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Fuzzy logics form a natural generalization of classical logic, in which truth values consist of some linearly ordered set, usually taken to be the real interval $[0, 1]$. They have a wide variety of applications, as they provide a reasonable model of certain very common vagueness phenomena. Both their propositional and first-order versions are well-studied by now (see, e.g., [8]). Clearly, for many interesting applications (see, e.g., [5] and Section 5.5.2 in Chapter I of [6]), propositional and first-order fuzzy logics do not suffice, and one has to use higher-order versions. These are much less developed (see, e.g., [16] and [6]), especially from the proof-theoretic perspective. Evidently, higher-order fuzzy logics deserve a proof-theoretic study, with the aim of providing a basis for automated deduction methods, as well as a complimentary point of view in the investigation of these logics.

The proof theory of propositional fuzzy logics is the main subject of [11]. There, an essential tool to develop well-behaved proof calculi for fuzzy logics is the transition from (Gentzen-style) sequents, to *hypersequents*. The latter, that are usually nothing more than disjunctions of sequents, turn to be an adequate proof-theoretic framework for the fundamental fuzzy logics. In particular, propositional Gödel logic (the logic interpreting conjunction as minimum, and disjunction as maximum) is easily captured by a cut-free hypersequent calculus called HG (introduced in [1]). The derivation rules of HG are the standard hypersequent versions of the sequent rules of Gentzen's LJ for intuitionistic logic, and they are augmented by the *communication rule* that allows "exchange of information" between two hypersequents [2]. In [3], it was shown that HIF, the extension of HG with the natural hypersequent versions of LJ's sequent rules for the first-order quantifiers, is sound and (cut-free) complete for standard first-order Gödel logic.[1] As a corollary, one obtains Herbrand theorem for the prenex fragment of this logic (see [11]).

In this work, we study the extension of HIF with usual rules for *second-order* quantifiers. These consist of the single-conclusion hypersequent version of the rules for introducing second-order quantifiers in the ordinary sequent calculus for classical logic (see, e.g., [7, 15]). We denote by HIF^2 the extension of (the cut-free fragment of) HIF with these rules. Our main result is that HIF^2 is sound and complete for second-order Gödel logic. Since we do not include the cut rule in HIF^2, this automatically implies the admissibility of cut, which makes this calculus a suitable possible basis for automated theorem proving. It should be noted that like in the case of second-order classical logic, the obtained calculus characterizes *Henkin-style* second-order Gödel logic. Thus second-order quantifiers range over a domain that is directly specified in the second-order structure, and it admits full comprehension (this is a domain of *fuzzy sets* in the case of fuzzy logics). This is in contrast to what is called the *standard semantics*, where second-order quantifiers range over *all* subsets of the universe. Hence HIF^2 is practically a system for two-sorted first-order Gödel logic together with the comprehension axioms (see also [4]).

While the soundness of HIF^2 is straightforward, proving its (cut-free) completeness turns out to be relatively involved. This is similar to the case of second-order classical logic, where the completeness of the *cut-free* sequent calculus was open for

[1] Note that Gödel logic is the only fundamental fuzzy logic whose first-order version is recursively axiomatizable [11].

several years, and known as *Takeuti's conjecture* [14].[2] While usual syntactic arguments for cut-elimination dramatically fail for the rules of second-order quantifiers, Takeuti's conjecture was initially verified by a semantic proof. This was accomplished in two steps. First, the completeness was proved with respect to three-valued non-deterministic semantics (this was done by Schütte in [12]). Then, it was left to show that from every three-valued non-deterministic counter-model, one can extract a usual (two-valued) counter-model, without losing comprehension (this was done first by Tait in [13]). Basically, we take a similar approach. First, we present a non-deterministic semantics for HIF^2 with generalized truth values. Then, we use this semantics to derive completeness with respect to the ordinary semantics. We also note that the main ideas behind the non-deterministic semantics that we use here were laid down in [9], where a proof-theoretic framework for adding non-deterministic connectives to propositional Gödel logic was suggested. In addition, the completeness proof for this semantics is an adaptation of the semantic proof in [10] of cut-admissibility in HIF.

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[2] More precisely, Takeuti's conjecture concerned full type-theory, namely, the completeness of the cut-free sequent calculus that includes rules for quantifiers of any finite arity. However, the proof for second-order fragment was the main breakthrough.

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