Finite-valued Semantics for Canonical Labelled Calculi

Anna Zamansky   Ori Lahav

Vienna University of Technology   Tel Aviv University

Compositional Meaning in Logic [GeTFun 1.0], Unilog 2013

Finite-valued Semantics for Canonical Labelled Calculi

Anna Zamansky Ori Lahav

Vienna University of Technology Tel Aviv University

Compositional Meaning in Logic [GeTFun 1.0], Unilog 2013

Matthias Baaz, Ori Lahav, and Anna Zamansky, *Finite-valued Semantics for Canonical Labelled Calculi*, **Journal of Automated Reasoning**, 2013.

The Big Picture

- Goals:
  - Characterization of important proof-theoretic properties of calculi: *cut-admissibility, the subformula property, invertibility of rules,...*
  - Understanding the dependencies between them
  - Tighten the relations between proof-theory and semantics
- Tool: Non-deterministic semantics
  - Goes back to [Schütte 1960], [Tait 1966]
  - Formalized and studied in [Avron,Lev 2001]
- Framework: Canonical labelled sequent calculi
  - Labelled = many-sided

Labelled Sequent Calculi

- A propositional language $\mathcal{L}$
- A finite set of labels $\mathcal{C}$ $\quad \mathcal{C} \subseteq \{\blacksquare, \blacksquare, \blacksquare, \blacksquare, \ldots\}$
- Labelled formula $:= \square : A \quad$ where $A \in Frm_{\mathcal{L}}$ and $\square \in \mathcal{C}$
- Sequent $:=$ a finite set of labelled formulas

Labelled Sequent Calculi

- A propositional language $\mathcal{L}$
- A finite set of labels $\mathcal{C}$ $\qquad \mathcal{C} \subseteq \{\textcolor{blue}{\blacksquare}, \textcolor{red}{\blacksquare}, \textcolor{orange}{\blacksquare}, \textcolor{green}{\blacksquare}, \ldots\}$
- Labelled formula$:= \Box : A$ $\qquad$ where $A \in Frm_{\mathcal{L}}$ and $\Box \in \mathcal{C}$
- Sequent$:=$ a finite set of labelled formulas

$$\mathcal{C} = \{\textcolor{blue}{\blacksquare}, \textcolor{red}{\blacksquare}, \textcolor{orange}{\blacksquare}, \textcolor{green}{\blacksquare}\} \qquad \{\textcolor{red}{\blacksquare} : p_1, \textcolor{green}{\blacksquare} : \neg p_1\}$$

Labelled Sequent Calculi

- A propositional language $\mathcal{L}$
- A finite set of labels $\mathcal{C}$ $\qquad \mathcal{C} \subseteq \{{\color{blue}\blacksquare}, {\color{red}\blacksquare}, {\color{orange}\blacksquare}, {\color{green}\blacksquare}, \ldots\}$
- Labelled formula:= $\square : A$ $\qquad$ where $A \in Frm_{\mathcal{L}}$ and $\square \in \mathcal{C}$
- Sequent:= a finite set of labelled formulas

$$\mathcal{C} = \{{\color{blue}\blacksquare}, {\color{red}\blacksquare}, {\color{orange}\blacksquare}, {\color{green}\blacksquare}\} \qquad \{{\color{red}\blacksquare} : p_1, {\color{green}\blacksquare} : \neg p_1\}$$

$$\dfrac{\dfrac{\{{\color{red}\blacksquare} : p_1\}}{\{{\color{red}\blacksquare} : \neg p_1, {\color{blue}\blacksquare} : \neg p_1\}} \qquad \dfrac{\{{\color{red}\blacksquare} : p_1\}}{\{{\color{blue}\blacksquare} : \neg p_1, {\color{orange}\blacksquare} : \neg p_1\}}}{\{{\color{blue}\blacksquare} : \neg p_1\}}$$

Labelled Sequent Calculi

- A propositional language $\mathcal{L}$
- A finite set of labels $\mathcal{C}$ $\qquad \mathcal{C} \subseteq \{{\color{blue}\blacksquare}, {\color{red}\blacksquare}, {\color{orange}\blacksquare}, {\color{green}\blacksquare}, \ldots\}$
- Labelled formula:= $\square : A \quad$ where $A \in Frm_{\mathcal{L}}$ and $\square \in \mathcal{C}$
- Sequent:= a finite set of labelled formulas

$$\mathcal{C} = \{{\color{blue}\blacksquare}, {\color{red}\blacksquare}, {\color{orange}\blacksquare}, {\color{green}\blacksquare}\} \qquad \{{\color{red}\blacksquare} : p_1, {\color{green}\blacksquare} : \neg p_1\}$$

$$\frac{\dfrac{\{{\color{red}\blacksquare} : p_1\}}{\{{\color{red}\blacksquare} : \neg p_1, {\color{blue}\blacksquare} : \neg p_1\}} \qquad \dfrac{\{{\color{red}\blacksquare} : p_1\}}{\{{\color{blue}\blacksquare} : \neg p_1, {\color{orange}\blacksquare} : \neg p_1\}}}{\{{\color{blue}\blacksquare} : \neg p_1\}}$$

$$p_1, p_1 \supset p_2 \Rightarrow p_2 \quad \leftrightsquigarrow \quad \{{\color{red}\blacksquare} : p_1, {\color{red}\blacksquare} : p_1 \supset p_2, {\color{blue}\blacksquare} : p_2\}$$

Canonical Labelled Calculi

1. All standard structural rules
   (exchange, contraction, weakening)
2. A finite set of primitive rules
3. A finite set of canonical logical rules

Primitive Rules

Manipulate labels. Have the form ($\Box$'s are replaced by labels)

$$\frac{\{\Box : A, \ldots, \Box : A\} \cup s \quad \ldots \quad \{\Box : A, \ldots, \Box : A\} \cup s}{\{\Box : A, \ldots, \Box : A\} \cup s}$$

Primitive Rules

Manipulate labels. Have the form ($\square$'s are replaced by labels)

$$\frac{\{\square : A, \ldots, \square : A\} \cup s \quad \ldots \quad \{\square : A, \ldots, \square : A\} \cup s}{\{\square : A, \ldots, \square : A\} \cup s}$$

Examples:

$$\frac{\{{\color{red}\blacksquare} : A\} \cup s \quad \{{\color{blue}\blacksquare} : A\} \cup s}{\{{\color{orange}\blacksquare} : A, {\color{green}\blacksquare} : A\} \cup s}$$

$$\frac{\{{\color{red}\blacksquare} : A\} \cup s \quad \{{\color{blue}\blacksquare} : A\} \cup s}{s}$$

$$\frac{}{\{{\color{red}\blacksquare} : A, {\color{blue}\blacksquare} : A\} \cup s}$$

Canonical Rules

- "Ideal" logical introduction rules [Avron, Lev 2001]:
  - Introduce *exactly one connective*.
  - The active formulas are *immediate subformulas* of the principal formula.
  - The application is *context-independent*.

$$\frac{\Gamma \Rightarrow A, \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \supset B \Rightarrow \Delta}$$

Canonical Rules

- "Ideal" logical introduction rules [Avron, Lev 2001]:
  - Introduce *exactly one connective*.
  - The active formulas are *immediate subformulas* of the principal formula.
  - The application is *context-independent*.

$$\frac{\Gamma \Rightarrow A, \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \supset B \Rightarrow \Delta}$$

- In Labelled Calculi [Avron, Zamansky 2009]:

$$\frac{\{{\color{blue}\blacksquare} : A\} \cup s \quad \{{\color{red}\blacksquare} : B\} \cup s}{\{{\color{red}\blacksquare} : A \supset B\} \cup s}$$

Canonical Rules

- "Ideal" logical introduction rules [Avron, Lev 2001]:
  - Introduce *exactly one connective*.
  - The active formulas are *immediate subformulas* of the principal formula.
  - The application is *context-independent*.

$$\frac{\Gamma \Rightarrow A, \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \supset B \Rightarrow \Delta}$$

- In Labelled Calculi [Avron, Zamansky 2009]:

$$\frac{\{{\color{blue}\blacksquare} : A\} \cup s \quad \{{\color{red}\blacksquare} : B\} \cup s}{\{{\color{red}\blacksquare} : A \supset B\} \cup s}$$

  - May introduce a connective with more than one label.

$$\frac{\{{\color{red}\blacksquare} : A, {\color{blue}\blacksquare} : B\} \cup s \quad \{{\color{orange}\blacksquare} : B, {\color{red}\blacksquare} : C, {\color{blue}\blacksquare} : C\} \cup s}{\{{\color{red}\blacksquare} : \heartsuit(A, B, C), {\color{blue}\blacksquare} : \heartsuit(A, B, C)\} \cup s}$$

Canonical Labelled Calculi

1. All standard structural rules
   (exchange, contraction, weakening)
2. A finite set of primitive rules
3. A finite set of canonical logical rules

Semantics

Intuition

- The value of A determines which of the labelled formulas ■ : A, ■ : A, ■ : $A, \ldots$ is true.
- In general, there are $2^{|\mathcal{C}|}$ possible options.
- Primitive rules forbid some of them.
- Logical rules are used to determine the values of compound formulas.

Semantics

Intuition

- The value of A determines which of the labelled formulas $\blacksquare : A, \blacksquare : A, \blacksquare : A, \dots$ is true.
- In general, there are $2^{|\mathcal{C}|}$ possible options.
- Primitive rules forbid some of them.
- Logical rules are used to determine the values of compound formulas.

Formalization

- The set of truth-values $\mathcal{T}_{\mathbf{G}} \subseteq P(\mathcal{C})$ is determined according to the primitive rules of **G**.
- A valuation $v : Frm_{\mathcal{L}} \to \mathcal{T}_{\mathbf{G}}$ is a model of $\square : A$ if $\square \in v(A)$.
- A valuation is a model of a sequent s if it is a model of some labelled formula in s.

Example: Semantic Effect of Primitive Rules

$$\mathcal{C} = \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}$$

Example: Semantic Effect of Primitive Rules

$$\mathcal{C} = \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}$$

$$\mathcal{T}_{\mathbf{G}} = \{\{\ \}, \{{\color{red}\blacksquare}\}, \{{\color{blue}\blacksquare}\}, \{{\color{orange}\blacksquare}\}, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{red}\blacksquare}, {\color{orange}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$$

Example: Semantic Effect of Primitive Rules

$$\mathcal{C} = \{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}, \textcolor{orange}{\blacksquare}\}$$

$$\frac{\{\textcolor{orange}{\blacksquare} : A\} \cup s}{\{\textcolor{red}{\blacksquare} : A, \textcolor{blue}{\blacksquare} : A\} \cup s}\ r_1$$

$$\mathcal{T}_{\mathbf{G}} = \{\{\ \}, \{\textcolor{red}{\blacksquare}\}, \{\textcolor{blue}{\blacksquare}\}, \{\textcolor{orange}{\blacksquare}\}, \{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}\}, \{\textcolor{red}{\blacksquare}, \textcolor{orange}{\blacksquare}\}, \{\textcolor{blue}{\blacksquare}, \textcolor{orange}{\blacksquare}\}, \{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}, \textcolor{orange}{\blacksquare}\}\}$$

Example: Semantic Effect of Primitive Rules

$$\mathcal{C} = \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}$$

$$\frac{\{{\color{orange}\blacksquare} : A\} \cup s}{\{{\color{red}\blacksquare} : A, {\color{blue}\blacksquare} : A\} \cup s}\ r_1$$

$$\mathcal{T}_{\mathbf{G}} = \{\{\ \}, \{{\color{red}\blacksquare}\}, \{{\color{blue}\blacksquare}\}, \cancel{\{{\color{orange}\blacksquare}\}}, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{red}\blacksquare}, {\color{orange}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$$

Example: Semantic Effect of Primitive Rules

$$\mathcal{C} = \{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}, \textcolor{orange}{\blacksquare}\}$$

$$\frac{\{\textcolor{orange}{\blacksquare} : A\} \cup s}{\{\textcolor{red}{\blacksquare} : A, \textcolor{blue}{\blacksquare} : A\} \cup s}\ r_1 \qquad \frac{\{\textcolor{red}{\blacksquare} : A\} \cup s \quad \{\textcolor{orange}{\blacksquare} : A\} \cup s}{s}\ r_2$$

$$\mathcal{T}_{\mathbf{G}} = \{\{\ \}, \{\textcolor{red}{\blacksquare}\}, \{\textcolor{blue}{\blacksquare}\}, \xcancel{\{\textcolor{orange}{\blacksquare}\}}, \{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}\}, \{\textcolor{red}{\blacksquare}, \textcolor{orange}{\blacksquare}\}, \{\textcolor{blue}{\blacksquare}, \textcolor{orange}{\blacksquare}\}, \{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}, \textcolor{orange}{\blacksquare}\}\}$$

Example: Semantic Effect of Primitive Rules

$$\mathcal{C} = \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}$$

$$\frac{\{{\color{orange}\blacksquare} : A\} \cup s}{\{{\color{red}\blacksquare} : A, {\color{blue}\blacksquare} : A\} \cup s}\ r_1 \qquad \frac{\{{\color{red}\blacksquare} : A\} \cup s \quad \{{\color{orange}\blacksquare} : A\} \cup s}{s}\ r_2$$

$$\mathcal{T}_{\mathbf{G}} = \{\{\ \}, \{{\color{red}\blacksquare}\}, \{{\color{blue}\blacksquare}\}, \xcancel{\{{\color{orange}\blacksquare}\}}, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \xcancel{\{{\color{red}\blacksquare}, {\color{orange}\blacksquare}\}}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}, \xcancel{\{{\color{red}\blacksquare}, {\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}}\}$$

Example: Semantic Effect of Primitive Rules

$$\mathcal{C} = \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}$$

$$\frac{\{{\color{orange}\blacksquare} : A\} \cup s}{\{{\color{red}\blacksquare} : A, {\color{blue}\blacksquare} : A\} \cup s}\; r_1 \qquad\qquad \frac{\{{\color{red}\blacksquare} : A\} \cup s \quad \{{\color{orange}\blacksquare} : A\} \cup s}{s}\; r_2$$

$$\mathcal{T}_{\mathbf{G}} = \{\{\ \}, \{{\color{red}\blacksquare}\}, \{{\color{blue}\blacksquare}\}, \xcancel{\{{\color{orange}\blacksquare}\}}, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \xcancel{\{{\color{red}\blacksquare}, {\color{orange}\blacksquare}\}}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}, \xcancel{\{{\color{red}\blacksquare}, {\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}}\}$$

$$\mathcal{T}_{\mathbf{G}} = \{\{\ \}, \{{\color{red}\blacksquare}\}, \{{\color{blue}\blacksquare}\}, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$$

The Truth-Tables

The table for a connective is algorithmically extracted from its logical rules.

The Truth-Tables

The table for a connective is algorithmically extracted from its logical rules.

For example:

$$\frac{\{{\color{blue}\blacksquare} : A\} \cup s \quad \{{\color{red}\blacksquare} : B\} \cup s}{\{{\color{red}\blacksquare} : A \supset B\} \cup s}$$

$$\mathcal{T}_{\mathbf{G}} = \{\{{\color{red}\blacksquare}\}, \{{\color{blue}\blacksquare}\}\}$$

$$\frac{\{{\color{red}\blacksquare} : A, {\color{blue}\blacksquare} : B\} \cup s}{\{{\color{blue}\blacksquare} : A \supset B\} \cup s}$$

The Truth-Tables

The table for a connective is algorithmically extracted from its logical rules.

For example:

$$\mathcal{T}_{\mathbf{G}} = \{\{\textcolor{red}{\blacksquare}\}, \{\textcolor{blue}{\blacksquare}\}\}$$

$$\frac{\{\textcolor{blue}{\blacksquare} : A\} \cup s \quad \{\textcolor{red}{\blacksquare} : B\} \cup s}{\{\textcolor{red}{\blacksquare} : A \supset B\} \cup s}$$

$$\frac{\{\textcolor{red}{\blacksquare} : A, \textcolor{blue}{\blacksquare} : B\} \cup s}{\{\textcolor{blue}{\blacksquare} : A \supset B\} \cup s}$$

$\tilde{\supset}$	$\{\textcolor{red}{\blacksquare}\}$	$\{\textcolor{blue}{\blacksquare}\}$
$\{\textcolor{red}{\blacksquare}\}$		
$\{\textcolor{blue}{\blacksquare}\}$		

The Truth-Tables

The table for a connective is algorithmically extracted from its logical rules.

For example:

$$\mathcal{T}_{\mathbf{G}} = \{\{\textcolor{red}{\blacksquare}\}, \{\textcolor{blue}{\blacksquare}\}\}$$

$$\frac{\{\textcolor{blue}{\blacksquare} : A\} \cup s \quad \{\textcolor{red}{\blacksquare} : B\} \cup s}{\{\textcolor{red}{\blacksquare} : A \supset B\} \cup s}$$

$$\frac{\{\textcolor{red}{\blacksquare} : A, \textcolor{blue}{\blacksquare} : B\} \cup s}{\{\textcolor{blue}{\blacksquare} : A \supset B\} \cup s}$$

$\tilde{\supset}$	$\{\textcolor{red}{\blacksquare}\}$	$\{\textcolor{blue}{\blacksquare}\}$
$\{\textcolor{red}{\blacksquare}\}$	$\{\textcolor{blue}{\blacksquare}\}$	
$\{\textcolor{blue}{\blacksquare}\}$		

The Truth-Tables

The table for a connective is algorithmically extracted from its logical rules.

For example:

$$\mathcal{T}_{\mathbf{G}} = \{\{\textcolor{red}{\blacksquare}\}, \{\textcolor{blue}{\blacksquare}\}\}$$

$$\frac{\{\textcolor{blue}{\blacksquare} : A\} \cup s \quad \{\textcolor{red}{\blacksquare} : B\} \cup s}{\{\textcolor{red}{\blacksquare} : A \supset B\} \cup s}$$

$$\frac{\{\textcolor{red}{\blacksquare} : A, \textcolor{blue}{\blacksquare} : B\} \cup s}{\{\textcolor{blue}{\blacksquare} : A \supset B\} \cup s}$$

$\tilde{\supset}$	$\{\textcolor{red}{\blacksquare}\}$	$\{\textcolor{blue}{\blacksquare}\}$
$\{\textcolor{red}{\blacksquare}\}$	$\{\textcolor{blue}{\blacksquare}\}$	
$\{\textcolor{blue}{\blacksquare}\}$	$\{\textcolor{red}{\blacksquare}\}$	

The Truth-Tables

The table for a connective is algorithmically extracted from its logical rules.

For example:

$$\mathcal{T}_{\mathbf{G}} = \{\{\textcolor{red}{\blacksquare}\}, \{\textcolor{blue}{\blacksquare}\}\}$$

$$\frac{\{\textcolor{blue}{\blacksquare} : A\} \cup s \quad \{\textcolor{red}{\blacksquare} : B\} \cup s}{\{\textcolor{red}{\blacksquare} : A \supset B\} \cup s}$$

$$\frac{\{\textcolor{red}{\blacksquare} : A, \textcolor{blue}{\blacksquare} : B\} \cup s}{\{\textcolor{blue}{\blacksquare} : A \supset B\} \cup s}$$

$\tilde{\supset}$	$\{\textcolor{red}{\blacksquare}\}$	$\{\textcolor{blue}{\blacksquare}\}$
$\{\textcolor{red}{\blacksquare}\}$	$\{\textcolor{blue}{\blacksquare}\}$	$\{\textcolor{blue}{\blacksquare}\}$
$\{\textcolor{blue}{\blacksquare}\}$	$\{\textcolor{red}{\blacksquare}\}$	$\{\textcolor{blue}{\blacksquare}\}$

The Truth-Tables

The table for a connective is algorithmically extracted from its logical rules.

For example:

$$\mathcal{T}_{\mathbf{G}} = \{\{\textcolor{red}{\blacksquare}\}, \{\textcolor{blue}{\blacksquare}\}\}$$

$$\frac{\{\textcolor{blue}{\blacksquare} : A\} \cup s \quad \{\textcolor{red}{\blacksquare} : B\} \cup s}{\{\textcolor{red}{\blacksquare} : A \supset B\} \cup s}$$

$$\frac{\{\textcolor{red}{\blacksquare} : A, \textcolor{blue}{\blacksquare} : B\} \cup s}{\{\textcolor{blue}{\blacksquare} : A \supset B\} \cup s}$$

$\tilde{\supset}$	$\{\textcolor{red}{\blacksquare}\}$	$\{\textcolor{blue}{\blacksquare}\}$
$\{\textcolor{red}{\blacksquare}\}$	$\{\textcolor{blue}{\blacksquare}\}$	$\{\textcolor{blue}{\blacksquare}\}$
$\{\textcolor{blue}{\blacksquare}\}$	$\{\textcolor{red}{\blacksquare}\}$	$\{\textcolor{blue}{\blacksquare}\}$

A legal valuation should respect the table:
$v(\diamond(A_1, \ldots, A_n)) = \tilde{\diamond}(v(A_1), \ldots, v(A_n))$

Non-determinism

Non-determinism

- Non truth-functional connectives,
  e.g. primal implication [Gurevich, Neeman 2009]:

$$\mathcal{T}_{\mathbf{G}} = \{\{{\color{red}\blacksquare}\}, \{{\color{blue}\blacksquare}\}\}$$

$$\frac{\{{\color{blue}\blacksquare} : A\} \cup s \quad \{{\color{red}\blacksquare} : B\} \cup s}{\{{\color{red}\blacksquare} : A \supset B\} \cup s}$$

$$\frac{\{{\color{blue}\blacksquare} : B\} \cup s}{\{{\color{blue}\blacksquare} : A \supset B\} \cup s}$$

How to determine $\tilde{\supset}(\{{\color{red}\blacksquare}\}, \{{\color{red}\blacksquare}\})$?

Non-determinism

- More than one option satisfies the conclusion, e.g.

$$\mathcal{T}_{\mathbf{G}} = \{\{{\color{red}\blacksquare}\}, \{{\color{blue}\blacksquare}\}, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}\}$$

$$\frac{\{{\color{blue}\blacksquare} : A\} \cup s \quad \{{\color{red}\blacksquare} : B\} \cup s}{\{{\color{red}\blacksquare} : A \supset B\} \cup s}$$

$$\frac{\{{\color{red}\blacksquare} : A, {\color{blue}\blacksquare} : B\} \cup s}{\{{\color{blue}\blacksquare} : A \supset B\} \cup s}$$

How to determine $\tilde{\supset}(\{{\color{blue}\blacksquare}\}, \{{\color{red}\blacksquare}\})$?

Non-determinism

- More than one option satisfies the conclusion, e.g.

$$\frac{\{\textcolor{blue}{\blacksquare} : A\} \cup s \quad \{\textcolor{red}{\blacksquare} : B\} \cup s}{\{\textcolor{red}{\blacksquare} : A \supset B\} \cup s}$$

$$\mathcal{T}_{\mathbf{G}} = \{\{\textcolor{red}{\blacksquare}\}, \{\textcolor{blue}{\blacksquare}\}, \{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}\}\}$$

$$\frac{\{\textcolor{red}{\blacksquare} : A, \textcolor{blue}{\blacksquare} : B\} \cup s}{\{\textcolor{blue}{\blacksquare} : A \supset B\} \cup s}$$

How to determine $\tilde{\supset}(\{\textcolor{blue}{\blacksquare}\}, \{\textcolor{red}{\blacksquare}\})$?

Non-deterministic Truth-Tables [Avron, Lev 2001]

A table of an n-ary connective $\diamond$ is a function $\tilde{\diamond} : \mathcal{T}^n \to P^+(\mathcal{T})$.
A legal valuation satisfies: $v(\diamond(A_1, \ldots, A_n)) \in \tilde{\diamond}(v(A_1), \ldots, v(A_n))$

Example: Construction of a Non-deterministic Truth-Table

$\mathcal{C} = \{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}, \textcolor{orange}{\blacksquare}\}$ $\mathcal{T}_{\mathbf{G}} = \{\emptyset, \{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}\}, \{\textcolor{blue}{\blacksquare}, \textcolor{orange}{\blacksquare}\}\}$ $\circ$ is a binary connective

Example: Construction of a Non-deterministic Truth-Table

$\mathcal{C} = \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}$ $\mathcal{T}_{\mathbf{G}} = \{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$ $\circ$ is a binary connective

$\widetilde{\circ}$	$\emptyset$	$\{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}$	$\{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}$
$\emptyset$	$\{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$	$\{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$	$\{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$
$\{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}$	$\{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$	$\{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$	$\{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$
$\{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}$	$\{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$	$\{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$	$\{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$

Example: Construction of a Non-deterministic Truth-Table

$\mathcal{C} = \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}$ $\mathcal{T}_{\mathbf{G}} = \{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$ $\circ$ is a binary connective

$$\frac{\{{\color{red}\blacksquare} : A\} \cup s \quad \{{\color{red}\blacksquare} : B\} \cup s}{\{{\color{red}\blacksquare} : A \circ B\} \cup s}$$

$\widetilde{\circ}$	$\emptyset$	$\{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}$	$\{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}$
$\emptyset$	$\{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$	$\{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$	$\{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$
$\{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}$	$\{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$	$\{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$	$\{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$
$\{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}$	$\{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$	$\{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$	$\{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$

Example: Construction of a Non-deterministic Truth-Table

$\mathcal{C} = \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}$ $\qquad$ $\mathcal{T}_{\mathbf{G}} = \{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$ $\qquad$ $\circ$ is a binary connective

$$\frac{\{{\color{red}\blacksquare} : A\} \cup s \quad \{{\color{red}\blacksquare} : B\} \cup s}{\{{\color{red}\blacksquare} : A \circ B\} \cup s}$$

$\widetilde{\circ}$	$\emptyset$	$\{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}$	$\{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}$
$\emptyset$	$\{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$	$\{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$	$\{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$
$\{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}$	$\{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$	{ ~~$\emptyset$~~, $\{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}$, ~~$\{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}$~~ }	$\{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$
$\{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}$	$\{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$	$\{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$	$\{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$

Example: Construction of a Non-deterministic Truth-Table

$\mathcal{C} = \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}$ $\qquad$ $\mathcal{T}_{\mathbf{G}} = \{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$ $\qquad$ $\circ$ is a binary connective

$$\frac{\{{\color{red}\blacksquare} : A\} \cup s \quad \{{\color{red}\blacksquare} : B\} \cup s}{\{{\color{red}\blacksquare} : A \circ B\} \cup s} \qquad \frac{\{{\color{red}\blacksquare} : A\} \cup s \quad \{{\color{orange}\blacksquare} : B\} \cup s}{\{{\color{red}\blacksquare} : A \circ B, {\color{blue}\blacksquare} : A \circ B\} \cup s}$$

$\widetilde{\circ}$	$\emptyset$	$\{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}$	$\{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}$
$\emptyset$	$\{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$	$\{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$	$\{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$
$\{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}$	$\{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$	$\{\cancel{\emptyset}, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \cancel{\{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}}\}$	$\{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$
$\{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}$	$\{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$	$\{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$	$\{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$

Example: Construction of a Non-deterministic Truth-Table

$\mathcal{C} = \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}$ $\qquad$ $\mathcal{T}_{\mathbf{G}} = \{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$ $\qquad$ $\circ$ is a binary connective

$$\frac{\{{\color{red}\blacksquare} : A\} \cup s \quad \{{\color{red}\blacksquare} : B\} \cup s}{\{{\color{red}\blacksquare} : A \circ B\} \cup s} \qquad \frac{\{{\color{red}\blacksquare} : A\} \cup s \quad \{{\color{orange}\blacksquare} : B\} \cup s}{\{{\color{red}\blacksquare} : A \circ B, {\color{blue}\blacksquare} : A \circ B\} \cup s}$$

$\widetilde{\circ}$	$\emptyset$	$\{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}$	$\{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}$
$\emptyset$	$\{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$	$\{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$	$\{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$
$\{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}$	$\{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$	$\{\cancel{\emptyset}, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \cancel{\{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}}\}$	$\{\cancel{\emptyset}, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$
$\{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}$	$\{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$	$\{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$	$\{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$

Example: Construction of a Non-deterministic Truth-Table

$\mathcal{C} = \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}$ $\quad \mathcal{T}_{\mathbf{G}} = \{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$ $\quad \circ$ is a binary connective

$$\frac{\{{\color{red}\blacksquare} : A\} \cup s \quad \{{\color{red}\blacksquare} : B\} \cup s}{\{{\color{red}\blacksquare} : A \circ B\} \cup s} \qquad \frac{\{{\color{red}\blacksquare} : A\} \cup s \quad \{{\color{orange}\blacksquare} : B\} \cup s}{\{{\color{red}\blacksquare} : A \circ B, {\color{blue}\blacksquare} : A \circ B\} \cup s} \qquad \frac{\{{\color{blue}\blacksquare} : A, {\color{orange}\blacksquare} : B\} \cup s}{\{{\color{blue}\blacksquare} : A \circ B \cup s}$$

$\widetilde{\circ}$	$\emptyset$	$\{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}$	$\{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}$
$\emptyset$	$\{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$	$\{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$	$\{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$
$\{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}$	$\{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$	$\{\cancel{\emptyset}, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \cancel{\{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}}\}$	$\{\cancel{\emptyset}, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$
$\{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}$	$\{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$	$\{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$	$\{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$

Example: Construction of a Non-deterministic Truth-Table

$\mathcal{C} = \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}$ $\quad$ $\mathcal{T}_{\mathbf{G}} = \{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$ $\quad$ $\circ$ is a binary connective

$$\frac{\{{\color{red}\blacksquare} : A\} \cup s \quad \{{\color{red}\blacksquare} : B\} \cup s}{\{{\color{red}\blacksquare} : A \circ B\} \cup s} \qquad \frac{\{{\color{red}\blacksquare} : A\} \cup s \quad \{{\color{orange}\blacksquare} : B\} \cup s}{\{{\color{red}\blacksquare} : A \circ B, {\color{blue}\blacksquare} : A \circ B\} \cup s} \qquad \frac{\{{\color{blue}\blacksquare} : A, {\color{orange}\blacksquare} : B\} \cup s}{\{{\color{blue}\blacksquare} : A \circ B \cup s}$$

$\tilde{\circ}$	$\emptyset$	$\{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}$	$\{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}$
$\emptyset$	$\{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$	$\{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$	$\{\cancel{\emptyset}, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$
$\{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}$	$\{\cancel{\emptyset}, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$	$\{\cancel{\emptyset}, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \cancel{\{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}}\}$	$\{\cancel{\emptyset}, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$
$\{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}$	$\{\cancel{\emptyset}, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$	$\{\cancel{\emptyset}, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$	$\{\cancel{\emptyset}, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$

Example: Construction of a Non-deterministic Truth-Table

$\mathcal{C} = \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}$ $\quad$ $\mathcal{T}_{\mathbf{G}} = \{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$ $\quad$ $\circ$ is a binary connective

$$\frac{\{{\color{red}\blacksquare} : A\} \cup s \quad \{{\color{red}\blacksquare} : B\} \cup s}{\{{\color{red}\blacksquare} : A \circ B\} \cup s} \qquad \frac{\{{\color{red}\blacksquare} : A\} \cup s \quad \{{\color{orange}\blacksquare} : B\} \cup s}{\{{\color{red}\blacksquare} : A \circ B, {\color{blue}\blacksquare} : A \circ B\} \cup s} \qquad \frac{\{{\color{blue}\blacksquare} : A, {\color{orange}\blacksquare} : B\} \cup s}{\{{\color{blue}\blacksquare} : A \circ B \cup s}$$

$\widetilde{\circ}$	$\emptyset$	$\{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}$	$\{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}$
$\emptyset$	$\{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$	$\{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$	$\{\cancel{\emptyset}, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$
$\{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}$	$\{\cancel{\emptyset}, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$	$\{\cancel{\emptyset}, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \cancel{\{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}}\}$	$\{\cancel{\emptyset}, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$
$\{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}$	$\{\cancel{\emptyset}, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$	$\{\cancel{\emptyset}, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$	$\{\cancel{\emptyset}, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$

$\widetilde{\circ}$	$\emptyset$	$\{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}$	$\{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}$
$\emptyset$	$\{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$	$\{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$	$\{\{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$
$\{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}$	$\{\{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$	$\{\{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}\}$	$\{\{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$
$\{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}$	$\{\{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$	$\{\{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$	$\{\{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$

What Can Go Wrong?

What Can Go Wrong?

- Contradictions between rules, e.g.

$$\mathcal{T}_{\mathbf{G}} = \{\{{\color{red}\blacksquare}\}, \{{\color{blue}\blacksquare}\}\} \qquad \frac{\{{\color{red}\blacksquare} : B\} \cup s}{\{{\color{red}\blacksquare} : A \diamond B\} \cup s} \qquad \frac{\{{\color{blue}\blacksquare} : A\} \cup s}{\{{\color{blue}\blacksquare} : A \diamond B\} \cup s}$$

How to determine $\tilde{\diamond}(\{{\color{blue}\blacksquare}\}, \{{\color{red}\blacksquare}\})$?

What Can Go Wrong?

- Contradictions between rules, e.g.

$$\mathcal{T}_{\mathbf{G}} = \{\{\color{red}\blacksquare\}, \{\color{blue}\blacksquare\}\} \qquad \frac{\{\color{red}\blacksquare : B\} \cup s}{\{\color{red}\blacksquare : A \diamond B\} \cup s} \qquad \frac{\{\color{blue}\blacksquare : A\} \cup s}{\{\color{blue}\blacksquare : A \diamond B\} \cup s}$$

How to determine $\tilde{\diamond}(\{\color{blue}\blacksquare\}, \{\color{red}\blacksquare\})$?

$\tilde{\diamond}$	$\{\color{red}\blacksquare\}$	$\{\color{blue}\blacksquare\}$
$\{\color{red}\blacksquare\}$	$\{\{\color{red}\blacksquare\}\}$	$\{\{\color{red}\blacksquare\}, \{\color{blue}\blacksquare\}\}$
$\{\color{blue}\blacksquare\}$	$\emptyset$	$\{\{\color{blue}\blacksquare\}\}$

$\{\color{blue}\blacksquare\}$ and $\{\color{red}\blacksquare\}$ cannot be used by the same valuation.

What Can Go Wrong?

- Contradictions between rules, e.g.

$$\mathcal{T}_{\mathbf{G}} = \{\{{\color{red}\blacksquare}\}, \{{\color{blue}\blacksquare}\}\} \qquad \frac{\{{\color{red}\blacksquare} : B\} \cup s}{\{{\color{red}\blacksquare} : A \diamond B\} \cup s} \qquad \frac{\{{\color{blue}\blacksquare} : A\} \cup s}{\{{\color{blue}\blacksquare} : A \diamond B\} \cup s}$$

How to determine $\tilde{\diamond}(\{{\color{blue}\blacksquare}\}, \{{\color{red}\blacksquare}\})$?

$\tilde{\diamond}$	$\{{\color{red}\blacksquare}\}$	$\{{\color{blue}\blacksquare}\}$
$\{{\color{red}\blacksquare}\}$	$\{\{{\color{red}\blacksquare}\}\}$	$\{\{{\color{red}\blacksquare}\}, \{{\color{blue}\blacksquare}\}\}$
$\{{\color{blue}\blacksquare}\}$	$\emptyset$	$\{\{{\color{blue}\blacksquare}\}\}$

$\{{\color{blue}\blacksquare}\}$ and $\{{\color{red}\blacksquare}\}$ cannot be used by the same valuation.

Partial Non-deterministic Truth-Tables

Allow empty entries: $\tilde{\diamond} : \mathcal{T}^n \to P(\mathcal{T})$.

The Semantic Framework

Partial Non-deterministic Matrices

A PNmatrix **M** for $\mathcal{L}$ and $\mathcal{C}$ consists of:

- A set $\mathcal{T}$ of truth-values.
- A function $\mathcal{D} : \mathcal{C} \to P(\mathcal{T})$ assigning a set of designated truth-values for every label.
- A partial non-deterministic truth-table $\tilde{\diamond} : \mathcal{T}^n \to P(\mathcal{T})$ for every n-ary connective of $\mathcal{L}$.

A valuation $v : Frm_{\mathcal{L}} \to \mathcal{T}$ is:

- a model (in **M**) of a sequent s if $v(A) \in \mathcal{D}(\Box)$ for some $\Box : A$ in s.
- **M**-legal if $v(\diamond(A_1, \ldots, A_n)) \in \tilde{\diamond}(v(A_1), \ldots, v(A_n))$ for every $\diamond(A_1, \ldots, A_n) \in Frm_{\mathcal{L}}$.

Main Result

Theorem

For every canonical labelled calculus $\mathbf{G}$*, there exists a strongly characteristic PNmatrix* $\mathbf{M_G}$ *(i.e.* $\Omega \vdash_{\mathbf{G}} s$ *iff every* $\mathbf{M_G}$*-legal valuation which is a model of every sequent in* Ω *is also a model of* s*).*

Moreover, we provide a uniform algorithm to obtain $\mathbf{M_G}$ *from* $\mathbf{G}$*.*

Main Result

Theorem

For every canonical labelled calculus $\mathbf{G}$*, there exists a strongly characteristic PNmatrix* $\mathbf{M_G}$ *(i.e.* $\Omega \vdash_{\mathbf{G}} s$ *iff every* $\mathbf{M_G}$*-legal valuation which is a model of every sequent in* Ω *is also a model of* s*).*

Moreover, we provide a uniform algorithm to obtain $\mathbf{M_G}$ *from* $\mathbf{G}$*.*

In many cases, the obtained semantics coincides with a known one:

- Propositional fragment of **LK**
- **LK** without cut [Girard 1987]
- **LK** without identity axiom [Hösli,Jäger 1994]
- Two-sided canonical systems [Avron,Lev 2001]
- Labelled calculi studied in [Baaz et al. 1998] and [Avron,Zamansky 2009]

Effectiveness

Theorem

Semantic consequence relations induced by PNmatrices are decidable.

Corollary

All canonical labelled calculi are decidable.

Effectiveness

Theorem

Semantic consequence relations induced by PNmatrices are decidable.

Proof Outline.

- Usual method: To decide whether s is valid in $\mathbf{M}$, check one-by-one all $\mathbf{M}$-legal partial valuations defined on the subformulas of s, and look for one which is not a model of s.
- Hidden assumption: All $\mathbf{M}$-legal partial valuations can be extended to full ones (semantic analyticity).
  But, it does not hold for PNmatrices (recall $\tilde{\diamond}(\{■\},\{■\}) = \emptyset$!).
- Lemma: It is decidable whether an $\mathbf{M}$-legal partial valuation can be extended to a full one.
- Solution: Check one-by-one all $\mathbf{M}$-legal partial valuations defined on the subformulas of s, and look for one which is both extendable and not a model of s.

Application - “Almost”-Canonical Calculi

Consider the following non-canonical calculus for the basic *LFI* called **BK**:

$$(\wedge\Rightarrow)\quad \frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta} \qquad (\Rightarrow\wedge)\quad \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}$$

$$(\vee\Rightarrow)\quad \frac{\Gamma, A \Rightarrow \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \vee B \Rightarrow \Delta} \qquad (\Rightarrow\vee)\quad \frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B}$$

$$(\supset\Rightarrow)\quad \frac{\Gamma \Rightarrow A, \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \supset B \Rightarrow \Delta} \qquad (\Rightarrow\supset)\quad \frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \supset B, \Delta}$$

$$(\Rightarrow\neg)\quad \frac{\Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg A}$$

$$(\circ\Rightarrow)\quad \frac{\Gamma \Rightarrow A, \Delta \quad \Gamma \Rightarrow \neg A, \Delta}{\Gamma, \circ A \Rightarrow \Delta} \qquad (\Rightarrow\circ)\quad \frac{\Gamma, A, \neg A \Rightarrow \Delta}{\Gamma \Rightarrow \circ A, \Delta}$$

$$(cut)\quad \frac{\Gamma, A \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta} \qquad (id)\quad \frac{}{\Gamma, A \Rightarrow \Delta, A} \qquad (weak)\quad \frac{\Gamma \Rightarrow \Delta}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

Application - "Almost"-Canonical Calculi

$$(\textcolor{red}{\blacksquare} : \wedge)\quad \frac{\{\textcolor{red}{\blacksquare} : A, \textcolor{red}{\blacksquare} : B\} \cup s}{\{\textcolor{red}{\blacksquare} : A \wedge B\} \cup s}$$

$$(\textcolor{blue}{\blacksquare} : \wedge)\quad \frac{\{\textcolor{blue}{\blacksquare} : A\} \cup s \quad \{\textcolor{blue}{\blacksquare} : B\} \cup s}{\{\textcolor{blue}{\blacksquare} : A \wedge B\} \cup s}$$

$$(\textcolor{red}{\blacksquare} : \vee)\quad \frac{\{\textcolor{red}{\blacksquare} : A\} \cup s \quad \{\textcolor{red}{\blacksquare} : B\} \cup s}{\{\textcolor{red}{\blacksquare} : A \vee B\} \cup s}$$

$$(\textcolor{blue}{\blacksquare} : \vee)\quad \frac{\{\textcolor{blue}{\blacksquare} : A, B\} \cup s}{\{\textcolor{blue}{\blacksquare} : A \vee B\} \cup s}$$

$$(\textcolor{red}{\blacksquare} : \supset)\quad \frac{\{\textcolor{red}{\blacksquare} : A\} \cup s \quad \{\textcolor{red}{\blacksquare} : B\} \cup s}{\{\textcolor{red}{\blacksquare} : A \supset B\} \cup s}$$

$$(\textcolor{blue}{\blacksquare} : \supset)\quad \frac{\{\textcolor{red}{\blacksquare} : A, \textcolor{blue}{\blacksquare} : B\} \cup s}{\{\textcolor{blue}{\blacksquare} : A \supset B\} \cup s}$$

$$(\textcolor{blue}{\blacksquare} : \neg)\quad \frac{\{\textcolor{red}{\blacksquare} : A\} \cup s}{\{\textcolor{blue}{\blacksquare} : \neg A\} \cup s}$$

$$(\textcolor{red}{\blacksquare} : \circ)\quad \frac{\{\textcolor{blue}{\blacksquare} : A\} \cup s \quad \{\textcolor{blue}{\blacksquare} : \neg A\} \cup s}{\{\textcolor{red}{\blacksquare} : \circ A\} \cup s}$$

$$(\textcolor{blue}{\blacksquare} : \circ)\quad \frac{\{\textcolor{red}{\blacksquare} : A, \textcolor{red}{\blacksquare} : \neg A\} \cup s}{\{\textcolor{blue}{\blacksquare} : \circ A\} \cup s}$$

$$(cut)\quad \frac{\{\textcolor{red}{\blacksquare} : A\} \cup s \quad \{\textcolor{blue}{\blacksquare} : A\} \cup s}{s} \qquad (id)\quad \frac{}{\{\textcolor{red}{\blacksquare} : A, \textcolor{blue}{\blacksquare} : A\} \cup s} \qquad (weak)\quad \frac{s}{s \cup s'}$$

Translation into a Canonical Labelled Calculus

- Add two labels: ${\color{red}\blacksquare_{\neg}}$ and ${\color{blue}\blacksquare_{\neg}}$.
- Replace the logical rules:

$$\frac{\{{\color{red}\blacksquare} : A\} \cup s}{\{{\color{blue}\blacksquare} : \neg A\} \cup s} \qquad \frac{\{{\color{blue}\blacksquare} : A\} \cup s \quad \{{\color{blue}\blacksquare} : \neg A\} \cup s}{\{{\color{red}\blacksquare} : \circ A\} \cup s} \qquad \frac{\{{\color{red}\blacksquare} : A, {\color{red}\blacksquare} : \neg A\} \cup s}{\{{\color{blue}\blacksquare} : \circ A\} \cup s}$$

by the rules:

$$\frac{\{{\color{red}\blacksquare} : A\} \cup s}{\{{\color{blue}\blacksquare_{\neg}} : A\} \cup s} \qquad \frac{\{{\color{blue}\blacksquare} : A\} \cup s \quad \{{\color{blue}\blacksquare_{\neg}} : A\} \cup s}{\{{\color{red}\blacksquare} : \circ A\} \cup s} \qquad \frac{\{{\color{red}\blacksquare} : A, {\color{red}\blacksquare_{\neg}} : A\} \cup s}{\{{\color{blue}\blacksquare} : \circ A\} \cup s}$$

- Add cut and axiom:

$$\frac{\{{\color{red}\blacksquare_{\neg}} : A\} \cup s \quad \{{\color{blue}\blacksquare_{\neg}} : A\} \cup s}{s} \qquad \frac{}{\{{\color{red}\blacksquare_{\neg}} : A, {\color{blue}\blacksquare_{\neg}} : A\} \cup s}$$

- Add extra logical rules:

$$\frac{\{{\color{red}\blacksquare_{\neg}} : A\} \cup s}{\{{\color{red}\blacksquare} : \neg A\} \cup s} \qquad \frac{\{{\color{blue}\blacksquare_{\neg}} : A\} \cup s}{\{{\color{blue}\blacksquare} : \neg A\} \cup s}$$

Translation into Canonical Labelled Calculi

- Now, we can use the previous method to obtain a PNmatrix for this calculus, and use it in a decision procedure.
- This translation is possible for every canonical calculus with additional logical rules of the form:

$$\frac{\Gamma, \Pi_1 \Rightarrow \Sigma_1, \Delta \quad \dots \quad \Gamma, \Pi_m \Rightarrow \Sigma_m, \Delta}{conc} \diamond$$

where:

- *conc* has one of the following forms (for some n-ary connective $\diamond$):
  - $\Gamma, \diamond(A_1, \dots, A_n) \Rightarrow \Delta$
  - $\Gamma \Rightarrow \diamond(A_1, \dots, A_n), \Delta$
  - $\Gamma, \star \diamond (A_1, \dots, A_n) \Rightarrow \Delta$ for some unary connective $\star$
  - $\Gamma \Rightarrow \star \diamond (A_1, \dots, A_n), \Delta$ for some unary connective $\star$.
- Π's and Σ's consist of A_i's and formulas of the form $\star A_i$ for some unary connective $\star$.

Cut-Admissibility in Canonical Labelled Calculi

A *cut* is a primitive rule of the form:

$$\frac{\{\Box : A, \ldots, \Box : A\} \cup s \quad \ldots \quad \{\Box : A, \ldots, \Box : A\} \cup s}{s}$$

Cut-Admissibility in Canonical Labelled Calculi

A *cut* is a primitive rule of the form:

$$\frac{\{\square : A, \ldots, \square : A\} \cup s \quad \ldots \quad \{\square : A, \ldots, \square : A\} \cup s}{s}$$

$$\frac{\{\textcolor{red}{\blacksquare} : A\} \cup s \quad \{\textcolor{blue}{\blacksquare} : A\} \cup s}{s}$$

$$\frac{\{\textcolor{red}{\blacksquare} : A, \textcolor{blue}{\blacksquare} : A\} \cup s \quad \{\textcolor{orange}{\blacksquare} : A\} \cup s \quad \{\textcolor{green}{\blacksquare} : A\} \cup s}{s}$$

- A is called the *cut-formula*.
- s is called the *cut-context*.

Cut-Admissibility in Canonical Labelled Calculi

A *cut* is a primitive rule of the form:

$$\frac{\{\Box : A, \ldots, \Box : A\} \cup s \quad \ldots \quad \{\Box : A, \ldots, \Box : A\} \cup s}{s}$$

Many-Sided Strong Cut-Admissibility

$\Omega \vdash_{\mathbf{G}} s \Longrightarrow$ there is a derivation of s from Ω in **G** in which: the cut-formula of each cut occurs either in Ω or in the cut-context.

Cut-Admissibility in Canonical Labelled Calculi

A *cut* is a primitive rule of the form:

$$\frac{\{\Box : A, \ldots, \Box : A\} \cup s \quad \ldots \quad \{\Box : A, \ldots, \Box : A\} \cup s}{s}$$

Many-Sided Strong Cut-Admissibility

$\Omega \vdash_{\mathbf{G}} s \Longrightarrow$ there is a derivation of s from Ω in **G** in which: the cut-formula of each cut occurs either in Ω or in the cut-context.

Theorem

A canonical labelled calculus **G** *enjoys many-sided strong cut-admissibility*

iff

$\mathbf{M_G}$ *does not include empty entries*

Summary

- We provided effective and modular semantic characterization for canonical labelled sequent calculi using partial non-deterministic matrices.
- Application: effective semantics for "almost"-canonical calculi via translation to canonical labelled calculi.
- Application: semantic characterization of proof-theoretic properties.

Thank you!