) \in \mathcal{U}^+(\mathcal{W})$ since $\psi_1 \diamond \psi_2 \in \mathcal{U}^-(\mathcal{W})$). This implies that $b : \star(\psi_1 \diamond \psi_2) \notin \Omega$, and so $v(\psi_1 \diamond \psi_2)^\star = b$. □

Corollary 1. *For every $\mathcal{L}_\mathcal{U}$-sequent s, $\vdash_G s$ iff $\vdash_{\mathcal{M}_G} s$.*

Proof. The claim follows by choosing $\mathcal{W} = wff_{\mathcal{L}_\mathcal{U}}$ in Thm. 2 and Thm. 3 (in this case $\mathcal{U}^+(\mathcal{W}) = \mathcal{U}^-(\mathcal{W}) = \mathcal{W}$). Note that an $\mathcal{M}_G$-legal $\mathcal{L}_\mathcal{U}$-valuation v w-satisfies an $\mathcal{L}_\mathcal{U}$-sequent iff $v \models_{\mathcal{M}_G} s$ (since $v(\star\psi)^0 = v(\psi)^\star$ for every $\mathcal{L}_\mathcal{U}$-formula $\star\psi$). □

4 Semantics at Work

Let us take stock of what we have achieved so far. Given a Hilbert calculus $H \in \mathbf{H}$ we introduced an equivalent sequent calculus $G_H \in \mathbf{G}$ and extracted a suitable semantics out of it (the PNmatrix $\mathcal{M}_{G_H}$). In this section we show how to use $\mathcal{M}_{G_H}$ to prove the decidability of H and to check whether G_H is analytic (in the sense defined below). If G_H is not analytic, $\mathcal{M}_{G_H}$ is used to define a family of cut-free calculi for H.

Corollary 2 (Decidability). *Given a Hilbert system $H \in \mathbf{H}$ and a finite set $\Gamma \cup \{\varphi\}$ of formulas, it is decidable whether $\Gamma \vdash_H \varphi$ or not.*

Proof. Follows by the soundness and completeness of $\mathcal{M}_{G_H}$ for G_H, Thm. 1, and the fact, proved in [6], that each logic characterized by a finite PNmatrix is decidable. □

Roughly speaking, a sequent calculus is analytic if whenever a sequent s is provable in it, it can also be proven using only the "syntactic material available within s". Usually this "material" is taken to consist of all subformulas occurring in s (in this case 'analyticity' is just another name for the global subformula property). However, weaker variants have also been considered in the literature, especially in modal logic. In this paper we use the following:

Definition 18. *A $\mathcal{U}$-extension G of LK^+ is* $\mathcal{U}$-analytic *if for every $\mathcal{L}_\mathcal{U}$-sequent s: $\vdash_G s$ implies that $\vdash_G^{\mathcal{U}^+(sub[s])} s$.*

Next, we show that $\mathcal{M}_G$ can be easily used to check whether G is $\mathcal{U}$-analytic.

Definition 19. *A PNmatrix $\mathcal{M}$ for $\mathcal{L}$ is called* proper *if $\mathcal{V}_\mathcal{M}$ is non-empty and $\diamond_\mathcal{M}(x_1, \ldots, x_n) \neq \emptyset$ for every n-ary connective $\diamond$ of $\mathcal{L}$ and $x_1, \ldots, x_n \in \mathcal{V}_\mathcal{M}$.*

Theorem 4. *A $\mathcal{U}$-extension G of LK^+ is $\mathcal{U}$-analytic iff $\mathcal{M}_G$ is proper.*

Proof. ($\Rightarrow$) Suppose that $\mathcal{M}_G$ is not proper. First, if $\mathcal{V}_{\mathcal{M}_G}$ is empty, then $\vdash_{\mathcal{M}_G} \emptyset$, and so (by Cor. 1), $\vdash_G \emptyset$. But, $\mathcal{U}^+(\emptyset) = \emptyset$, and clearly there is no derivation in G that does not contain any formula. It follows that G is not $\mathcal{U}$-analytic in this case. Otherwise, there exist either some $\triangleright \in \mathcal{U}$ and $u \in \mathcal{V}_{\mathcal{M}_G}$ such that $\triangleright_{\mathcal{M}_G}(u) = \emptyset$, or some $\diamond \in \{\wedge, \vee, \supset\}$ and $u_1, u_2 \in \mathcal{V}_{\mathcal{M}_G}$ such that $\diamond_{\mathcal{M}_G}(u_1, u_2) = \emptyset$. We consider here only the first case and leave the second to the reader. Define the

$\mathcal{L}_{\mathcal{U}}$-sequent $s = \{\overline{u^0} : p_1\} \cup \{\overline{u^\star} : \star p_1 \mid \star \in \mathcal{U}\}$ (where $\overline{t} = f$ and $\overline{f} = t$). We first prove that $\vdash_G s$. By Cor. 1 it suffices to show $\vdash_{\mathcal{M}_G} s$. Suppose otherwise, and let v be an $\mathcal{M}_G$-legal $\mathcal{L}_{\mathcal{U}}$-valuations such that $v \not\models_{\mathcal{M}_G} s$. Then, $v(p_1)^0 = u^0$ and $v(\star p_1)^0 = u^\star$ for every $\star \in \mathcal{U}$. Since v is $\mathcal{M}_G$-legal, we have that $v(p_1)^\star = u^\star$ for every $\star \in \mathcal{U}$. It follows that $v(p_1) = u$. Since v is $\mathcal{M}_G$-legal, we have $v(\triangleright p_1) \in \triangleright_{\mathcal{M}_G}(v(p_1))$. Clearly, this is not possible under the assumption that $\triangleright_{\mathcal{M}_G}(u) = \emptyset$. Next we claim that $\not\vdash_G^{\mathcal{U}^+(sub[s])} s$ (and so G is not $\mathcal{U}$-analytic). To see this, note that the $\{p_1\}$-valuation defined by $v(p_1) = u$ is an $\mathcal{M}_G$-legal $\mathcal{U}^-(sub[s])$-valuation that does not w-satisfy s. By Thm. 2, $\not\vdash_G^{\mathcal{U}^+(sub[s])} s$.
($\Leftarrow$) Assume that $\mathcal{M}_G$ is proper and $\not\vdash_G^{\mathcal{U}^+(sub[s])} s$ for some $\mathcal{L}_{\mathcal{U}}$-sequent s. We prove that $\not\vdash_G s$. By Thm. 3, there exists an $\mathcal{M}_G$-legal $\mathcal{U}^-(sub[s])$-valuation v that does not w-satisfy s. Being $\mathcal{M}_G$ proper, it is straightforward to extend v to a (full) $\mathcal{M}_G$-legal $\mathcal{L}_{\mathcal{U}}$-valuation v'. Note that $v' \not\models_{\mathcal{M}_G} s$ (since $v(\star\psi)^0 = v'(\psi)^\star$ for every $\mathcal{L}_{\mathcal{U}}$-formula $\star\psi$). Cor. 1 then entails that $\not\vdash_G s$. □

There are, however, calculi in **G** which are *not* $\mathcal{U}$-analytic. This is the case, e.g., for the extension of HCL^+ by axioms $(\mathbf{n_1})$, $(\mathbf{n^r_\wedge})$, $(\mathbf{b})$ and $(\mathbf{o^1_\wedge})$ (cf. Fig. 1). Its corresponding sequent calculus induces a PNmatrix which is not proper (this can be verified in the system *Paralyzer*), hence it is not $\{\neg, \circ\}$-analytic. When $G \in \mathbf{G}$ is not $\mathcal{U}$-analytic, we start by transforming $\mathcal{M}_G$ into a *finite family* of *proper* PNmatrices, which satisfy the following property:

Definition 20. *([6]) Let $\mathcal{M}$ and $\mathcal{N}$ be PNmatrices for $\mathcal{L}$. We say that $\mathcal{N}$ is a* simple refinement *of $\mathcal{M}$ if $\mathcal{V}_{\mathcal{N}} \subseteq \mathcal{V}_{\mathcal{M}}$, $\mathcal{D}_{\mathcal{N}} = \mathcal{D}_{\mathcal{M}} \cap \mathcal{V}_{\mathcal{N}}$, and $\diamond_{\mathcal{N}}(x_1, \ldots, x_n) \subseteq \diamond_{\mathcal{M}}(x_1, \ldots, x_n)$ for every n-ary connective $\diamond$ of $\mathcal{L}$ and $x_1, \ldots, x_n \in \mathcal{V}_{\mathcal{N}}$.*

Theorem 5. *For every finite PNmatrix $\mathcal{M}$ for $\mathcal{L}$, there exists $\mathcal{M}_1 \ldots \mathcal{M}_n$, finite proper simple refinements of $\mathcal{M}$, such that $\vdash_{\mathcal{M}} = \vdash_{\cap \mathcal{M}_i}$ for $i = 1, \ldots, n$.*

Proof (Outline). Let $\mathcal{M}$ be a PNmatrix for $\mathcal{L}$. Choose $\mathcal{M}_1, \ldots, \mathcal{M}_n$ to be all simple refinements of $\mathcal{M}$ which are proper PNmatrices. Based on the results in [6], we show that $\vdash_{\mathcal{M}} = \vdash_{\cap \mathcal{M}_i}$. ($\Rightarrow$) By Prop. 1 in [6], $\vdash_{\mathcal{M}} \subseteq \vdash_{\mathcal{N}}$ for every simple refinement $\mathcal{N}$ of $\mathcal{M}$. Therefore, $\vdash_{\mathcal{M}} \subseteq \vdash_{\cap \mathcal{M}_i}$.
($\Leftarrow$) Suppose that $\not\vdash_{\mathcal{M}} s$. Thus $v \not\models_{\mathcal{M}} s$ for some $\mathcal{M}$-legal $\mathcal{L}$-valuation v. Thm. 1 in [6] ensures that there exists some $\mathcal{M}_i$, such that v is $\mathcal{M}_i$-legal. The fact that $v \not\models_{\mathcal{M}} s$ easily entails that $v \not\models_{\mathcal{M}_i} s$, and so $\not\vdash_{\mathcal{M}_i} s$. □

We can now apply the method of [3], which produces a cut-free sequent calculus G which is sound and complete for any proper PNmatrix $\mathcal{M}$, whose set of designated truth values ($\mathcal{D}_{\mathcal{M}}$) is a non-empty proper subset of the set of its truth values ($\mathcal{V}_{\mathcal{M}}$), provided that its language satisfies the following slightly reformulated condition of [3]:

Definition 21. *Let $\mathcal{M}$ be a proper PNmatrix for $\mathcal{L}$. We say that $\mathcal{L}$ is* sufficiently expressive *for $\mathcal{M}$ if for any $x \in \mathcal{V}_{\mathcal{M}}$, there exists a set $\mathcal{S}$ of $\mathcal{L}$-sequents, each of which has the form $\{b : \psi\}$, for some $b \in \{f, t\}$ and $\psi \in wff_{\mathcal{L}}$ in which p_1 is the only atomic variable, such that the following condition holds:*

– *For any $\mathcal{M}$-legal $\mathcal{L}$-valuation v and $\varphi \in wff_{\mathcal{L}}$, $v(\varphi) = x$ iff v satisfies every $\mathcal{L}$-sequent in $\sigma(\mathcal{S})$ for $\mathcal{M}$ for any $\mathcal{L}$-substitution σ such that $\sigma(p_1) = \varphi$.*

Corollary 3. *Let $G \in \mathbf{G}$ be a $\mathcal{U}$-extension of LK^+ that is not $\mathcal{U}$-analytic. We can construct a family of cut-free sequent calculi F_G, such that for every sequent s: $\vdash_G s$ iff $\vdash_{G'} s$ for every $G' \in \mathsf{F}_G$.*

Proof. We start by constructing $\mathcal{M}_G$. If $\mathcal{D}_{\mathcal{M}_G} = \emptyset$ or $\mathcal{D}_{\mathcal{M}_G} = \mathcal{V}_{\mathcal{M}_G}$, $\mathcal{M}_G$ has a trivial corresponding cut-free calculus. For the rest of the cases, the claim follows by Thm. 5 using the method of [3]. Note that $\mathcal{L}_{\mathcal{U}}$ is sufficiently expressive for any simple refinement of $\mathcal{M}_G$. Indeed, for $x \in \mathcal{V}_{\mathcal{M}_G}$, define $\mathcal{S}_x = \{x^0 : p_1\} \cup \{x^\star : \star p_1 \mid \star \in \mathcal{U}\}$. Let $\mathcal{M}$ be a simple refinement of $\mathcal{M}_G$ and let v be an $\mathcal{M}$-legal $\mathcal{L}_{\mathcal{U}}$-valuation. The required condition is met by the fact that for every $\star \in \mathcal{U}$ and $\theta \in wff_{\mathcal{L}_{\mathcal{U}}}$, $v(\star\theta)^0 = v(\theta)^\star$ (by condition (i) in Def. 13). $\square$

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