Finite-valued Semantics for Canonical Labelled Calculi

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Abstract

We define a general family of *canonical labelled calculi*, of which many previously studied sequent and labelled calculi are particular instances. We then provide a uniform and modular method to obtain finite-valued semantics for every canonical labelled calculus by introducing the notion of *partial non-deterministic matrices*. The semantics is applied to provide simple *decidable* semantic criteria for two crucial syntactic properties of these calculi: (strong) analyticity and cut-admissibility. Finally, we demonstrate an application of this framework for a large family of paraconsistent logics.

1 Introduction

A useful semantics is an important property of formal calculi. In addition to providing real insights into their underlying logic, such semantics should also be *effective* in the sense of naturally inducing a decision procedure for its calculus. Another desirable property of such semantics is the possibility to apply it for characterizing important syntactic properties of the calculi, which are hard to establish by other means. Analyticity and cut-admissibility are two crucial cases in point.

Recently some systematic methods for constructing such semantics for various calculi have been formulated. In [1] and [2] two families of *labelled sequent calculi* have been studied in this context.[1] [1] considers labelled calculi with generalized forms of cuts and identity axioms and a restricted form of logical rules, and provides some necessary and sufficient conditions for such calculi to have a characteristic finite-valued matrix. In [2] labelled calculi with a less restrictive form of logical rules (but a more restrictive form of cuts and axioms) are considered. The calculi of [2], satisfying a certain coherence condition, have a semantic characterization using a natural generalization of the usual finite-valued matrix called *non-deterministic matrices* ([4]). The semantics provided in [1, 2] for these families of labelled calculi is effective in the above sense, that is the question of whether a sequent s follows in some (non-deterministic) matrix from a set of sequents $\mathcal{S}$, can be reduced to considering legal *partial* valuations, defined on the subformulas of $\mathcal{S} \cup \{s\}$. This naturally induces a decision procedure for such logics.

In this paper we show that the class of labelled calculi that have a finite-valued effective semantics is substantially larger than all the families of calculi considered in the literature in this context. We start by defining a general class of fully-structural and propositional labelled calculi, called *canonical labelled calculi*, of which the labelled calculi of [1, 2] are particular examples. In addition to the weakening rule, canonical labelled calculi have rules of two forms: primitive rules and introduction rules. The former operate on labels and do not mention any connectives, where the generalized cuts and axioms of [1] are specific instances of such rules. As for the latter, each such rule introduces one logical connective of the language. To provide semantics for all of these calculi in a systematic and modular way, we generalize the notion of non-deterministic matrices to *partial non-deterministic matrices* (PNmatrices), in which empty

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[1] A remark is in order here on the relationship between the labelled calculi studied here and the general framework of labelled deductive systems (LDS) from [3]. Both frameworks consider consequence relations between labelled formulas. Methodologically, however, they have different aims: [3] constructs a system for a given logic defined in semantic terms, while we define a semantics for a given labelled system. Moreover, in LDS *anything* is allowed to serve as labels, while we assume a finite set of labels. In this sense, our labelled calculi are a particular instance of LDS.

sets of options are allowed in the truth tables of logical connectives. Although applicable to a much wider range of calculi, the semantic framework of finite PNmatrices shares the following attractive property with both usual and non-deterministic matrices: any calculus that has a characteristic PNmatrix is decidable. Moreover, as opposed to the results in [1, 2], *no* conditions are required for a canonical labelled calculus to have a characteristic PNmatrix: *all* such calculi have one, and so *all* of them are decidable. We then apply PNmatrices to provide simple *decidable* characterizations of the crucial syntactic properties of strong analyticity and strong cut-admissibility in canonical labelled calculi. Finally, we demonstrate how the theory of labelled canonical calculi developed here can be exploited to provide effective semantics also for a variety of logics induced by calculi which are not canonical. One such example is calculi for paraconsistent logics known as C-systems.

We note that this paper is an extension of [5] in two main aspects: it includes a new section on the applications of the theory of labelled canonical calculi (Section 7), and contains full proofs, examples and explanations that were not included in the previous version.

2 Preliminaries

In what follows $\mathcal{L}$ denotes an arbitrary propositional language, and $\pounds$ denotes a finite non-empty set of labels. We assume that $p_1, p_2, \ldots$ are the atomic formulas of any propositional language. We denote by $Frm_{\mathcal{L}}$ the set of all wffs of $\mathcal{L}$. We usually use φ, ψ as metavariables for formulas, Γ, Δ for finite sets of formulas, l for labels, and L for sets of labels.

Definition 2.1. A *labelled formula* (over $\mathcal{L}$ and $\pounds$) is an expression of the form $l : \psi$, where $l \in \pounds$ and $\psi \in Frm_{\mathcal{L}}$. A *sequent* (over $\mathcal{L}$ and $\pounds$) is a finite set of labelled formulas (over $\mathcal{L}$ and $\pounds$). An *n-clause* (over $\pounds$) is a sequent (over $\pounds$) consisting only of atomic formulas from $\{p_1, \ldots, p_n\}$.

Given a set $L \subseteq \pounds$, we write $(L : \psi)$ instead of (the sequent) $\{l : \psi \mid l \in L\}$. Given a labelled formula γ, we denote by $frm[\gamma]$ the (ordinary) formula appearing in γ, and by $sub[\gamma]$ the set of subformulas of the formula $frm[\gamma]$. frm and sub are extended to sets of labelled formulas and to sets of sets of labelled formulas in the obvious way.

Remark 2.2. The usual (two-sided) sequent notation $\psi_1, \ldots, \psi_n \Rightarrow \varphi_1, \ldots, \varphi_m$ can be interpreted as $\{f : \psi_1, \ldots, f : \psi_n, t : \varphi_1, \ldots, t : \varphi_m\}$, i.e. a sequent in the sense of Definition 2.1 over $\pounds = \{t, f\}$. Below we shall sometimes refer to sequents over $\{t, f\}$ as *two-sided sequents*. Similarly, we shall refer to sequent systems employing sequents over $\{t, f\}$ as two-sided sequent systems.

Definition 2.3. A *substitution* (for $\mathcal{L}$) is a function $\sigma : Frm_{\mathcal{L}} \to Frm_{\mathcal{L}}$, which satisfies $\sigma(\diamond(\psi_1, \ldots, \psi_n)) = \diamond(\sigma(\psi_1), \ldots, \sigma(\psi_n))$ for every n-ary connective $\diamond$ of $\mathcal{L}$. A substitution is extended to labelled formulas, sequents, etc. in the obvious way.

3 Canonical Labelled Systems

In this section we define the family of *canonical labelled systems*. This is a general family of labelled systems, which includes many natural subclasses of previously studied calculi. These include the system **LK** for classical logic, the canonical sequent calculi of [4], the signed calculi of [2], the labelled calculi of [1] and the semi-canonical systems of [6].

All canonical labelled systems have in common the *weakening* rule. In addition, they include rules of two types: *primitive rules* and *introduction rules*. Each rule of the latter type introduces exactly one logical connective, while rules of the former type operate on labels and do not mention any logical connectives.

Definition 3.1 (Weakening)**.** The *weakening* rule allows to infer $s \cup s'$ from s for every two sequents s and s'.

Definition 3.2 (Primitive Rules)**.** A *primitive rule for* $\pounds$ is an expression of the form $\{L_1, \ldots, L_n\}/L$ where $n \geq 0$ and $L_1, \ldots, L_n, L \subseteq \pounds$. An *application of a primitive rule* $\{L_1, \ldots, L_n\}/L$ is any inference step of the following form:

$$\frac{(L_1 : \psi) \cup s_1 \quad \ldots \quad (L_n : \psi) \cup s_n}{(L : \psi) \cup s_1 \cup \ldots \cup s_n}$$

where ψ is a formula, and s_i is a sequent for every $1 \leq i \leq n$.

Example 3.3. Suppose $\pounds = \{a, b, c\}$. Consider the primitive rule $\{\{a\}, \{b\}\}/\{b, c\}$. This rule allows to infer $(\{b, c\} : \psi) \cup s_1 \cup s_2$ from $\{a : \psi\} \cup s_1$ and $\{b : \psi\} \cup s_2$ for every two sequents s_1, s_2 and formula ψ.

Definition 3.4. A primitive rule for $\pounds$ of the form $\emptyset/L$ is called a *canonical axiom.* Its applications provide all axioms of the form $(L : \psi)$.

Example 3.5. Axiom schemas of two-sided sequent calculi usually have the form $\psi \Rightarrow \psi$. Using the notation from Remark 2.2, it can be presented as the canonical axiom $\emptyset/\{t, f\}$.

Note that applications of canonical axioms do not include context formulas. However, using the weakening rule (which is available in every system we consider), it is possible to derive $(L : \psi) \cup s$ from $(L : \psi)$ for every sequent s.

Definition 3.6. A primitive rule for $\pounds$ of the form $\{L_1, \ldots, L_n\}/\emptyset$ is called a *canonical cut.* Its applications allow to infer $s_1 \cup \ldots \cup s_n$ from the sequents $(L_i : \psi) \cup s_i$ for every $1 \leq i \leq n$ (the formula ψ is called the *cut-formula*).

Example 3.7. Applications of the cut rule for two-sided sequent calculi are usually presented by the following schema:

$$\frac{\Gamma_1 \Rightarrow \psi, \Delta_1 \quad \Gamma_2, \psi \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}$$

Using the notation from Remark 2.2, the corresponding canonical cut has the form $\{\{t\}, \{f\}\}/\emptyset$.

Definition 3.8 (Introduction Rules)**.** A *canonical introduction rule* for an n-ary connective $\diamond$ of $\mathcal{L}$ and $\pounds$ is an expression of the form $\mathcal{Q}/(L : \diamond(p_1, \ldots, p_n))$, where $\mathcal{Q}$ is a finite set of n-clauses (see Definition 2.1) (called *premises*), and L is a non-empty subset of $\pounds$. An *application* of a canonical introduction rule $\{c_1, \ldots, c_m\}/(L : \diamond(p_1, \ldots, p_n))$ is any inference step of the following form:[2]

$$\frac{\sigma(c_1) \cup s_1 \quad \ldots \quad \sigma(c_m) \cup s_m}{(L : \sigma(\diamond(p_1, \ldots, p_n))) \cup s_1 \cup \ldots \cup s_m}$$

where σ is a substitution, and s_i is a sequent for every $1 \leq i \leq m$.

Example 3.9. The introduction rules for the classical conjunction in **LK** are usually presented as follows:

$$\frac{\Gamma, \psi, \varphi \Rightarrow \Delta}{\Gamma, \psi \wedge \varphi \Rightarrow \Delta} \qquad \frac{\Gamma_1 \Rightarrow \Delta_1, \psi \quad \Gamma_2 \Rightarrow \Delta_2, \varphi}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2, \psi \wedge \varphi}$$

Using the notation from Remark 2.2, the canonical representation of the schemas above is:

$$r_1 = \{\{f : p_1, f : p_2\}\}/\{f : p_1 \wedge p_2\} \qquad r_2 = \{\{t : p_1\}, \{t : p_2\}\}/\{t : p_1 \wedge p_2\}$$

Their applications have the forms:

$$\frac{\{f : \psi, f : \varphi\} \cup s}{\{f : \psi \wedge \varphi\} \cup s} \qquad \frac{\{t : \psi\} \cup s_1 \quad \{t : \varphi\} \cup s_2}{\{t : \psi \wedge \varphi\} \cup s_1 \cup s_2}$$

Definition 3.10 (Canonical Labelled Systems)**.** A *canonical labelled system* **G** for $\mathcal{L}$ and $\pounds$ includes the weakening rule, a finite set of primitive rules for $\pounds$, and a finite set of introduction rules for the connectives of $\mathcal{L}$. We say that a sequent s *is derivable* in a canonical labelled system **G** from a set of sequents $\mathcal{S}$, denoted by $\mathcal{S} \vdash_{\mathbf{G}} s$, if there exists a derivation in **G** of s from $\mathcal{S}$.

[2]Note the full separation between a rule and its application; $p_1, \ldots, p_n$ appearing in the rule serve as schematic variables, which are replaced by actual formulas of the language in the application of the rule.

Notation 3.11. Given a canonical labelled system $\mathbf{G}$ for $\mathcal{L}$ and $\pounds$, we denote by $\mathsf{P}_{\mathbf{G}}$ the set of primitive rules of $\mathbf{G}$. For every connective $\diamond$ of $\mathcal{L}$, denote by $\mathsf{R}^{\diamond}_{\mathbf{G}}$ the set of canonical introduction rules for $\diamond$ of $\mathbf{G}$.

Example 3.12. The system **LK** can be represented as a canonical labelled system for the language of classical logic and $\{t, f\}$ (see Remark 2.2 and Examples 3.5 to 3.9).

Henceforth, to improve readability, we shall sometimes omit the parentheses from the set appearing before the "/" symbol in primitive rules and canonical introduction rules.

Example 3.13. For $\pounds = \{a, b, c\}$, the canonical labelled system $\mathbf{G}_{abc}$ includes the primitive rules $\emptyset/\{a, b\}, \emptyset/\{b, c\}, \emptyset/\{a, c\}$, and $\{a, b, c\}/\emptyset$. It also has the following canonical introduction rules for a ternary connective $\circ$:

$$\{a : p_1, c : p_2\}, \{a : p_3, b : p_2\}/(\{a, c\} : \circ(p_1, p_2, p_3))$$

$$\{c : p_2\}, \{a : p_3, b : p_3\}, \{c : p_1\}/(\{b, c\} : \circ(p_1, p_2, p_3))$$

Their applications are of the forms:

$$\frac{\{a : \psi_1, c : \psi_2\} \cup s_1 \quad \{a : \psi_3, b : \psi_2\} \cup s_2}{(\{a, c\} : \circ(\psi_1, \psi_2, \psi_3)) \cup s_1 \cup s_2}$$

$$\frac{\{c : \psi_2\} \cup s_1 \quad \{a : \psi_3, b : \psi_3\} \cup s_2 \quad \{c : \psi_1\} \cup s_3}{(\{b, c\} : \circ(\psi_1, \psi_2, \psi_3)) \cup s_1 \cup s_2 \cup s_3}$$

Note that the canonical labelled calculi studied here are substantially more general than the signed calculi of [2] and the labelled calculi of [1], as the primitive rules of both of these families of calculi include only canonical cuts and axioms. Moreover, in the latter only introduction rules which introduce a singleton are allowed, which is not the case for the calculus in Example 3.13. In the former, all systems have $\emptyset/\pounds$ as their only axiom, and the set of cuts is always assumed to be $\{\{l_1\}, \{l_2\}/\emptyset \mid l_1 \neq l_2\}$ (again leaving the calculus in Example 3.13 out of scope).

4 Partial Non-deterministic Matrices

Non-deterministic matrices (Nmatrices) ([4, 2]) provide a natural generalization of the notion of a standard many-valued matrix. These are structures, in which the truth value of a complex formula is chosen non-deterministically out of a *non-empty* set of options, which is determined by the truth values of its subformulas. In this paper we introduce a further generalization of the concept of an Nmatrix, in which this set of options is allowed to be *empty*. Intuitively, empty sets of options correspond to forbidding some combinations of truth values. As we shall see, this will allow us to characterize a wider class of calculi than that obtained by applying usual Nmatrices. However, as shown in the sequel, the property of *effectiveness* is preserved in PNmatrices, and like finite-valued matrices and Nmatrices, (calculi characterized by) finite-valued PNmatrices are decidable.

4.1 Introducing PNmatrices

Definition 4.1. A *partial non-deterministic matrix* (*PNmatrix* for short) $\mathcal{M}$ for $\mathcal{L}$ and $\pounds$ consists of: (i) a set $\mathcal{V}_{\mathcal{M}}$ of truth values, (ii) a function $\mathcal{D}_{\mathcal{M}} : \pounds \to P(\mathcal{V}_{\mathcal{M}})$ assigning a set of (designated) truth values to the labels of $\pounds$, and (iii) a function (called truth table) $\diamond_{\mathcal{M}} : \mathcal{V}_{\mathcal{M}}{}^n \to P(\mathcal{V}_{\mathcal{M}})$ for every n-ary connective $\diamond$ of $\mathcal{L}$. We say that $\mathcal{M}$ is *finite* if so is $\mathcal{V}_{\mathcal{M}}$.

Definition 4.2. Let $\mathcal{M}$ be a PNmatrix for $\mathcal{L}$ and $\pounds$.

1. An $\mathcal{M}$-legal $\mathcal{L}$-valuation is a function $v : \mathit{Frm}_{\mathcal{L}} \to \mathcal{V}_{\mathcal{M}}$ satisfying $v(\diamond(\psi_1, \ldots, \psi_n)) \in \diamond_{\mathcal{M}}(v(\psi_1), \ldots, v(\psi_n))$ for every compound formula $\diamond(\psi_1, \ldots, \psi_n) \in \mathit{Frm}_{\mathcal{L}}$.
2. Let v be an $\mathcal{M}$-legal $\mathcal{L}$-valuation. A sequent s is *true* in v for $\mathcal{M}$ (denoted by $v \models_{\mathcal{M}} s$) if $v(\psi) \in \mathcal{D}_{\mathcal{M}}(l)$ for some $l : \psi \in s$. A set $\mathcal{S}$ of sequents is *true* in v for $\mathcal{M}$ (denoted by $v \models_{\mathcal{M}} \mathcal{S}$) if $v \models_{\mathcal{M}} s$ for every $s \in \mathcal{S}$.

3. Given a set of sequents $\mathcal{S}$ and a single sequent s, $\mathcal{S} \vdash_{\mathcal{M}} s$ if for every $\mathcal{M}$-legal $\mathcal{L}$-valuation v, $v \models_{\mathcal{M}} s$ whenever $v \models_{\mathcal{M}} \mathcal{S}$.

It should be stressed that the relation induced by $\mathcal{M}$ defined in the last item above is between a set of sequents and a sequent. However, the usual notion of a consequence relation between a set of formulas and a formula can be fully characterized in terms of the former relation in the following way:

Definition 4.3. Assume that the set of labels $\pounds$ contains a distinguished label t.[3] Let $\mathcal{M}$ be a PNmatrix for $\mathcal{L}$ and $\pounds$. The *consequence relation* $\vdash^{frm}_{\mathcal{M}}$ between sets of formulas and formulas which is induced by $\mathcal{M}$ is defined as follows: $\mathcal{T} \vdash^{frm}_{\mathcal{M}} \varphi$ if $\{\{t : \psi\} \mid \psi \in \mathcal{T}\} \vdash_{\mathcal{M}} \{t : \varphi\}$.

Remark 4.4. Note that to determine whether $\mathcal{T} \vdash^{frm}_{\mathcal{M}} \varphi$, only $\mathcal{D}_{\mathcal{M}}(t)$ is used, while the values assigned by $\mathcal{D}_{\mathcal{M}}$ to all labels in $\pounds \setminus \{t\}$ are immaterial. Thus when one is only interested in the consequence relation between sets of formulas and formulas, it suffices to define $\mathcal{D}_{\mathcal{M}}$ as a *set* of designated truth values, as it is usually done for many-values matrices and Nmatrices. On the other hand, when the focus is on the consequence relation between sets of sequents and sequents, we need to determine when $v \models_{\mathcal{M}} s$ also for sequents with labels in $\pounds \setminus \{t\}$. In general, we can have different conditions for each label, and for this reason $\mathcal{D}_{\mathcal{M}}$ is defined to be a function from $\pounds$ to $P(\mathcal{V}_{\mathcal{M}})$.

We now define a special subclass of PNmatrices, in which no empty sets of truth values are allowed in the truth tables of the logical connectives. This corresponds to the case of Nmatrices.

Definition 4.5. We say that a PNmatrix $\mathcal{M}$ for $\mathcal{L}$ and $\pounds$ is *proper* if $\mathcal{V}_{\mathcal{M}}$ is non-empty and $\diamond_{\mathcal{M}}(x_1, \ldots, x_n)$ is non-empty for every n-ary connective $\diamond$ of $\mathcal{L}$ and $x_1, \ldots, x_n \in \mathcal{V}_{\mathcal{M}}$.

The usual concept of Nmatrices can be thought of as proper PNmatrices for $\mathcal{L}$ and $\pounds$, where $\pounds$ is a singleton (and so $\mathcal{D}_{\mathcal{M}}$ is taken to be a *set* of truth values). In this paper, however, we do not require that the set of designated truth values (for every $l \in \pounds$) is a non-empty proper subset of $\mathcal{V}_{\mathcal{M}}$.

Example 4.6. Let $\pounds = \{a, b\}$ and suppose that $\mathcal{L}$ consists of one unary connective $\star$. Define the PNmatrices $\mathcal{M}_1$ and $\mathcal{M}_2$ as follows: $\mathcal{V}_{\mathcal{M}_1} = \mathcal{V}_{\mathcal{M}_2} = \{t, f\}$, $\mathcal{D}_{\mathcal{M}_1}(a) = \mathcal{D}_{\mathcal{M}_2}(a) = \{t\}$ and $\mathcal{D}_{\mathcal{M}_1}(b) = \mathcal{D}_{\mathcal{M}_2}(b) = \{f\}$. The truth tables for $\star$ are defined as follows:

x	$\star_{\mathcal{M}_1}(x)$
t	$\{f\}$
f	$\{t, f\}$

x	$\star_{\mathcal{M}_2}(x)$
t	$\emptyset$
f	$\{t, f\}$

While both $\mathcal{M}_1$ and $\mathcal{M}_2$ are (finite) PNmatrices, only $\mathcal{M}_1$ is proper. Note that in this case we have $\{\{a : p_1\}\} \vdash_{\mathcal{M}_2} \emptyset$, simply because there is no $\mathcal{M}_2$-legal $\mathcal{L}$-valuation that assigns t to p_1.

Finally, we extend the notion of *simple refinements* of Nmatrices (see, e.g. [7]) to the context of PNmatrices:

Definition 4.7. Let $\mathcal{M}$ and $\mathcal{N}$ be PNmatrices for $\mathcal{L}$ and $\pounds$. We say that $\mathcal{N}$ is a *simple refinement* of $\mathcal{M}$, denoted by $\mathcal{N} \subseteq \mathcal{M}$, if $\mathcal{V}_{\mathcal{N}} \subseteq \mathcal{V}_{\mathcal{M}}$, $\mathcal{D}_{\mathcal{N}}(l) = \mathcal{D}_{\mathcal{M}}(l) \cap \mathcal{V}_{\mathcal{N}}$ for every $l \in \pounds$, and $\diamond_{\mathcal{N}}(x_1, \ldots, x_n) \subseteq \diamond_{\mathcal{M}}(x_1, \ldots, x_n)$ for every n-ary connective $\diamond$ of $\mathcal{L}$ and $x_1, \ldots, x_n \in \mathcal{V}_{\mathcal{N}}$.

Proposition 4.8. Let $\mathcal{M}$ and $\mathcal{N}$ be PNmatrices for $\mathcal{L}$ and $\pounds$, such that $\mathcal{N} \subseteq \mathcal{M}$. Then: (1) Every $\mathcal{N}$-legal $\mathcal{L}$-valuation is also $\mathcal{M}$-legal; and (2) $\vdash_{\mathcal{M}} \subseteq \vdash_{\mathcal{N}}$.

Proof. (1) is trivial. For (2), let $\mathcal{S} \vdash_{\mathcal{M}} s$. Let v be an $\mathcal{N}$-legal $\mathcal{L}$-valuation, such that $v \models_{\mathcal{N}} \mathcal{S}$. For every $s' \in \mathcal{S}$, $v(\psi) \in \mathcal{D}_{\mathcal{N}}(l)$ for some $l : \psi \in s'$. Since $\mathcal{D}_{\mathcal{N}}(l) \subseteq \mathcal{D}_{\mathcal{M}}(l)$ for every $l \in \pounds$, we have that for every $s' \in \mathcal{S}$, $v(\psi) \in \mathcal{D}_{\mathcal{M}}(l)$ for some $l : \psi \in s'$. Thus $v \models_{\mathcal{M}} \mathcal{S}$, and since v is $\mathcal{M}$-legal (using (1)), $v \models_{\mathcal{M}} s$. Then $v(\psi) \in \mathcal{D}_{\mathcal{M}}(l)$ for some $l : \psi \in s$. Since $\mathcal{D}_{\mathcal{N}}(l) = \mathcal{D}_{\mathcal{M}}(l) \cap \mathcal{V}_{\mathcal{N}}$ and $v(\psi) \in \mathcal{V}_{\mathcal{N}}$ (since v is $\mathcal{N}$-legal), $v(\psi) \in \mathcal{D}_{\mathcal{N}}(l)$ and so $v \models_{\mathcal{N}} s$. □

[3] This obviously holds for two-sided sequents, where t is used for the right handside of the sequent.

4.2 Decidability

A desirable property for a denotational semantics is , its *effectiveness*. In other words, the question of whether some conclusion follows from a finite set of assumptions, should be decidable by considering some *computable* set of partial valuations defined on some *finite* set of "relevant" formulas. Usually, the "relevant" formulas are taken as all the subformulas occurring in the conclusion and the assumptions. Next, we show that the semantics induced by PNmatrices is effective in this sense. The notion of a partial valuation is defined similarly to that of $\mathcal{L}$-valuation (Definition 4.2):

Definition 4.9. Let $\mathcal{M}$ be a PNmatrix for $\mathcal{L}$ and $\pounds$, and let $\mathcal{F} \subseteq \mathit{Frm}_{\mathcal{L}}$ be closed under subformulas. An $\mathcal{M}$-legal $\mathcal{F}$-valuation is a function $v : \mathcal{F} \to \mathcal{V}_{\mathcal{M}}$ satisfying $v(\diamond(\psi_1, \dots, \psi_n)) \in \diamond_{\mathcal{M}}(v(\psi_1), \dots, v(\psi_n))$ for every formula $\diamond(\psi_1, \dots, \psi_n) \in \mathcal{F}$. $\models_{\mathcal{M}}$ is defined for $\mathcal{F}$-valuations exactly as for $\mathcal{L}$-valuations. We say that an $\mathcal{M}$-legal $\mathcal{F}$-valuation is *extendable in* $\mathcal{M}$ if it can be extended to an $\mathcal{M}$-legal $\mathcal{L}$-valuation.

In proper PNmatrices, all partial valuations are extendable:

Proposition 4.10. Let $\mathcal{M}$ be a *proper* PNmatrix for $\mathcal{L}$ and $\pounds$, and let $\mathcal{F} \subseteq \mathit{Frm}_{\mathcal{L}}$ be closed under subformulas. Then any $\mathcal{M}$-legal $\mathcal{F}$-valuation is *extendable in* $\mathcal{M}$.

Proof. The proof goes exactly like the one for Nmatrices in [8]. Note that the non-emptiness of $\mathcal{V}_{\mathcal{M}}$ is needed in order to extend the empty valuation. Clearly, the different definition of $\mathcal{D}_{\mathcal{M}}$ is immaterial here. □

However, this is not the case for arbitrary PNmatrices:

Example 4.11. Consider the PNmatrix $\mathcal{M}_2$ from Example 4.6. Let v be the $\mathcal{M}_2$-legal $\{p_1\}$-valuation defined by $v(p_1) = t$. Obviously, there is no $\mathcal{M}_2$-legal $\mathcal{L}$-valuation that extends v (as there is no way to assign a truth value to $\star p_1$). Thus v is not extendable in $\mathcal{M}_2$.

However, a simple criterion for extendability can be obtained:

Theorem 4.12. Let $\mathcal{M}$ be a PNmatrix for $\mathcal{L}$ and $\pounds$ and $\mathcal{F} \subseteq \mathit{Frm}_{\mathcal{L}}$ be closed under subformulas. An $\mathcal{M}$-legal $\mathcal{F}$-valuation v is extendable in $\mathcal{M}$ iff v is $\mathcal{N}$-legal for some proper PNmatrix $\mathcal{N} \subseteq \mathcal{M}$.

Proof. ($\Leftarrow$) Suppose that there is some proper PNmatrix $\mathcal{N} \subseteq \mathcal{M}$, such that v is $\mathcal{N}$-legal. By Proposition 4.10, there exists an $\mathcal{N}$-legal $\mathcal{L}$-valuation v' that extends v. By Proposition 4.8, v' is $\mathcal{M}$-legal. Thus v is extendable in $\mathcal{M}$.

($\Rightarrow$) Let v' be an $\mathcal{M}$-legal $\mathcal{L}$-valuation that extends v. Define the PNmatrix $\mathcal{N}$ as follows: $\mathcal{V}_{\mathcal{N}} = \mathsf{Image}(v')$; for every $l \in \pounds$, $\mathcal{D}_{\mathcal{N}}(l) = \mathcal{D}_{\mathcal{M}}(l) \cap \mathcal{V}_{\mathcal{N}}$; and for every n-ary connective $\diamond$ of $\mathcal{L}$, and $x_1, \dots, x_n \in \mathcal{V}_{\mathcal{N}}$, $\diamond_{\mathcal{N}}(x_1, \dots, x_n) = \diamond_{\mathcal{M}}(x_1, \dots, x_n) \cap \mathcal{V}_{\mathcal{N}}$. Clearly, we have $\mathcal{N} \subseteq \mathcal{M}$. We show that $\mathcal{N}$ is proper. Obviously, $\mathcal{V}_{\mathcal{N}}$ is non-empty. Let $\diamond$ be an n-ary connective of $\mathcal{L}$, and let $x_1, \dots, x_n \in \mathcal{V}_{\mathcal{N}}$. Since $\mathcal{V}_{\mathcal{N}} = \mathsf{Image}(v')$, there are some $\psi_1, \dots, \psi_n \in \mathit{Frm}_{\mathcal{L}}$, such that $v'(\psi_i) = x_i$ for every $1 \leq i \leq n$. Since v' is an $\mathcal{M}$-legal $\mathcal{L}$-valuation, $v'(\diamond(\psi_1, \dots, \psi_n)) \in \diamond_{\mathcal{M}}(x_1, \dots, x_n)$. Thus $v'(\diamond(\psi_1, \dots, \psi_n)) \in \diamond_{\mathcal{N}}(x_1, \dots, x_n)$, and so $\diamond_{\mathcal{N}}(x_1, \dots, x_n) \neq \emptyset$. It remains to show that v is $\mathcal{N}$-legal. Let $\diamond(\psi_1, \dots, \psi_n) \in \mathcal{F}$. Since v' is an $\mathcal{M}$-legal $\mathcal{L}$-valuation, $v'(\diamond(\psi_1, \dots, \psi_n)) \in \diamond_{\mathcal{M}}(v'(\psi_1), \dots, v'(\psi_n))$. By definition $v'(\diamond(\psi_1, \dots, \psi_n)) = v(\diamond(\psi_1, \dots, \psi_n))$, and $v'(\psi_i) = v(\psi_i)$ for every $1 \leq i \leq n$. The construction of $\diamond_{\mathcal{N}}$ then ensures that $v(\diamond(\psi_1, \dots, \psi_n)) \in \diamond_{\mathcal{N}}(v(\psi_1), \dots, v(\psi_n))$. □

Corollary 4.13. Given a finite PNmatrix $\mathcal{M}$ for $\mathcal{L}$ and $\pounds$, a finite set $\mathcal{F} \subseteq \mathit{Frm}_{\mathcal{L}}$ closed under subformulas, and a function $v : \mathcal{F} \to \mathcal{V}_{\mathcal{M}}$, it is decidable whether v is an $\mathcal{M}$-legal $\mathcal{F}$-valuation which is extendable in $\mathcal{M}$.

Proof. Checking whether v is an $\mathcal{M}$-legal $\mathcal{F}$-valuation is straightforward. To verify that it is extendable in $\mathcal{M}$, we go over all (finite) proper PNmatrices $\mathcal{N} \subseteq \mathcal{M}$ (there is a finite number of them since $\mathcal{M}$ is finite), and check whether v is $\mathcal{N}$-legal for some such $\mathcal{N}$. We return a positive answer iff we have found some $\mathcal{N} \subseteq \mathcal{M}$ such that v is $\mathcal{N}$-legal. The correctness is guaranteed by Theorem 4.12. □

Corollary 4.14. Given a finite PNmatrix $\mathcal{M}$ for $\mathcal{L}$ and $\pounds$, a finite set $\mathcal{S}$ of sequents, and a sequent s, the question whether $\mathcal{S} \vdash_{\mathcal{M}} s$ is decidable.

Proof. Using Corollary 4.13, it is possible to enumerate all functions $v : sub[\mathcal{S} \cup \{s\}] \to \mathcal{V}_{\mathcal{M}}$, and check if one of them is an $\mathcal{M}$-legal $sub[\mathcal{S} \cup \{s\}]$-valuation extendable in $\mathcal{M}$, such that $v \models_{\mathcal{M}} \mathcal{S}$ but $v \not\models_{\mathcal{M}} s$. We claim that $\mathcal{S} \vdash_{\mathcal{M}} s$ iff such a function is not found. To see this, note that if $\mathcal{S} \not\vdash_{\mathcal{M}} s$, then by definition there exists an $\mathcal{M}$-legal $\mathcal{L}$-valuation v' such that $v' \models_{\mathcal{M}} \mathcal{S}$ but $v' \not\models_{\mathcal{M}} s$. Its restriction v to $sub[\mathcal{S} \cup \{s\}]$ is an $\mathcal{M}$-legal $sub[\mathcal{S} \cup \{s\}]$-valuation extendable in $\mathcal{M}$, such that $v \models_{\mathcal{M}} \mathcal{S}$ but $v \not\models_{\mathcal{M}} s$. On the other hand, if there exists an $\mathcal{M}$-legal $sub[\mathcal{S} \cup \{s\}]$-valuation v extendable in $\mathcal{M}$, such that $v \models_{\mathcal{M}} \mathcal{S}$ but $v \not\models_{\mathcal{M}} s$, then for any of its $\mathcal{M}$-legal extensions v', we have $v' \models_{\mathcal{M}} \mathcal{S}$ but $v' \not\models_{\mathcal{M}} s$. Consequently, $\mathcal{S} \not\vdash_{\mathcal{M}} s$ in this case. □

In the literature of Nmatrices (see e.g. [8]) effectiveness is usually identified with the property given in Proposition 4.10.[4] In this case Corollary 4.13 trivially holds: to check that v is an extendable $\mathcal{M}$-legal $\mathcal{F}$-valuation, it suffices to check that it is $\mathcal{M}$-legal, as extendability is a priori guaranteed for Nmatrices. However, the results above show that this property is not a necessary condition for decidability. To guarantee the latter, instead of requiring that *all* partial valuations are extendable, it is sufficient to have an algorithm that establishes which of them are.

Remark 4.15. As done for ordinary matrices (see, e.g. [9]) it is also possible to define $\vdash_F$, the consequence relation induced by *a family $\mathcal{F}$ of proper PNmatrices* to be $\bigcap_{\mathcal{N} \in F} \vdash_{\mathcal{N}}$. A PNmatrix can then be thought of as a succinct presentation of a family of proper PNmatrices: following the proof of Theorem 4.12, the consequence relation induced by a PNmatrix $\mathcal{M}$ can be shown to be equivalent to the relation induced by the family of all the proper PNmatrices $\mathcal{N}$, such that $\mathcal{N} \subseteq \mathcal{M}$. Conversely, for every family of proper PNmatrices it is possible to construct an equivalent PNmatrix.

4.3 Minimality

In the next section, we show that the framework of PNmatrices provides a semantic way of characterizing canonical labelled systems. A natural question in this context is how one can obtain *minimal* such characterizations. Next we provide lower bounds on the number of truth values that are needed to characterize $\vdash_{\mathcal{M}}$ of some PNmatrix $\mathcal{M}$ satisfying a separability condition defined below. Moreover, we provide a method to extract from a given (separable) PNmatrix an equivalent PNmatrix with the *minimal* number of truth values.

Definition 4.16. Let $\mathcal{M}$ be a PNmatrix for $\mathcal{L}$ and $\pounds$.

1. A truth value $x \in \mathcal{V}_{\mathcal{M}}$ is called *useful in* $\mathcal{M}$ if $x \in \mathcal{V}_{\mathcal{N}}$ for some proper PNmatrix $\mathcal{N} \subseteq \mathcal{M}$.
2. The PNmatrix $R[\mathcal{M}]$ is the simple refinement of $\mathcal{M}$, defined as follows: $\mathcal{V}_{R[\mathcal{M}]}$ consists of all truth values in $\mathcal{V}_{\mathcal{M}}$ which are useful in $\mathcal{M}$; for every $l \in \pounds$, $\mathcal{D}_{R[\mathcal{M}]}(l) = \mathcal{D}_{\mathcal{M}}(l) \cap \mathcal{V}_{R[\mathcal{M}]}$; and for every n-ary connective $\diamond$ of $\mathcal{L}$ and $x_1, \dots, x_n \in \mathcal{V}_{R[\mathcal{M}]}$, $\diamond_{R[\mathcal{M}]}(x_1, \dots, x_n) = \diamond_{\mathcal{M}}(x_1, \dots, x_n) \cap \mathcal{V}_{R[\mathcal{M}]}$.

Proposition 4.17. Let $\mathcal{M}$ be a PNmatrix for $\mathcal{L}$ and $\pounds$, and let v be an $\mathcal{M}$-legal $\mathcal{L}$-valuation. Then: (1) For every formula ψ, $v(\psi)$ is useful in $\mathcal{M}$; and (2) Every $\mathcal{M}$-legal $\mathcal{L}$-valuation is also $R[\mathcal{M}]$-legal.

Proof. (2) easily follows from (1). For (1), note that v is trivially extendable in $\mathcal{M}$, and so Theorem 4.12 entails that there is some proper PNmatrix $\mathcal{N} \subseteq \mathcal{M}$, such that v is $\mathcal{N}$-legal. By definition, $v(\psi) \in \mathcal{V}_{\mathcal{N}}$ for every formula ψ. Thus $v(\psi)$ is useful in $\mathcal{M}$ for every formula ψ. □

Corollary 4.18. $\vdash_{\mathcal{M}} = \vdash_{R[\mathcal{M}]}$ for every PNmatrix $\mathcal{M}$.

Proof. One direction follows from Proposition 4.8, simply because $R[\mathcal{M}]$ is a simple refinement of $\mathcal{M}$ by definition. The converse is easily established using Proposition 4.17. □

Definition 4.19. Let $\mathcal{M}$ be a PNmatrix for $\mathcal{L}$ and $\pounds$. We say that two truth values $x_1, x_2 \in \mathcal{V}_{\mathcal{M}}$ are *separable* in $\mathcal{M}$ for $l \in \pounds$ if $x_1 \in \mathcal{D}_{\mathcal{M}}(l) \Leftrightarrow x_2 \notin \mathcal{D}_{\mathcal{M}}(l)$ holds. $\mathcal{M}$ is called *separable* if every pair of truth values in $\mathcal{V}_{\mathcal{M}}$ are separable in $\mathcal{M}$ for some $l \in \pounds$.

[4] This property is sometimes called (semantic) *analyticity*. Note that in this paper the term 'analyticity' refers to a *proof-theoretic* property (see Definition 6.1).

We are now ready to obtain a *lower bound* on the number of truth values needed to characterize $\vdash_{\mathcal{M}}$ for a given separable PNmatrix $\mathcal{M}$:

Theorem 4.20. Let $\mathcal{M}$ be a separable PNmatrix for $\mathcal{L}$ and $\pounds$. If $\vdash_{\mathcal{M}}=\vdash_{\mathcal{N}}$ for some PNmatrix $\mathcal{N}$ for $\mathcal{L}$ and $\pounds$, then $\mathcal{N}$ contains at least $|\mathcal{V}_{R[\mathcal{M}]}|$ truth values.

Proof. Let $\mathcal{N}$ be a PNmatrix for $\mathcal{L}$ and $\pounds$ with $|\mathcal{V}_{\mathcal{N}}| < |\mathcal{V}_{R[\mathcal{M}]}|$. We show that $\vdash_{\mathcal{N}} \neq \vdash_{\mathcal{M}}$. For every $y \in \mathcal{V}_{\mathcal{N}}$, define $V_y = \{x \in \mathcal{V}_{\mathcal{M}} \mid \forall l \in \pounds. y \in \mathcal{D}_{\mathcal{N}}(l) \Leftrightarrow x \in \mathcal{D}_{\mathcal{M}}(l)\}$. Since $\mathcal{M}$ is separable, V_y is either singleton or empty for every $y \in \mathcal{V}_{\mathcal{N}}$. Since $|\mathcal{V}_{\mathcal{N}}| < |\mathcal{V}_{R[\mathcal{M}]}|$ (and $\mathcal{V}_{R[\mathcal{M}]} \subseteq \mathcal{V}_{\mathcal{M}}$), there exists some $x_0 \in \mathcal{V}_{R[\mathcal{M}]}$, such that $x_0 \notin V_y$ for every $y \in \mathcal{V}_{\mathcal{N}}$. Let $L = \{l \in \pounds \mid x_0 \in \mathcal{D}_{\mathcal{M}}(l)\}$. Let $\mathcal{S}$ be the set of all 1-clauses of the form $\{l : p_1\}$ for $l \in L$, and s be the 1-clause $(\pounds \setminus L : p_1)$. We claim that $\mathcal{S} \vdash_{\mathcal{N}} s$. Suppose otherwise. Then there exists an $\mathcal{N}$-legal $\mathcal{L}$-valuation v such that $v \models_{\mathcal{N}} \mathcal{S}$, but $v \not\models_{\mathcal{N}} s$. Thus $v(p_1) \in \mathcal{D}_{\mathcal{N}}(l)$ for every $l \in L$, and $v(p_1) \notin \mathcal{D}_{\mathcal{N}}(l)$ for every $l \notin L$. But, it then follows that $x_0 \in V_{v(p_1)}$, and this contradicts our assumption concerning x_0.

On the other hand, it is easy to see that $\mathcal{S} \not\vdash_{\mathcal{M}} s$. Indeed, consider an $\mathcal{M}$-legal $\{p_1\}$-valuation v that assigns x_0 to p_1. Since x_0 is useful in $\mathcal{M}$, there exists some proper PNmatrix $\mathcal{N} \subseteq \mathcal{M}$ such that $x_0 \in \mathcal{V}_{\mathcal{N}}$. v is trivially an $\mathcal{N}$-legal $\{p_1\}$-valuation, and so by Theorem 4.12, v is extendable in $\mathcal{M}$. Let v' be an $\mathcal{M}$-legal $\mathcal{L}$-valuation which extends v. Clearly, $v' \models_{\mathcal{M}} \mathcal{S}$, but $v' \not\models_{\mathcal{M}} s$. ☐

5 Finite PNmatrices for Canonical Labelled Systems

Definition 5.1. We say that a PNmatrix $\mathcal{M}$ (for $\mathcal{L}$ and $\pounds$) is *characteristic* for a canonical labelled system $\mathbf{G}$ (for $\mathcal{L}$ and $\pounds$) if $\vdash_{\mathcal{M}}=\vdash_{\mathbf{G}}$.

Next we provide a systematic way to obtain a characteristic PNmatrix $\mathcal{M}_{\mathbf{G}}$ for every canonical labelled system $\mathbf{G}$. The intuitive idea is as follows: the primitive rules of $\mathbf{G}$ determine the set of the truth values of $\mathcal{M}_{\mathbf{G}}$, while the introduction rules for the logical connectives dictate their corresponding truth tables. The semantics based on PNmatrices is thus *modular*: each rule corresponds to a certain semantic condition, and the semantics of a system is obtained by joining the semantic effects of each of its derivation rules.

Definition 5.2. Let $r = \{L_1, \ldots, L_n\}/L_0$ be a primitive rule for $\pounds$. Define:

$$r^* = \{L \subseteq \pounds \mid L_i \cap L = \emptyset \textit{ for some } 1 \leq i \leq n \textit{ or } L_0 \cap L \neq \emptyset\}$$

Example 5.3. For an axiom $r = \emptyset/L_0$, we have $r^* = \{L \subseteq \pounds \mid L_0 \cap L \neq \emptyset\}$. For a cut $r = \{L_1, \ldots, L_n\}/\emptyset$, $r^* = \{L \subseteq \pounds \mid L_i \cap L = \emptyset \textit{ for some } 1 \leq i \leq n\}$. In particular, continuing Examples 3.5 and 3.7 (for $\pounds = \{t, f\}$), $r^* = \{\{t\}, \{f\}, \{t, f\}\}$ for the classical axiom, and $r^* = \{\emptyset, \{t\}, \{f\}\}$ for the classical cut.

Example 5.4. Suppose $\pounds = \{a, b, c, d\}$. For a primitive rule $r = \{\{a, b\}, \{c\}\}/\{d\}$, r^* consists of all subsets of $\pounds$ except for $\{a, c\}$,$\{b, c\}$, and $\{a, b, c\}$.

Definition 5.5. Let $\diamond$ be an n-ary connective, and let $r = \mathcal{Q}/(L_0 : \diamond(p_1, \ldots, p_n))$ be a canonical introduction rule for $\diamond$ and $\pounds$. For every $L_1, \ldots, L_n \subseteq \pounds$, define:

$$r^*[L_1, \ldots, L_n] = \begin{cases} \{L \subseteq \pounds \mid L_0 \cap L \neq \emptyset\} & \forall c \in \mathcal{Q}.((L_1 : p_1) \cup \ldots \cup (L_n : p_n)) \cap c \neq \emptyset \\ P(\pounds) & \text{otherwise} \end{cases}$$

Example 5.6. Let $\pounds = \{t, f\}$. Recall the usual introduction rules for conjunction from Example 3.9. By Definition 5.5:

$$r_1^*[L_1, L_2] = \begin{cases} \{\{f\}, \{t, f\}\} & f \in L_1 \cup L_2 \\ P(\{t, f\}) & \textit{otherwise} \end{cases}$$

$$r_2^*[L_1, L_2] = \begin{cases} \{\{t\}, \{t, f\}\} & t \in L_1 \cap L_2 \\ P(\{t, f\}) & \textit{otherwise} \end{cases}$$

Definition 5.7 (The PNmatrix $\mathcal{M}_{\mathbf{G}}$)**.** Let $\mathbf{G}$ be a canonical labelled system for $\mathcal{L}$ and $\pounds$. The PNmatrix $\mathcal{M}_{\mathbf{G}}$ (for $\mathcal{L}$ and $\pounds$) is defined by:

1. $\mathcal{V}_{\mathcal{M}_\mathbf{G}} = \{L \subseteq \pounds \mid L \in r^* \textit{ for every } r \in \mathsf{P}_\mathbf{G}\}$.

2. For every $l \in \pounds$, $\mathcal{D}_{\mathcal{M}_\mathbf{G}}(l) = \{L \in \mathcal{V}_{\mathcal{M}_\mathbf{G}} \mid l \in L\}$.

3. For every n-ary connective $\diamond$ of $\mathcal{L}$ and $L_1, \ldots, L_n \in \mathcal{V}_{\mathcal{M}_\mathbf{G}}$:

$$\diamond_{\mathcal{M}_\mathbf{G}}(L_1, \ldots, L_n) = \{L \in \mathcal{V}_{\mathcal{M}_\mathbf{G}} \mid L \in r^*[L_1, \ldots, L_n] \textit{ for every } r \in \mathsf{R}^\diamond_\mathbf{G}\}$$

Example 5.8. Let $\pounds = \{t, f\}$ and consider the system $\mathbf{G}_\wedge$ whose primitive rules include only the classical axiom, and the classical cut (see Examples 3.5 and 3.7), and whose only introduction rules are the two usual rules for conjunction (see Example 3.9). By Example 5.3 and the construction above, $\mathcal{V}_{\mathcal{M}_{\mathbf{G}_\wedge}} = \{\{t\}, \{f\}\}$, $\mathcal{D}_{\mathcal{M}_{\mathbf{G}_\wedge}}(t) = \{\{t\}\}$, and $\mathcal{D}_{\mathcal{M}_{\mathbf{G}_\wedge}}(f) = \{\{f\}\}$. Using Example 5.6, we obtain the following interpretation of $\wedge$:

$\wedge_{\mathcal{M}_{\mathbf{G}_\wedge}}$	$\{t\}$	$\{f\}$
$\{t\}$	$\{t\}$	$\{f\}$
$\{f\}$	$\{f\}$	$\{f\}$

Now, consider the system $\mathbf{G}_T$, which has the same primitive rules, but the following introduction rules:

$$r_1 = \{\{f : p_2\}\}/\{f : p_1 T p_2\} \qquad r_2 = \{\{t : p_1\}\}/\{t : p_1 T p_2\}$$

(these are equivalent to the "Tonk" introduction rules of [10]). $\mathcal{V}_{\mathcal{M}_{\mathbf{G}_T}}$ and $\mathcal{D}_{\mathcal{M}_{\mathbf{G}_T}}$ are the same as $\mathcal{V}_{\mathcal{M}_{\mathbf{G}_\wedge}}$ and $\mathcal{D}_{\mathcal{M}_{\mathbf{G}_\wedge}}$, and T has the following truth table:

$T_{\mathcal{M}_{\mathbf{G}_T}}$	$\{t\}$	$\{f\}$
$\{t\}$	$\{t\}$	$\emptyset$
$\{f\}$	$\{t, f\}$	$\{f\}$

Note that the resulting PNmatrix in this case is not proper.

Example 5.9. Let $\pounds = \{a, b, c\}$, and assume that $\mathcal{L}$ consists of one unary connective $\star$. Let us start with the system $\mathbf{G}_0$, the primitive rules of which include the canonical axiom $\emptyset/\{a, b, c\}$ and the canonical cuts $\{a\}, \{c\}/\emptyset$ and $\{a\}, \{b\}/\emptyset$, while $\mathbf{G}_0$ has no introduction rules. Here we have $\mathcal{V}_{\mathcal{M}_{\mathbf{G}_0}} = \{\{a\}, \{b\}, \{c\}, \{b, c\}\}$, $\mathcal{D}_{\mathcal{M}_{\mathbf{G}_0}}(a) = \{\{a\}\}$, $\mathcal{D}_{\mathcal{M}_{\mathbf{G}_0}}(b) = \{\{b\}, \{b, c\}\}$ and $\mathcal{D}_{\mathcal{M}_{\mathbf{G}_0}}(c) = \{\{c\}, \{b, c\}\}$. $\star_{\mathcal{M}_{\mathbf{G}_0}}$ is given in the table below (it has the maximal level of non-determinism). One can now obtain a system $\mathbf{G}_1$ by adding the rule $\{a : p_1\}/(\{b, c\} : \star p_1)$. This leads to a refinement of the truth table, described below. Finally, one can obtain the system $\mathbf{G}_2$ by adding $\{b : p_1\}/\{a : \star p_1\}$, resulting in another refinement of truth table, also described below.

x	$\star_{\mathcal{M}_{\mathbf{G}_0}}(x)$	$\star_{\mathcal{M}_{\mathbf{G}_1}}(x)$	$\star_{\mathcal{M}_{\mathbf{G}_2}}(x)$
$\{a\}$	$\{\{a\},\{b\},\{c\},\{b,c\}\}$	$\{\{b\},\{c\},\{b,c\}\}$	$\{\{b\},\{c\},\{b,c\}\}$
$\{b\}$	$\{\{a\},\{b\},\{c\},\{b,c\}\}$	$\{\{a\},\{b\},\{c\},\{b,c\}\}$	$\{\{a\}\}$
$\{c\}$	$\{\{a\},\{b\},\{c\},\{b,c\}\}$	$\{\{a\},\{b\},\{c\},\{b,c\}\}$	$\{\{a\},\{b\},\{c\},\{b,c\}\}$
$\{b,c\}$	$\{\{a\},\{b\},\{c\},\{b,c\}\}$	$\{\{a\},\{b\},\{c\},\{b,c\}\}$	$\{\{a\}\}$

Soundness and completeness

To establish the soundness and completeness of $\mathcal{M}_\mathbf{G}$ for each canonical labelled system $\mathbf{G}$, the usual approach would be to show that $\mathcal{S} \vdash_\mathbf{G} s$ iff $\mathcal{S} \vdash_{\mathcal{M}_\mathbf{G}} s$. However, here we use stronger notions of soundness and completeness. Later, this will allow us to characterize strong analyticity and strong cut-admissibility in canonical labelled calculi (Sections 6.1 and 6.2).

Definition 5.10. Let $\mathbf{G}$ be a canonical labelled system for $\mathcal{L}$ and $\pounds$, and let $\mathcal{F} \subseteq \mathit{Frm}_\mathcal{L}$ be closed under subformulas. We write $\mathcal{S} \vdash^\mathcal{F}_\mathbf{G} s$ if there exists a derivation in $\mathbf{G}$ of a sequent s from a set $\mathcal{S}$ of sequents consisting only of (sequents consisting of) formulas from $\mathcal{F}$.

Definition 5.11 (Analytic Soundness)**.** A PNmatrix $\mathcal{M}$ for $\mathcal{L}$ and $\pounds$ is *analytically sound* for a canonical labelled system $\mathbf{G}$ (for $\mathcal{L}$ and $\pounds$) if for every $\mathcal{F} \subseteq \mathit{Frm}_\mathcal{L}$ closed under subformulas, set $\mathcal{S}$ of sequents, and sequent s such that $sub[\mathcal{S} \cup \{s\}] \subseteq \mathcal{F}$ and $\mathcal{S} \vdash^\mathcal{F}_\mathbf{G} s$: if $v \models_{\mathcal{M}_\mathbf{G}} \mathcal{S}$ for an $\mathcal{M}_\mathbf{G}$-legal $\mathcal{F}$-valuation v, then also $v \models_{\mathcal{M}_\mathbf{G}} s$.

Intuitively, analytic soundness means that if we are able to prove s from $\mathcal{S}$ in $\mathbf{G}$ using only the "material" available in $\mathcal{F}$, then for every $\mathcal{M}_{\mathbf{G}}$-legal valuation defined on the formulas in this "material", if it satisfies $\mathcal{S}$, it must also satisfy s. Obviously, by taking $\mathcal{F} = Frm_{\mathcal{L}}$, we obtain the usual notion of soundness (i.e. $\mathcal{S} \vdash_{\mathbf{G}} s$ implies $\mathcal{S} \vdash_{\mathcal{M}_{\mathbf{G}}} s$).

Theorem 5.12. For a canonical labelled system $\mathbf{G}$ (for $\mathcal{L}$ and $\pounds$), $\mathcal{M}_G$ is analytically sound for $\mathbf{G}$.

Proof. Let $\mathcal{F} \subseteq Frm_{\mathcal{L}}$ be closed under subformulas, $\mathcal{S}$ be a set of sequents, and s be a sequent. Assume that $sub[\mathcal{S} \cup \{s\}] \subseteq \mathcal{F}$ and that $\mathcal{S} \vdash_{\mathbf{G}}^{\mathcal{F}} s$. Let v be an $\mathcal{M}_{\mathbf{G}}$-legal $\mathcal{F}$-valuation. Suppose that $v \models_{\mathcal{M}_{\mathbf{G}}} \mathcal{S}$. Using induction on the length of P, we show that $v \models_{\mathcal{M}_{\mathbf{G}}} s$ for every sequent s occurring in P. This trivially holds for the sequents of $\mathcal{S}$. We show that this property is also preserved by applications of the rules of $\mathbf{G}$. This obviously holds for the weakening rule. We show it holds for primitive rules and for canonical introduction rules as well:

- Suppose $(L : \psi) \cup s_1 \cup \ldots \cup s_n$ is derived from the sequents $(L_1 : \psi) \cup s_1, \ldots, (L_n : \psi) \cup s_n$ using the primitive rule $r = \{L_1, \ldots, L_n\}/L$. Assume that $v \models_{\mathcal{M}_{\mathbf{G}}} (L_i : \psi) \cup s_i$ for every $1 \leq i \leq n$. We show that $v \models_{\mathcal{M}_{\mathbf{G}}} (L : \psi) \cup s_1 \cup \ldots \cup s_n$. By definition it suffices to show that there exists some $l : \varphi \in (L : \psi) \cup s_1 \cup \ldots \cup s_n$ such that $v(\varphi) \in \mathcal{D}_{\mathcal{M}_{\mathbf{G}}}(l)$. If there exists some $l : \varphi \in s_1 \cup \ldots \cup s_n$ such that $v(\varphi) \in \mathcal{D}_{\mathcal{M}_{\mathbf{G}}}(l)$, then we are done. Assume otherwise. Then our assumption entails that $v \models_{\mathcal{M}_{\mathbf{G}}} (L_i : \psi)$ for every $1 \leq i \leq n$, and so for every $1 \leq i \leq n$, there exists some $l \in L_i$ such that $v(\psi) \in \mathcal{D}_{\mathcal{M}_{\mathbf{G}}}(l)$. The definition of $\mathcal{D}_{\mathcal{M}_{\mathbf{G}}}$ entails that for every $1 \leq i \leq n$, there exists some $l \in L_i$ such that $l \in v(\psi)$. In other words, for every $1 \leq i \leq n$, $L_i \cap v(\psi) \neq \emptyset$. Since v is $\mathcal{M}_{\mathbf{G}}$-legal, $v(\psi) \in \mathcal{V}_{\mathcal{M}_{\mathbf{G}}}$. In particular, $v(\psi) \in r^*$, and so $v(\psi) \cap L \neq \emptyset$. Hence there exists some $l_0 \in L$, such that $l_0 \in v(\psi)$. It follows that $v(\psi) \in \mathcal{D}_{\mathcal{M}_{\mathbf{G}}}(l_0)$. Thus, in this case $v \models_{\mathcal{M}_{\mathbf{G}}} (L : \psi)$.

- Suppose $(L_0 : \sigma(\diamond(p_1, \ldots, p_n))) \cup s_1 \cup \ldots \cup s_m$ is derived from the sequents $\sigma(c_1) \cup s_1, \ldots, \sigma(c_m) \cup s_m$ using the canonical introduction rule $r = \{c_1, \ldots, c_m\}/(L_0 : \diamond(p_1, \ldots, p_n))$. Assume that $v \models_{\mathcal{M}_{\mathbf{G}}} \sigma(c_i) \cup s_i$ for every $1 \leq i \leq m$. We show that $v \models_{\mathcal{M}_{\mathbf{G}}} (L_0 : \sigma(\diamond(p_1, \ldots, p_n))) \cup s_1 \cup \ldots \cup s_m$. By definition it suffices to show that there exists some $l : \varphi \in (L_0 : \sigma(\diamond(p_1, \ldots, p_n))) \cup s_1 \cup \ldots \cup s_m$ such that $v(\varphi) \in \mathcal{D}_{\mathcal{M}_{\mathbf{G}}}(l)$. If there exists some $l : \varphi \in s_1 \cup \ldots \cup s_m$ such that $v(\varphi) \in \mathcal{D}_{\mathcal{M}_{\mathbf{G}}}(l)$, then we are done. Assume otherwise. Then our assumption entails that $v \models_{\mathcal{M}_{\mathbf{G}}} \sigma(c_i)$ for every $1 \leq i \leq m$. Thus for every $1 \leq i \leq m$ there exists some $l : p \in c_i$, such that $v(\sigma(p)) \in \mathcal{D}_{\mathcal{M}_{\mathbf{G}}}(l)$. The definition of $\mathcal{D}_{\mathcal{M}_{\mathbf{G}}}$ entails that for every $1 \leq i \leq m$ there exists some $l : p \in c_i$, such that $l \in v(\sigma(p))$. Let $L_j = v(\sigma(p_j))$ for every $1 \leq j \leq n$. It follows that $((L_1 : p_1) \cup \ldots \cup (L_n : p_n)) \cap c_k \neq \emptyset$ for every $1 \leq k \leq m$. Thus $r^*[L_1, \ldots, L_n] = \{L \subseteq \pounds \mid L \cap L_0 \neq \emptyset\}$. Since v is $\mathcal{M}_{\mathbf{G}}$-legal and $\sigma(\diamond(p_1, \ldots, p_n)) \in \mathcal{F}$, $v(\sigma(\diamond(p_1, \ldots, p_n))) \in r^*[v(\sigma(p_1)), \ldots, v(\sigma(p_n))]$. Hence, $v(\sigma(\diamond(p_1, \ldots, p_n))) \cap L_0 \neq \emptyset$. Thus there exists some $l \in L_0$, such that $l \in v(\sigma(\diamond(p_1, \ldots, p_n)))$. It follows that $v(\sigma(\diamond(p_1, \ldots, p_n))) \in \mathcal{D}_{\mathcal{M}_{\mathbf{G}}}(l)$. Thus, in this case $v \models_{\mathcal{M}_{\mathbf{G}}} (L_0 : \sigma(\diamond(p_1, \ldots, p_n)))$. □

We now turn to completeness.

Definition 5.13. Let $\mathbf{G}$ be a canonical labelled system for $\mathcal{L}$ and $\pounds$, let $\mathcal{F} \subseteq Frm_{\mathcal{L}}$ be closed under subformulas, and let $\mathcal{C} \subseteq Frm_{\mathcal{L}}$. We write $\mathcal{S} \vdash_{\mathbf{G}}^{\langle \mathcal{F}, \mathcal{C} \rangle} s$ if there exists a derivation P in $\mathbf{G}$ of a sequent s from a set $\mathcal{S}$ of sequents such that:

1. P consists only of (sequents consisting of) formulas from $\mathcal{F}$.
2. Only formulas from $\mathcal{C}$ serve as cut-formulas in P (see Definition 3.6).

Notation 5.14. We denote by $\mathbf{G_{cf}}$ the canonical labelled system obtained from a canonical labelled system $\mathbf{G}$ by discarding all the canonical cut rules of $\mathbf{G}$.

Definition 5.15 (Cut-Admissible Completeness)**.** A PNmatrix $\mathcal{M}$ for $\mathcal{L}$ and $\pounds$ is *cut-admissible complete* for a canonical labelled system $\mathbf{G}$ (for $\mathcal{L}$ and $\pounds$) if for every $\mathcal{F} \subseteq Frm_{\mathcal{L}}$ closed under subformulas, $\mathcal{C} \subseteq Frm_{\mathcal{L}}$, set $\mathcal{S}$ of sequents, and sequent s such that $sub[\mathcal{S} \cup \{s\}] \subseteq \mathcal{F}$ and $\mathcal{S} \not\vdash_{\mathbf{G}}^{\langle \mathcal{F}, \mathcal{C} \rangle} s$: there exists an $\mathcal{M}_{\mathbf{G_{cf}}}$-legal $\mathcal{F}$-valuation v, such that (i) $v \models_{\mathcal{M}_{\mathbf{G_{cf}}}} \mathcal{S}$, (ii) $v \not\models_{\mathcal{M}_{\mathbf{G_{cf}}}} s$, and (iii) $v(\psi) \in \mathcal{V}_{\mathcal{M}_{\mathbf{G}}}$ for every $\psi \in \mathcal{C} \cap \mathcal{F}$.

Note that the set of truth values $\mathcal{V}_{\mathcal{M}_{\mathbf{G}}}$ is a subset of the set of truth values $\mathcal{V}_{\mathcal{M}_{\mathbf{G_{cf}}}}$. Thus intuitively, cut-admissible completeness means the following. Suppose that for each $\mathcal{M}_{\mathbf{G_{cf}}}$-legal $\mathcal{F}$-valuation v, which assigns to formulas of $\mathcal{C} \cap \mathcal{F}$ only values from $\mathcal{V}_{\mathcal{M}_{\mathbf{G}}}$, it holds that $v \models_{\mathcal{M}_{\mathbf{G_{cf}}}} \mathcal{S}$ implies $v \models_{\mathcal{M}_{\mathbf{G_{cf}}}} s$. Then s is provable from $\mathcal{S}$ in $\mathbf{G}$ using only the "syntactic material" in $\mathcal{F}$, and applying cuts only on formulas from $\mathcal{C}$. As we show below, by taking $\mathcal{F} = \mathcal{C} = \mathit{Frm}_{\mathcal{L}}$, we get the usual notion of completeness (i.e. $\mathcal{S} \vdash_{\mathcal{M}_{\mathbf{G}}} s$ implies $\mathcal{S} \vdash_{\mathbf{G}} s$).

Theorem 5.16. For a canonical labelled system $\mathbf{G}$ (for $\mathcal{L}$ and $\pounds$), $\mathcal{M}_G$ is cut-admissible complete for $\mathbf{G}$.

Proof. Let $\mathcal{F} \subseteq \mathit{Frm}_{\mathcal{L}}$ be closed under subformulas, $\mathcal{C} \subseteq \mathit{Frm}_{\mathcal{L}}$, $\mathcal{S}$ be a set of sequents, and s be a sequent. Assume that $sub[\mathcal{S} \cup \{s\}] \subseteq \mathcal{F}$ and that $\mathcal{S} \not\vdash_{\mathbf{G}}^{\langle \mathcal{F}, \mathcal{C} \rangle} s$. We construct an $\mathcal{M}_{\mathbf{G_{cf}}}$-legal $\mathcal{F}$-valuation v such that $v(\psi) \in \mathcal{V}_{\mathcal{M}_{\mathbf{G}}}$ for every $\psi \in \mathcal{C} \cap \mathcal{F}$, and $v \models_{\mathcal{M}_{\mathbf{G_{cf}}}} \mathcal{S}$ but $v \not\models_{\mathcal{M}_{\mathbf{G_{cf}}}} s$. Call a set Ω of labelled formulas *maximal* if it satisfies the following conditions:

1. $\mathit{frm}[\Omega] \subseteq \mathcal{F}$.
2. $\mathcal{S} \not\vdash_{\mathbf{G}}^{\langle \mathcal{F}, \mathcal{C} \rangle} s'$ for every sequent $s' \subseteq \Omega$.
3. For every labelled formula $l : \psi \notin \Omega$ for $\psi \in \mathcal{F}$, there exists a sequent $s' \subseteq \Omega$ such that $\mathcal{S} \vdash_{\mathbf{G}}^{\langle \mathcal{F}, \mathcal{C} \rangle} s' \cup \{l : \psi\}$.

Let Ω be a maximal set extending s. An existence of such a set is ensured by the next lemma.

Lemma: Let s' be a set of labelled formulas, such that $\mathit{frm}[s'] \subseteq \mathcal{F}$. If $\mathcal{S} \not\vdash_{\mathbf{G}}^{\langle \mathcal{F}, \mathcal{C} \rangle} s'$, then there exists a maximal set Ω such that $s' \subseteq \Omega$.
Proof: Let $\gamma_1, \gamma_2, \ldots$ be an enumeration of all labelled formulas, such that $\mathit{frm}[\gamma_i] \in \mathcal{F}$ and $\gamma_i \notin s'$ for every $i \geq 1$. We recursively define an (infinite) sequence of sets of labelled formulas $\Omega_0, \Omega_1, \ldots$, as follows. Let $\Omega_0 = s'$. For $k \geq 1$, let $\Omega_k = \Omega_{k-1}$ iff there exists a sequent $s'' \subseteq \Omega_{k-1}$ such that $\mathcal{S} \vdash_{\mathbf{G}}^{\langle \mathcal{F}, \mathcal{C} \rangle} s'' \cup \{\gamma_k\}$. Otherwise, let $\Omega_k = \Omega_{k-1} \cup \{\gamma_k\}$. Finally, let $\Omega = \bigcup_{k \geq 0} \Omega_k$. It is easy to verify that Ω has all required properties.

Next, let v be a function from $\mathcal{F}$ to $P(\pounds)$ defined by $v(\psi) = \{l \in \pounds \mid l : \psi \notin \Omega\}$ for every $\psi \in \mathcal{F}$. We claim that:

(A) For every sequent c, such that $\mathit{frm}[c] \subseteq \mathcal{F}$: there exists a sequent $s' \subseteq \Omega$ such that $\mathcal{S} \vdash_{\mathbf{G}}^{\langle \mathcal{F}, \mathcal{C} \rangle} c \cup s'$ iff $(v(\psi) : \psi) \cap c \neq \emptyset$ for some $\psi \in \mathcal{F}$.

(B) v is an $\mathcal{M}_{\mathbf{G_{cf}}}$-legal $\mathcal{F}$-valuation.

(C) $v(\psi) \in \mathcal{V}_{\mathcal{M}_{\mathbf{G}}}$ for every $\psi \in \mathcal{C} \cap \mathcal{F}$.

(D) $v \models_{\mathcal{M}_{\mathbf{G_{cf}}}} \mathcal{S}$.

(E) $v \not\models_{\mathcal{M}_{\mathbf{G_{cf}}}} s$.

Proof of (A): Let c be a sequent such that $\mathit{frm}[c] \subseteq \mathcal{F}$. Suppose that there exists a sequent $s' \subseteq \Omega$ such that $\mathcal{S} \vdash_{\mathbf{G}}^{\langle \mathcal{F}, \mathcal{C} \rangle} c \cup s'$. The maximality of Ω entails that $c \not\subseteq \Omega$. Thus there exists a signed formula $l : \psi \in c$ such that $l : \psi \notin \Omega$. The construction of v entails that $l \in v(\psi)$, and so $(v(\psi) : \psi) \cap c \neq \emptyset$. For the converse, assume that $(v(\psi) : \psi) \cap c \neq \emptyset$ for some $\psi \in \mathcal{F}$. Hence there exists some $l \in v(\psi)$ such that $l : \psi \in c$. By definition, $l : \psi \notin \Omega$. The maximality of Ω entails that there exists a sequent $s' \subseteq \Omega$ such that $\mathcal{S} \vdash_{\mathbf{G}}^{\langle \mathcal{F}, \mathcal{C} \rangle} s' \cup \{l : \psi\}$. Using weakening, we obtain $\mathcal{S} \vdash_{\mathbf{G}}^{\langle \mathcal{F}, \mathcal{C} \rangle} c \cup s'$.

Proof of (B): We first show that $v(\psi) \in \mathcal{V}_{\mathcal{M}_{\mathbf{G_{cf}}}}$ for every $\psi \in \mathcal{F}$. Thus we prove that for every formula $\psi \in \mathcal{F}$, $v(\psi) \in r^*$ for every rule $r \in \mathsf{P}_{\mathbf{G_{cf}}}$. Let $\psi \in \mathcal{F}$, and let $r = \{L_1, \ldots, L_n\}/L$ be a primitive rule of $\mathbf{G_{cf}}$. To see that $v(\psi) \in r^*$, we show that if $L_i \cap v(\psi) \neq \emptyset$ for every $1 \leq i \leq n$, then $L \cap v(\psi) \neq \emptyset$. Suppose that $L_i \cap v(\psi) \neq \emptyset$ for every $1 \leq i \leq n$. (A) entails that for every $1 \leq i \leq n$, there exists some sequent $s_i \subseteq \Omega$ such that $\mathcal{S} \vdash_{\mathbf{G}}^{\langle \mathcal{F}, \mathcal{C} \rangle} (L_i : \psi) \cup s_i$. Using the rule r (which is not a canonical cut), we obtain $\mathcal{S} \vdash_{\mathbf{G}}^{\langle \mathcal{F}, \mathcal{C} \rangle} (L : \psi) \cup s_1 \cup \ldots \cup s_n$. (A) again entails that $L \cap v(\psi) \neq \emptyset$.

Next, we show that $v(\diamond(\psi_1,\ldots,\psi_n)) \in \diamond_{\mathcal{M}_{\mathbf{G_{cf}}}}(v(\psi_1),\ldots,v(\psi_n))$ for every formula $\diamond(\psi_1,\ldots,\psi_n) \in \mathcal{F}$. Thus we prove that for every formula $\diamond(\psi_1,\ldots,\psi_n) \in \mathcal{F}$, $v(\diamond(\psi_1,\ldots,\psi_n)) \in r^*[v(\psi_1),\ldots,v(\psi_n)]$ for every rule $r \in \mathsf{R}^{\diamond}_{\mathbf{G}}$. Let $\diamond(\psi_1,\ldots,\psi_n) \in \mathcal{F}$, and let $r = \mathcal{Q}/(L : \diamond(p_1,\ldots,p_n))$ be a rule in $\mathsf{R}^{\diamond}_{\mathbf{G}}$. To see that $v(\diamond(\psi_1,\ldots,\psi_n)) \in r^*[v(\psi_1),\ldots,v(\psi_n)]$, we show that if $((v(\psi_1) : p_1) \cup \ldots \cup (v(\psi_n) : p_n)) \cap c \neq \emptyset$ for every $c \in \mathcal{Q}$, then $v(\diamond(\psi_1,\ldots,\psi_n)) \cap L \neq \emptyset$. Suppose that $((v(\psi_1) : p_1) \cup \ldots \cup (v(\psi_n) : p_n)) \cap c \neq \emptyset$ for every $c \in \mathcal{Q}$. Let σ be a substitution assigning ψ_i to p_i for every $1 \leq i \leq n$. Thus $((v(\psi_1) : \psi_1) \cup \ldots \cup (v(\psi_n) : \psi_n)) \cap \sigma(c) \neq \emptyset$ for every $c \in \mathcal{Q}$. Hence for every $c \in \mathcal{Q}$, there exists some $1 \leq i \leq n$, such that $(v(\psi_i) : \psi_i) \cap \sigma(c) \neq \emptyset$. (A) entails that for every $c \in \mathcal{Q}$, there exists some sequent $s_c \subseteq \Omega$ such that $\mathcal{S} \vdash^{\langle \mathcal{F},\mathcal{C}\rangle}_{\mathbf{G}} \sigma(c) \cup s_c$. By applying r, we obtain $\mathcal{S} \vdash^{\langle \mathcal{F},\mathcal{C}\rangle}_{\mathbf{G}} (L : \diamond(\psi_1,\ldots,\psi_n)) \cup \bigcup_{c\in\mathcal{Q}} s_c$. Since $\bigcup_{c\in\mathcal{Q}} s_c \subseteq \Omega$, (A) entails that $v(\diamond(\psi_1,\ldots,\psi_n)) \cap L \neq \emptyset$.

Proof of (C): Let $\psi \in \mathcal{C} \cap \mathcal{F}$. We show that $v(\psi) \in \mathcal{V}_{\mathcal{M}_{\mathbf{G}}}$. Following (B), it suffices to show that $v(\psi) \in r^*$ for every canonical cut r of $\mathbf{G}$. Let $r = \{L_1,\ldots,L_n\}/\emptyset$ be a canonical cut of $\mathbf{G}$. To have $v(\psi) \in r^*$, it suffices to prove that $L_i \cap v(\psi) = \emptyset$ for some $1 \leq i \leq n$. Suppose otherwise. (A) entails that for every $1 \leq i \leq n$, there exists some sequent $s_i \subseteq \Omega$ such that $\mathcal{S} \vdash^{\langle \mathcal{F},\mathcal{C}\rangle}_{\mathbf{G}} (L_i : \psi) \cup s_i$. Using the canonical cut r (with the cut-formula $\psi \in \mathcal{C}$), we obtain $\mathcal{S} \vdash^{\langle \mathcal{F},\mathcal{C}\rangle}_{\mathbf{G}} s_1 \cup \ldots \cup s_n$. This contradicts the fact that $s_1 \cup \ldots \cup s_n \subseteq \Omega$.

Proof of (D): Let $s' \in \mathcal{S}$. Clearly, $\mathcal{S} \vdash^{\langle \mathcal{F},\mathcal{C}\rangle}_{\mathbf{G}} s'$. By (A), $(v(\psi) : \psi) \cap s' \neq \emptyset$ for some $\psi \in \mathcal{F}$. Thus there exists some $l \in v(\psi)$ such that $l : \psi \in s'$. Since $l \in v(\psi)$, we have $v(\psi) \in \mathcal{D}_{\mathcal{M}_{\mathbf{G_{cf}}}}(l)$. Hence, $v \models_{\mathcal{M}_{\mathbf{G_{cf}}}} s'$. Consequently, $v \models_{\mathcal{M}_{\mathbf{G_{cf}}}} \mathcal{S}$.

Proof of (E): Since $s \subseteq \Omega$, $l \notin v(\psi)$ for every $l : \psi \in s$. It follows that for every $l : \psi \in s$, $v(\psi) \notin \mathcal{D}_{\mathcal{M}_{\mathbf{G_{cf}}}}(l)$. Therefore, $v \not\models_{\mathcal{M}_{\mathbf{G_{cf}}}} s$.

Finally, properties (B)-(E) show that v is an $\mathcal{M}_{\mathbf{G_{cf}}}$-legal $\mathcal{F}$-valuation with all required properties. ☐

By the above results, usual soundness and completeness easily follow:

Corollary 5.17. Let $\mathbf{G}$ be a canonical labelled system for $\mathcal{L}$ and $\pounds$, and let $\mathcal{F} \subseteq \mathit{Frm}_{\mathcal{L}}$ be closed under subformulas. Let $\mathcal{S}$ be a set of sequents, and s be a sequent, such that $\mathit{sub}[\mathcal{S} \cup \{s\}] \subseteq \mathcal{F}$. If $v \models_{\mathcal{M}_{\mathbf{G}}} \mathcal{S}$ implies $v \models_{\mathcal{M}_{\mathbf{G}}} s$ for every $\mathcal{M}_{\mathbf{G}}$-legal $\mathcal{F}$-valuation v, then $\mathcal{S} \vdash^{\mathcal{F}}_{\mathbf{G}} s$.

Proof. Suppose that for every $\mathcal{M}_{\mathbf{G}}$-legal $\mathcal{F}$-valuation v, $v \models_{\mathcal{M}_{\mathbf{G}}} s$ whenever $v \models_{\mathcal{M}_{\mathbf{G}}} \mathcal{S}$. We prove that $v \models_{\mathcal{M}_{\mathbf{G_{cf}}}} \mathcal{S}$ implies $v \models_{\mathcal{M}_{\mathbf{G_{cf}}}} s$ for every $\mathcal{M}_{\mathbf{G_{cf}}}$-legal $\mathcal{F}$-valuation v such that $v(\psi) \in \mathcal{V}_{\mathcal{M}_{\mathbf{G}}}$ for every $\psi \in \mathcal{F}$. Theorem 5.16 then implies that $\mathcal{S} \vdash^{\mathcal{F}}_{\mathbf{G}} s$ (choose $\mathcal{C} = \mathcal{F}$). Let v be an $\mathcal{M}_{\mathbf{G_{cf}}}$-legal $\mathcal{F}$-valuation such that $v(\psi) \in \mathcal{V}_{\mathcal{M}_{\mathbf{G}}}$ for every $\psi \in \mathcal{F}$, and $v \models_{\mathcal{M}_{\mathbf{G_{cf}}}} \mathcal{S}$. The fact that $v(\psi) \in \mathcal{V}_{\mathcal{M}_{\mathbf{G}}}$ for every $\psi \in \mathcal{F}$, easily entails that v is also an $\mathcal{M}_{\mathbf{G}}$-legal $\mathcal{F}$-valuation. Similarly, $v \models_{\mathcal{M}_{\mathbf{G}}} \mathcal{S}$. Our assumption entails that $v \models_{\mathcal{M}_{\mathbf{G}}} s$. It follows that $v \models_{\mathcal{M}_{\mathbf{G_{cf}}}} s$. ☐

Corollary 5.18 (Soundness and Completeness)**.** For every canonical labelled system $\mathbf{G}$, $\mathcal{M}_{\mathbf{G}}$ is characteristic PNmatrix for $\mathbf{G}$.

Proof. Follows directly from Theorem 5.12 and Corollary 5.17 (by choosing $\mathcal{F} = \mathit{Frm}_{\mathcal{L}}$). ☐

Decidability is automatically obtained by the above results.

Corollary 5.19 (Decidability)**.** Given a canonical labelled system $\mathbf{G}$, a finite set $\mathcal{S}$ of sequents, and a sequent s, the question whether $\mathcal{S} \vdash_{\mathbf{G}} s$ is decidable. In particular: the question whether a given canonical labelled system $\mathbf{G}$ is consistent (i.e. $\not\vdash_{\mathbf{G}} \emptyset$) is decidable.

Proof. Follows directly by Corollary 5.18 and Corollary 4.14. ☐

$\mathcal{M}_{\mathbf{G}}$ provides a semantic characterization for $\mathbf{G}$, however it may not be a minimal one. For a minimal semantic representation, we should consider the equivalent PNmatrix $R[\mathcal{M}_{\mathbf{G}}]$:

Corollary 5.20 (Minimality)**.** For every canonical labelled system $\mathbf{G}$, $R[\mathcal{M}_{\mathbf{G}}]$ is a minimal (in terms of number of truth values) characteristic PNmatrix for $\mathbf{G}$.

Proof. The claim follows by Theorem 4.20 from the fact that $\mathcal{M}_{\mathbf{G}}$ is separable for every system $\mathbf{G}$. ☐

6 Proof-Theoretic Consequences

In this section we apply the general soundness and completeness results to provide *decidable* semantic criteria for syntactic properties of canonical labelled calculi that are not easily generally characterized by other means. We focus on the notions of *analyticity* and *cut-admissibility*, extended to the context of reasoning with assumptions.

6.1 Strong Analyticity

Strong analyticity is a crucial property of a useful (propositional) calculus, as it usually implies its consistency and decidability. Intuitively, a calculus is *strongly analytic* if whenever a sequent s is provable in it from a set of assumptions $\mathcal{S}$, then s can be proven using only the formulas available within $\mathcal{S}$ and s.

Definition 6.1. A canonical labelled system $\mathbf{G}$ is *strongly analytic* if $\mathcal{S} \vdash_{\mathbf{G}} s$ implies that $\mathcal{S} \vdash_{\mathbf{G}}^{\mathcal{F}} s$ for $\mathcal{F} = sub[\mathcal{S} \cup \{s\}]$ (i.e. there exists a derivation in $\mathbf{G}$ of s from $\mathcal{S}$ consisting solely of formulas from $sub[\mathcal{S} \cup \{s\}]$).

Below we provide a *decidable* semantic characterization of strong analyticity of canonical labelled calculi:

Theorem 6.2 (Characterization of Strong Analyticity)**.** Let $\mathbf{G}$ be a canonical labelled system for $\mathcal{L}$ and $\pounds$. Suppose that $\mathbf{G}$ does not include the (trivial) primitive rule $\emptyset/\emptyset$. Then, $\mathbf{G}$ is strongly analytic iff $\mathcal{M}_{\mathbf{G}}$ is proper.

Proof. Suppose that $\mathcal{M}_{\mathbf{G}}$ is proper. Assume that $\mathcal{S} \not\vdash_{\mathbf{G}}^{\mathcal{F}} s$ for $\mathcal{F} = sub[\mathcal{S} \cup \{s\}]$. We show that $\mathcal{S} \not\vdash_{\mathbf{G}} s$. By Corollary 5.17, there exists some $\mathcal{M}_{\mathbf{G}}$-legal $\mathcal{F}$-valuation v, such that $v \models_{\mathcal{M}_{\mathbf{G}}} \mathcal{S}$ but $v \not\models_{\mathcal{M}_{\mathbf{G}}} s$. By Proposition 4.10, v is extendable to a (full) $\mathcal{M}_{\mathbf{G}}$-legal $\mathcal{L}$-valuation v'. Clearly, $v' \models_{\mathcal{M}_{\mathbf{G}}} \mathcal{S}$ but $v' \not\models_{\mathcal{M}_{\mathbf{G}}} s$, and so $\mathcal{S} \not\vdash_{\mathcal{M}_{\mathbf{G}}} s$. By the soundness of $\mathcal{M}_{\mathbf{G}}$ for $\mathbf{G}$ (Corollary 5.18), $\mathcal{S} \not\vdash_{\mathbf{G}} s$.

For the converse, suppose that $\mathcal{M}_{\mathbf{G}}$ is not proper. If $\mathcal{V}_{\mathcal{M}_{\mathbf{G}}}$ is empty, then $\vdash_{\mathcal{M}_{\mathbf{G}}} \emptyset$ (since there are no $\mathcal{M}_{\mathbf{G}}$-legal $\mathcal{L}$-valuations), and so (by Corollary 5.18) $\vdash_{\mathbf{G}} \emptyset$. Clearly, without using a rule of the form $\emptyset/\emptyset$, there is no derivation in $\mathbf{G}$ that does not contain any formula. It follows that $\mathbf{G}$ is not strongly analytic in this case. Otherwise $\mathcal{V}_{\mathcal{M}_{\mathbf{G}}}$ is non-empty. Thus $\diamond_{\mathcal{M}_{\mathbf{G}}}(L_1, \ldots, L_n) = \emptyset$ for some n-ary connective $\diamond$ of $\mathcal{L}$ and $L_1, \ldots, L_n \in \mathcal{V}_{\mathcal{M}_{\mathbf{G}}}$. For every $1 \leq i \leq n$, let $\mathcal{S}_i$ be the set of all clauses of the form $\{l : p_i\}$ where $l \in L_i$, and let $s_i = \{l : p_i \mid l \notin L_i\}$. We claim that $\mathcal{S}_1 \cup \ldots \cup \mathcal{S}_n \vdash_{\mathbf{G}} s_1 \cup \ldots \cup s_n$, but there does not exist a derivation of $s_1 \cup \ldots \cup s_n$ from $\mathcal{S}_1 \cup \ldots \cup \mathcal{S}_n$ in $\mathbf{G}$ that consists solely of formulas from $\{p_1, \ldots, p_n\}$. For the latter, note that for the $\mathcal{M}_{\mathbf{G}}$-legal $\{p_1, \ldots, p_n\}$-valuation v assigning L_i to p_i for every $1 \leq i \leq n$, we have $v \models_{\mathcal{M}_{\mathbf{G}}} \mathcal{S}_1 \cup \ldots \cup \mathcal{S}_n$ but $v \not\models_{\mathcal{M}_{\mathbf{G}}} s_1 \cup \ldots \cup s_n$. Thus the claim follows by Theorem 5.12. For the former, note that every $\mathcal{M}_{\mathbf{G}}$-legal $\mathcal{L}$-valuation v' for which $v' \models_{\mathcal{M}_{\mathbf{G}}} \mathcal{S}_1 \cup \ldots \cup \mathcal{S}_n$ but $v' \not\models_{\mathcal{M}_{\mathbf{G}}} s_1 \cup \ldots \cup s_n$, we must have $v'(p_i) = L_i$ for every $1 \leq i \leq n$. But then $v'(\diamond(p_1, \ldots, p_n))$ should be an element of the empty set. Since such an $\mathcal{L}$-valuation does not exist, Corollary 5.18 entails that $\mathcal{S}_1 \cup \ldots \cup \mathcal{S}_n \vdash_{\mathbf{G}} s_1 \cup \ldots \cup s_n$. □

Corollary 6.3. The question whether a given canonical labelled system is strongly analytic is decidable.

6.2 Strong Cut-Admissibility

As the property of strong analyticity is sometimes difficult to establish, it is traditional in proof theory to investigate the property of *cut-admissibility*, which means that whenever s is provable in $\mathbf{G}$, it has a cut-free derivation in $\mathbf{G}$. In this paper we investigate a stronger notion of this property, defined as follows for labelled calculi:

Definition 6.4. A canonical labelled system $\mathbf{G}$ enjoys *strong cut-admissibility* if whenever $\mathcal{S} \vdash_{\mathbf{G}} s$, there exists a derivation in $\mathbf{G}$ of s from $\mathcal{S}$ in which only formulas from $frm[\mathcal{S}]$ serve as cut-formulas.

Due to the special form of primitive and introduction rules of canonical calculi (which, except for canonical cuts, enjoy the subformula property), the above property guarantees strong analyticity:

Proposition 6.5. If a canonical labelled system enjoys strong cut-admissibility, then it is strongly analytic.

Although for two-sided canonical sequent calculi the notions of strong analyticity and strong cut-admissibility coincide (see [2]), this is not the case for general canonical labelled calculi, for which the converse of Proposition 6.5 does not necessarily hold, as shown by the following example:

Example 6.6. Let $\pounds = \{a, b, c\}$, and assume that $\mathcal{L}$ consists of a unary connective $\star$. Let $\mathbf{G}$ be the canonical labelled system for $\mathcal{L}$ and $\pounds$, the primitive rules of which include only the canonical cuts $\{a\}, \{b\}/\emptyset$, $\{a\}, \{c\}/\emptyset$, and $\{b\}, \{c\}/\emptyset$, and its only introduction rules are $\{a : p_1\}/(\{a, b\} : \star p_1)$ and $\{a : p_1\}/(\{b, c\} : \star p_1)$. To see that this system is strongly analytic, by Theorem 6.2, it suffices to construct $\mathcal{M}_{\mathbf{G}}$ and check that it is proper. The construction proceeds as follows: $\mathcal{V}_{\mathcal{M}_{\mathbf{G}}} = \{\emptyset, \{a\}, \{b\}, \{c\}\}$, $\mathcal{D}_{\mathcal{M}_{\mathbf{G}}}(l) = \{l\}$ for $l \in \{a, b, c\}$, and the truth table for $\star$ is the following:

x	$\star_{\mathcal{M}_{\mathbf{G}}}(x)$
$\emptyset$	$\{\emptyset, \{a\}, \{b\}, \{c\}\}$
$\{a\}$	$\{\{b\}\}$
$\{b\}$	$\{\emptyset, \{a\}, \{b\}, \{c\}\}$
$\{c\}$	$\{\emptyset, \{a\}, \{b\}, \{c\}\}$

This is a proper PNmatrix, and so $\mathbf{G}$ is strongly analytic. However, it impossible to derive the sequent $\{b : \star p_1\}$ from the singleton set $\{\{a : p_1\}\}$ using only p_1 as a cut-formula. This is possible by applying the two introduction rules of $\mathbf{G}$ and then using the cut $\{a\}, \{c\}/\emptyset$ (with $\star p_1$ as the cut-formula). Thus although this system is strongly analytic, it does not enjoy strong cut-admissibility.

The intuitive explanation is that non-eliminable applications of canonical cuts (like the one in the above example) are not harmful for strong analyticity because they enjoy the subformula property. Thus, the equivalence between strong analyticity and cut-admissibility can be restored if we enforce the following property:

Definition 6.7. A canonical labelled system $\mathbf{G}$ for $\mathcal{L}$ and $\pounds$ is *cut-saturated* if for every canonical cut $\{L_1, \ldots, L_n\}/\emptyset$ of $\mathbf{G}$ and $l \in \pounds$, $\mathbf{G}$ contains the primitive rule $\{L_1, \ldots, L_n\}/\{l\}$.

Proposition 6.8. For every canonical labelled system $\mathbf{G}$, there is an equivalent cut-saturated canonical labelled system $\mathbf{G}'$ (i.e. $\vdash_{\mathbf{G}}=\vdash_{\mathbf{G}'}$).

Example 6.9. Revisiting the system from Example 6.6, we observe that $\mathbf{G}$ is not cut-saturated. To obtain a cut-saturated equivalent system $\mathbf{G}'$, we add (among others) the three primitive rules: $r_1 = \{a\}, \{b\}/\{c\}$, $r_2 = \{a\}, \{c\}/\{b\}$, and $r_3 = \{b\}, \{c\}/\{a\}$. Clearly, each of these rules can be simulated in $\mathbf{G}$ by applying cut and weakening. Note that the addition of these rules does not affect the corresponding PNmatrix, i.e. $\mathcal{M}_{\mathbf{G}} = \mathcal{M}_{\mathbf{G}'}$. However, we can now derive $\{b : \star p_1\}$ from $\{\{a : p_1\}\}$ without any cuts using the two introduction rules for $\star$ and the new rule r_2. Moreover, by Theorem 6.12 below, $\mathbf{G}'$ *does* enjoy strong cut-admissibility.

We are now ready to provide a decidable semantic characterization of strong cut-admissibility. For that we use the following lemmas:

Lemma 6.10. Let $\mathbf{G}$ be a canonical labelled system for $\mathcal{L}$ and $\pounds$. For every n-ary connective $\diamond$ of $\mathcal{L}$, and every $L_1, \ldots, L_n, L'_1, \ldots, L'_n \in \mathcal{V}_{\mathcal{M}_{\mathbf{G}}}$ such that $L_i \subseteq L'_i$ for every $1 \leq i \leq n$, we have $\diamond_{\mathcal{M}_{\mathbf{G}}}(L'_1, \ldots, L'_n) \subseteq \diamond_{\mathcal{M}_{\mathbf{G}}}(L_1, \ldots, L_n)$.

Proof. It suffices to note that for every $L_1, \ldots, L_n, L'_1, \ldots, L'_n \in \mathcal{V}_{\mathcal{M}_{\mathbf{G}}}$ such that $L_i \subseteq L'_i$ for every $1 \leq i \leq n$, and $r \in \mathsf{R}^{\diamond}_{\mathbf{G}}$, we have: $r^*[L'_1, \ldots, L'_n] \subseteq r^*[L_1, \ldots, L_n]$. □

Lemma 6.11. Let $\mathbf{G}$ be a cut-saturated canonical labelled system for $\mathcal{L}$ and $\pounds$. $\mathcal{V}_{\mathcal{M}_{\mathbf{G_{cf}}}} = \mathcal{V}_{\mathcal{M}_{\mathbf{G}}} \cup \{\pounds\}$, $\mathcal{D}_{\mathcal{M}_{\mathbf{G_{cf}}}}(l) = \mathcal{D}_{\mathcal{M}_{\mathbf{G}}}(l) \cup \{\pounds\}$ for every $l \in \pounds$, and $\diamond_{\mathcal{M}_{\mathbf{G_{cf}}}}(L_1, \ldots, L_n) = \diamond_{\mathcal{M}_{\mathbf{G}}}(L_1, \ldots, L_n) \cup \{\pounds\}$ for every n-ary connective $\diamond$ of $\mathcal{L}$ and every $L_1, \ldots, L_n \in \mathcal{V}_{\mathcal{M}_{\mathbf{G}}}$.

Proof. It suffices to show that $\mathcal{V}_{\mathcal{M}_{\mathbf{G_{cf}}}} = \mathcal{V}_{\mathcal{M}_{\mathbf{G}}} \cup \{\pounds\}$, and the other claims easily follow. Since $\mathsf{P}_{\mathbf{G_{cf}}} \subseteq \mathsf{P}_{\mathbf{G}}$, we obviously have $\mathcal{V}_{\mathcal{M}_{\mathbf{G}}} \subseteq \mathcal{V}_{\mathcal{M}_{\mathbf{G_{cf}}}}$. In addition, since there are no canonical cuts in $\mathbf{G_{cf}}$, $\pounds \in r^*$ for every primitive rule r of $\mathbf{G_{cf}}$, and thus $\pounds \in \mathcal{V}_{\mathcal{M}_{\mathbf{G_{cf}}}}$. Now, let $L \in \mathcal{V}_{\mathcal{M}_{\mathbf{G_{cf}}}}$, and suppose that $L \notin \mathcal{V}_{\mathcal{M}_{\mathbf{G}}}$. We show that $L = \pounds$. Since $L \notin \mathcal{V}_{\mathcal{M}_{\mathbf{G}}}$, $L \notin r^*$ for some $r \in \mathsf{P}_{\mathbf{G}}$. The fact that $L \in \mathcal{V}_{\mathcal{M}_{\mathbf{G_{cf}}}}$ entails that r must be a canonical cut, and thus $r = \{L_1, \ldots, L_n\}/\emptyset$ for some $L_1, \ldots, L_n \subseteq \pounds$. Consequently, since $L \notin \mathcal{V}_{\mathcal{M}_{\mathbf{G}}}$, we have $L_i \cap L \neq \emptyset$ for every $1 \leq i \leq n$. Since $\mathbf{G}$ is cut-saturated, for every $l \in \pounds$, $\mathbf{G_{cf}}$ includes the primitive rule $r_l = \{L_1, \ldots, L_n\}/\{l\}$. The fact that $L \notin \mathcal{V}_{\mathcal{M}_{\mathbf{G_{cf}}}}$ then entails that $L \in r_l^*$ for every $l \in \pounds$. It follows that $\{l\} \cap L \neq \emptyset$ for every $l \in \pounds$, and thus $L = \pounds$. □

Theorem 6.12 (Characterization of Strong Cut-Admissibility)**.** Let $\mathbf{G}$ be a cut-saturated canonical labelled system for $\mathcal{L}$ and $\pounds$. Suppose that $\mathbf{G}$ does not include the (trivial) primitive rule $\emptyset/\emptyset$. Then, $\mathbf{G}$ enjoys strong cut-admissibility iff $\mathcal{M}_{\mathbf{G}}$ is proper.

Proof. Suppose that $\mathcal{M}_{\mathbf{G}}$ is not proper. By Theorem 6.2, $\mathbf{G}$ is not strongly analytic. By Proposition 6.5, it does not enjoy strong cut-admissibility.

Now suppose that $\mathcal{M}_{\mathbf{G}}$ is proper. Assume that there does not exist a derivation of a sequent s from a set $\mathcal{S}$ of sequents in $\mathbf{G}$ in which only formulas from $frm[\mathcal{S}]$ serve as cut-formulas. We show that $\mathcal{S} \not\vdash_{\mathbf{G}} s$. By choosing $\mathcal{F} = Frm_{\mathcal{L}}$ and $\mathcal{C} = frm[\mathcal{S}]$ in Theorem 5.16, we obtain that there exists some $\mathcal{M}_{\mathbf{G_{cf}}}$-legal $\mathcal{L}$-valuation v' assigning only values from $\mathcal{V}_{\mathcal{M}_{\mathbf{G}}}$ to the formulas in $frm[\mathcal{S}]$, such that $v' \models_{\mathcal{M}_{\mathbf{G_{cf}}}} \mathcal{S}$ but $v' \not\models_{\mathcal{M}_{\mathbf{G_{cf}}}} s$. By Lemma 6.11, v' is a function from $Frm_{\mathcal{L}}$ to $\mathcal{V}_{\mathcal{M}_{\mathbf{G}}} \cup \{\pounds\}$. We construct a function $v : Frm_{\mathcal{L}} \to \mathcal{V}_{\mathcal{M}_{\mathbf{G}}}$, such that v is an $\mathcal{M}_{\mathbf{G}}$-legal $\mathcal{L}$-valuation; $v(\psi) \subseteq v'(\psi)$ for every $\psi \in Frm_{\mathcal{L}}$; and $v(\psi) = v'(\psi)$ whenever $v'(\psi) \in \mathcal{V}_{\mathcal{M}_{\mathbf{G}}}$. In particular, it is easy to verify that we will have $v \models_{\mathcal{M}_{\mathbf{G}}} \mathcal{S}$, and $v \not\models_{\mathcal{M}_{\mathbf{G}}} s$, and consequently (by Corollary 5.18) $\mathcal{S} \not\vdash_{\mathbf{G}} s$. The construction of v is done by recursion on the build-up of formulas. First, for atomic formulas, if $v'(p) \in \mathcal{V}_{\mathcal{M}_{\mathbf{G}}}$, we choose $v(p) = v'(p)$. Otherwise, $v'(p) = \pounds$, and we (arbitrarily) choose $v(p)$ to be an element of $\mathcal{V}_{\mathcal{M}_{\mathbf{G}}}$ ($\mathcal{V}_{\mathcal{M}_{\mathbf{G}}}$ is non-empty since $\mathcal{M}_{\mathbf{G}}$ is proper). Now, let $\diamond$ be an n-ary connective of $\mathcal{L}$, and suppose $v(\psi_i)$ was defined for every $1 \leq i \leq n$. We choose $v(\diamond(\psi_1, \ldots, \psi_n))$ to be equal to $v'(\diamond(\psi_1, \ldots, \psi_n))$ if the latter is in $\mathcal{V}_{\mathcal{M}_{\mathbf{G}}}$. Otherwise, $v'(\diamond(\psi_1, \ldots, \psi_n)) = \pounds$, and we choose $v(\diamond(\psi_1, \ldots, \psi_n))$ to be some element of $\diamond_{\mathcal{M}_{\mathbf{G}}}(v(\psi_1), \ldots, v(\psi_n))$ (such an element exists since $\mathcal{M}_{\mathbf{G}}$ is proper). Obviously, $v(\psi) \subseteq v'(\psi)$ for every $\psi \in Frm_{\mathcal{L}}$, and $v(\psi) = v'(\psi)$ whenever $v'(\psi) \in \mathcal{V}_{\mathcal{M}_{\mathbf{G}}}$. To see that v is an $\mathcal{M}_{\mathbf{G}}$-legal $\mathcal{L}$-valuation, suppose (for contradiction) that $v(\diamond(\psi_1, \ldots, \psi_n)) \notin \diamond_{\mathcal{M}_{\mathbf{G}}} (v(\psi_1), \ldots, v(\psi_n))$ for some formula $\diamond(\psi_1, \ldots, \psi_n)$. By Lemma 6.11, $v(\diamond(\psi_1, \ldots, \psi_n)) \notin \diamond_{\mathcal{M}_{\mathbf{G_{cf}}}} (v(\psi_1), \ldots, v(\psi_n))$. Now, since $v(\psi) \subseteq v'(\psi)$ for every $\psi \in Frm_{\mathcal{L}}$, Lemma 6.10 entails that $v(\diamond(\psi_1, \ldots, \psi_n)) \notin \diamond_{\mathcal{M}_{\mathbf{G_{cf}}}} (v'(\psi_1), \ldots, v'(\psi_n))$. Consequently, since v' is an $\mathcal{M}_{\mathbf{G_{cf}}}$-legal $\mathcal{L}$-valuation, $v'(\diamond(\psi_1, \ldots, \psi_n)) \neq v(\diamond(\psi_1, \ldots, \psi_n))$. It follows that $v'(\diamond(\psi_1, \ldots, \psi_n)) = \pounds$. By definition, $v(\diamond(\psi_1, \ldots, \psi_n)) \in \diamond_{\mathcal{M}_{\mathbf{G}}}(v(\psi_1), \ldots, v(\psi_n))$, reaching a contradiction. □

Corollary 6.13. Let $\mathbf{G}$ be a cut-saturated canonical labelled system for $\mathcal{L}$ and $\pounds$. Suppose that $\mathbf{G}$ does not include the (trivial) primitive rule $\emptyset/\emptyset$. Then the following statements concerning $\mathbf{G}$ are equivalent: (i) $\mathcal{M}_{\mathbf{G}}$ is proper, (ii) $\mathbf{G}$ is strongly analytic, and (iii) $\mathbf{G}$ enjoys strong cut-admissibility.

Remark 6.14. When $\mathbf{G}$ is not cut-saturated, (i) and (ii) in the corollary above are equivalent to the following property: (iii') whenever $\mathcal{S} \vdash_{\mathbf{G}} s$, there exists a derivation in $\mathbf{G}$ of s from $\mathcal{S}$ in which for every application of a canonical cut deriving a sequent of the form $s_1 \cup \ldots \cup s_n$ from the sequents $(L_1 : \psi) \cup s_1, \ldots, (L_n : \psi) \cup s_n$, we have $\psi \in frm[\mathcal{S} \cup \{s_1, \ldots, s_n\}]$.

Remark 6.15. In [11, 4] a necessary and sufficient condition for cut-admissibility is provided using a simple syntactic condition of *coherence* for canonical Gentzen-type systems, which are particular instances of (two-sided) labelled canonical calculi. It should be noted that in these systems cut-admissibility is equivalent to strong cut-admissibility, and so coherence of such system is equivalent to the existence of a characteristic proper PNmatrix.

7 An Application: Quasi-Canonical Systems

The theory of canonical labelled calculi developed here can be applied for characterization of logics, which are not necessarily given in terms of canonical labelled systems. In this section we demonstrate one such application for the class of *quasi-canonical systems*, defined in [7]. Many well-known logics are induced by quasi-canonical systems, including a variety of paraconsistent logics. Quasi-canonical systems are two-sided sequent calculi, which are not necessarily canonical in the sense of this paper, as they may have more complex introduction rules than those allowed in two-sided canonical systems. However, we show that they can still be translated to *canonical labelled systems* (with more than two labels). The translation easily implies decidability for all quasi-canonical systems, and also allows for providing finite-valued PNmatrices semantics for logics they induce.

7.1 Quasi-Canonical Systems

Quasi-canonical systems, defined in [7], are two-sided systems, which in addition to the usual weakening rule, identity axiom and cut, include also logical rules with the following properties (the language is assumed to have a unary connective $\neg$):

1. Exactly one formula is introduced in the conclusion of the rule, on exactly one of its two sides.
2. The formula being introduced is either of the form $\diamond(p_1, \dots, p_n)$ or of the form $\neg \diamond (p_1, \dots, p_n)$.
3. All formulas in the premises of a rule introducing an n-ary connective, belong to the set $\{p_1, \dots, p_n, \neg p_1, \dots, \neg p_n\}$.
4. There are no restrictions on the side formulas in the rule application (i.e. every context is legitimate).

Of course, the rules described above are not canonical introduction rules in the sense of Definition 3.8, due to the following two "violations": (i) the introduced formula can be not only $\diamond(p_1, \dots, p_n)$, but also $\neg \diamond (p_1, \dots, p_n)$, and (ii) the premises may contain not only atomic formulas p_i, but also $\neg p_i$. Hence the results obtained in this paper for canonical labelled calculi do not directly apply. However, below we show that these results can still be exploited by translating quasi-canonical calculi into equivalent (in the sense defined below) *canonical* labelled calculi.

As noted above, the language of quasi-canonical systems is assumed in [7] to include a unary connective $\neg$. This restriction can be lifted by considering quasi-canonical rules defined with respect to any finite set of unary connectives. Thus in what follows we assume that $\mathcal{L}$ contains some fixed set $\mathcal{U}$ of unary connectives. The notion of such generalized quasi-canonical rule can then be formalized in our terms as follows:

Definition 7.1. An *n-quasi-clause* is a two-sided sequent consisting only of formulas of the form p_i or $\star p_i$ for $1 \leq i \leq n$ and $\star \in \mathcal{U}$.

Definition 7.2. A *(generalized) quasi-canonical rule* for an n-ary connective $\diamond$ of $\mathcal{L}$ is an expression of the form $\mathcal{Q}/s$, where $\mathcal{Q}$ is a set of n-quasi-clauses and s has the form: $\{l : \star\diamond(p_1, \dots, p_n)\}$ or $\{l : \diamond(p_1, \dots, p_n)\}$ ($l \in \{t, f\}$ and $\star \in \mathcal{U}$). An application of a quasi-canonical rule $\{q_1, \dots, q_m\}/s$ is an inference step of the form:

$$\frac{\sigma(q_1) \cup s_1 \quad \dots \quad \sigma(q_m) \cup s_m}{\sigma(s) \cup s_1 \cup \dots \cup s_m}$$

where σ is a substitution, and s_i is a sequent for every $1 \leq i \leq m$.

Example 7.3. Suppose that $\neg \in \mathcal{U}$. The following rules from [12] are quasi-canonical rules for $\wedge$:

$$\{\neg p_1 \Rightarrow, \neg p_2 \Rightarrow\}/\neg(p_1 \wedge p_2) \Rightarrow \qquad \{\Rightarrow \neg p_1\}/ \Rightarrow \neg(p_1 \wedge p_2) \quad \{\Rightarrow \neg p_2\}/ \Rightarrow \neg(p_1 \wedge p_2)$$

Their applications have the forms:

$$\frac{\Gamma, \neg\psi \Rightarrow \Delta \quad \Gamma, \neg\varphi \Rightarrow \Delta}{\Gamma, \neg(\psi \wedge \varphi) \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \neg\psi}{\Gamma \Rightarrow \Delta, \neg(\psi \wedge \varphi)} \qquad \frac{\Gamma \Rightarrow \Delta, \neg\varphi}{\Gamma \Rightarrow \Delta, \neg(\psi \wedge \varphi)}$$

Definition 7.4. A quasi-canonical system for $\mathcal{L}$ is a two-sided sequent system consisting of the weakening rule (see Definition 3.1), the standard identity axiom and the cut rule (see Examples 3.5 and 3.7), and an arbitrary finite set of quasi-canonical rules. As before, we say that a (two-sided) sequent s *is derivable* in a quasi-canonical system $\mathbf{G}$ from a set of (two-sided) sequents $\mathcal{S}$ (and denote it by $\mathcal{S} \vdash_{\mathbf{G}} s$) if there exists a derivation in $\mathbf{G}$ of s from $\mathcal{S}$.

While in this paper we have been interested in derivability of sequents from sets of sequents, in quasi-canonical systems one is usually interested in its *logic*, i.e. the consequence relation between sets of formulas and formulas which is induced by the system. This relation can be defined as follows:

Definition 7.5. Let $\mathbf{G}$ be a quasi-canonical system for $\mathcal{L}$. The *consequence relation* $\vdash^{frm}_{\mathbf{G}}$ between sets of formulas and formulas which is induced by $\mathbf{G}$ is defined as follows: $\mathcal{T} \vdash^{frm}_{\mathbf{G}} \varphi$ if $\{\{t : \psi\} \mid \psi \in \mathcal{T}\} \vdash_{\mathbf{G}} \{t : \varphi\}$.

Various well-known logics are induced by quasi-canonical systems. This includes a large family of paraconsistent logics known as C-systems ([13]), for which cut-free quasi-canonical systems were proposed in [7], and various other paraconsistent extensions of positive classical logic studied in [12].

7.2 Translation of Quasi-Canonical Systems to Canonical Labelled Systems

Next we provide a translation of a given quasi-canonical calculus $\mathbf{G}$ into a canonical labelled one $\mathbf{G}^L$, which is equivalent to the original one in a sense defined below. The idea is to "encode" the information related to the connectives from $\mathcal{U}$ in the labels of $\mathbf{G}^L$, so that connectives from $\mathcal{U}$ "violating" canonicity are removed. To this end, we use $\{t, f\} \cup \{t_\star \mid \star \in \mathcal{U}\} \cup \{f_\star \mid \star \in \mathcal{U}\}$ as the set of labels. We denote this set by $\pounds_{\mathcal{U}}$.

Definition 7.6. For a labelled formula $l : \psi$ where $l \in \{t, f\}$, $T(l : \psi)$ is a labelled formula over $\pounds_{\mathcal{U}}$, defined as follows: $T(l : \psi) = l_\star : \varphi$ if $\psi = \star\varphi$ for some $\star \in \mathcal{U}$, and otherwise $T(l : \psi) = l : \psi$. T is extended to two-sided sequents and sets of two-sided sequents in the obvious way (e.g. $T(s) = \{T(l : \psi) \mid l : \psi \in s\}$).

Example 7.7. Suppose that $\neg \in \mathcal{U}$. Then $T(\neg\neg p_1 \Rightarrow \neg p_1, p_3) = \{t : p_3, t_\neg : p_1, f_\neg : \neg p_1\}$.

Notation 7.8. Given a labelled formula γ, we denote by $lab[\gamma]$ the label appearing in γ. lab is extended to sequents in the obvious way.

Definition 7.9. Let $\mathbf{G}$ be a quasi-canonical system for $\mathcal{L}$. $\mathbf{G}^L$ is the canonical labelled system for $\mathcal{L}$ and $\pounds_{\mathcal{U}}$ with the following rules (except for weakening):

- Primitive rules:
  - The canonical axioms $\emptyset/\{t, f\}$ and $\emptyset/\{t_\star, f_\star\}$ for every $\star \in \mathcal{U}$.
  - The canonical cuts $\{t\}, \{f\}/\emptyset$ and $\{t_\star\}, \{f_\star\}/\emptyset$ for every $\star \in \mathcal{U}$.
  - $\{lab[T(q)] \mid q \in \mathcal{Q}\}/\{l_\star\}$ for every quasi-canonical rule $\mathcal{Q}/\{l : \star p_1\}$ of $\mathbf{G}$ ($l \in \{t, f\}$ and $\star \in \mathcal{U}$).
- Canonical introduction rules:
  - $\{l_\star : p_1\}/\{l : \star p_1\}$ every $l \in \{t, f\}$ and $\star \in \mathcal{U}$.
  - $T(\mathcal{Q})/\{l : \diamond(p_1, \dots, p_n)\}$ for every quasi-canonical rule $\mathcal{Q}/\{l : \diamond(p_1, \dots, p_n)\}$ of $\mathbf{G}$ where $l \in \{t, f\}$, and $\diamond$ is an n-ary connective of $\mathcal{L}$ which does not belong to $\mathcal{U}$.
  - $T(\mathcal{Q})/\{l_\star : \diamond(p_1, \dots, p_n)\}$ for every quasi-canonical rule $\mathcal{Q}/\{l : \star \diamond (p_1, \dots, p_n)\}$ of $\mathbf{G}$ where $l \in \{t, f\}$, $\diamond$ is an n-ary connective of $\mathcal{L}$, and $\star \in \mathcal{U}$.

Example 7.10. Suppose that $\mathcal{U} = \{\neg\}$, and that $\mathcal{L}$ consists of a unary connective $\neg$ and a binary connective $\wedge$. The system $PLK[\{(\neg\wedge \Rightarrow)\}]$ from [12] is a quasi-canonical system for $\mathcal{L}$, whose only quasi-canonical rules are $\{p_1 \Rightarrow\}/ \Rightarrow \neg p_1$, $\{\neg p_1 \Rightarrow\ ,\ \neg p_2 \Rightarrow\}/\neg(p_1 \wedge p_2) \Rightarrow$, and the two usual rules for $\wedge$ (see Example 3.9). We denote this system by $\mathbf{G}_0$. $\mathbf{G}_0^L$ is the canonical labelled system for $\mathcal{L}$ and $\pounds_{\mathcal{U}} = \{t, f, t_\neg, f_\neg\}$, with the following rules (except for weakening):

- Primitive rules:
  - The canonical axioms $\emptyset/\{t, f\}$ and $\emptyset/\{t_\neg, f_\neg\}$.
  - The canonical cuts $\{t\}, \{f\}/\emptyset$ and $\{t_\neg\}, \{f_\neg\}/\emptyset$.
  - $\{f\}/\{t_\neg\}$ (since $\mathbf{G}_0$ has a quasi-canonical rule $\{p_1 \Rightarrow\}/ \Rightarrow \neg p_1$, and $T(p_1 \Rightarrow) = \{f : p_1\}$).
- Canonical introduction rules:
  - $\{t_\neg : p_1\}/\{t : \neg p_1\}$ and $\{f_\neg : p_1\}/\{f : \neg p_1\}$.

- $\{f_\neg : p_1\}, \{f_\neg : p_2\}/\{f_\neg : p_1 \wedge p_2\}$ (since $\mathbf{G}_0$ has a quasi-canonical rule $\neg p_1 \Rightarrow\ ,\ \neg p_2 \Rightarrow /\neg(p_1 \wedge p_2) \Rightarrow$, and $T(\neg p_i \Rightarrow) = \{f_\neg : p_i\}$).
- The two usual rules for $\wedge$ (since these rules are already canonical).

The obtained labelled system $\mathbf{G}^L$ is equivalent to the original quasi-canonical system $\mathbf{G}$ in the following sense:

Theorem 7.11. For every quasi-canonical system $\mathbf{G}$, $\mathcal{S} \vdash_{\mathbf{G}} s$ iff $T(\mathcal{S}) \vdash_{\mathbf{G}^L} T(s)$.

Proof. See Appendix A. $\square$

A general decidability result for quasi-canonical systems easily follows:

Corollary 7.12 (Decidability)**.** Given a quasi-canonical system $\mathbf{G}$, a finite set $\mathcal{S}$ of sequents, and a sequent s, the question whether $\mathcal{S} \vdash_{\mathbf{G}} s$ is decidable.

Proof. Follows directly from Corollary 5.19 and Theorem 7.11. Note that the construction of $\mathbf{G}^L$ from $\mathbf{G}$ is obviously computable. $\square$

Next, we use the semantic framework of PNmatrices to provide finite-valued semantics for all logics induced by quasi-canonical systems.

Proposition 7.13. Let $\mathbf{G}$ be a quasi-canonical system for $\mathcal{L}$, and let v be an $\mathcal{M}_{\mathbf{G}^L}$-legal $\mathcal{L}$-valuation. For every two-sided sequent s: $v \models_{\mathcal{M}_{\mathbf{G}^L}} s$ iff $v \models_{\mathcal{M}_{\mathbf{G}^L}} T(s)$.

Proof. Suppose that $v \models_{\mathcal{M}_{\mathbf{G}^L}} T(s)$. By definition, there exists some formula ψ such that $l : \psi \in s$ and $l \in v(\psi)$ (where $l \in \pounds_{\mathcal{U}}$). If $l \in \{t, f\}$ then $l : \psi \in s$ as well, and clearly $v \models_{\mathcal{M}_{\mathbf{G}^L}} s$. Otherwise $l = t_\star$ or $l = f_\star$ for some $\star \in \mathcal{U}$. Suppose that $l = t_\star$ (the other case is symmetric). Thus $t : \star\psi \in s$. Since v is $\mathcal{M}_{\mathbf{G}^L}$-legal, $v(\star\psi) \in \star_{\mathcal{M}_{\mathbf{G}^L}}(v(\psi))$. The construction of $\mathcal{M}_{\mathbf{G}^L}$ ensures that $t \in v(\star\psi)$ (this follows from the fact that $\mathbf{G}^L$ includes the introduction rule $\{t_\star : p_1\}/\{t : \star p_1\}$). It follows that $v \models_{\mathcal{M}_{\mathbf{G}^L}} s$.

For the converse, suppose that $v \models_{\mathcal{M}_{\mathbf{G}^L}} s$. By definition, there exists some formula ψ such that $l : \psi \in s$ and $l \in v(\psi)$ (where $l \in \{t, f\}$). If ψ does not have a form $\star\varphi$ for $\star \in \mathcal{U}$, then $l : \psi \in T(s)$, and clearly $v \models_{\mathcal{M}_{\mathbf{G}^L}} T(s)$. Otherwise, $\psi = \star\varphi$ for some $\star \in \mathcal{U}$. Thus $l_\star : \varphi \in T(s)$. Since v is $\mathcal{M}_{\mathbf{G}^L}$-legal, $v(\psi) \in \star_{\mathcal{M}_{\mathbf{G}^L}}(v(\varphi))$. Suppose now that $l = t$ (the other case is symmetric). The construction of $\mathcal{M}_{\mathbf{G}^L}$ ensures that $t_\star \in v(\varphi)$ (this follows from the fact that $\mathbf{G}^L$ includes the canonical axiom $\emptyset/\{t_\star, f_\star\}$, the introduction rule $\{f_\star : p_1\}/\{f : \star p_1\}$, and the canonical cut $\{t\}, \{f\}/\emptyset$). It follows that $v \models_{\mathcal{M}_{\mathbf{G}^L}} T(s)$. $\square$

Corollary 7.14 (Soundness and Completeness)**.** For every quasi-canonical system $\mathbf{G}$, $\mathcal{T} \vdash_{\mathbf{G}}^{frm} \varphi$ iff $\mathcal{T} \vdash_{\mathcal{M}_{\mathbf{G}^L}}^{frm} \varphi$.

Proof. Easily follows from Proposition 7.13, Theorem 7.11, and Corollary 5.18. $\square$

Example 7.15. Let us revisit the system $\mathbf{G}_0$ from Example 7.10. The corresponding (proper) PNmatrix $\mathcal{M}_{\mathbf{G}_0^L}$, which is obtained from the canonical labelled calculus $\mathbf{G}_0^L$ constructed in that example, is defined as follows: $\mathcal{V}_{\mathcal{M}_{\mathbf{G}_0^L}} = \{\{t, t_\neg\}, \{t, f_\neg\}, \{f, t_\neg\}\}$, $\mathcal{D}_{\mathcal{M}_{\mathbf{G}_0^L}} = \{\{t, t_\neg\}, \{t, f_\neg\}\}$,[5] and $\neg$ and $\wedge$ have the following interpretations:

$\wedge_{\mathcal{M}_{\mathbf{G}_0^L}}$	$\{t, f_\neg\}$	$\{t, t_\neg\}$	$\{f, t_\neg\}$
$\{t, f_\neg\}$	$\{\{t, f_\neg\}\}$	$\{\{t, t_\neg\}, \{t, f_\neg\}\}$	$\{\{f, t_\neg\}\}$
$\{t, t_\neg\}$	$\{\{t, t_\neg\}, \{t, f_\neg\}\}$	$\{\{t, t_\neg\}, \{t, f_\neg\}\}$	$\{\{f, t_\neg\}\}$
$\{f, t_\neg\}$	$\{\{f, t_\neg\}\}$	$\{\{f, t_\neg\}\}$	$\{\{f, t_\neg\}\}$

$\neg_{\mathcal{M}_{\mathbf{G}_0^L}}$	
$\{t, f_\neg\}$	$\{\{f, t_\neg\}\}$
$\{t, t_\neg\}$	$\{\{t, t_\neg\}, \{t, f_\neg\}\}$
$\{f, t_\neg\}$	$\{\{t, t_\neg\}, \{t, f_\neg\}\}$

The PNmatrix $\mathcal{M}_{\mathbf{G}_0^L}$ is isomorphic to the Nmatrix given for the system $PLK[(\neg\wedge \Rightarrow)]$ in [12]. It is easy to check that this is the case for *all* the PNmatrices obtained by this procedure for the quasi-canonical systems considered in [12, 7].

[5] Note that we take $\mathcal{D}_{\mathcal{M}_{\mathbf{G}_0^L}}$ to be a set, as explained in Remark 4.4.

8 Related work

In addition to the aforementioned papers [1, 11, 4, 2], there are a number of works providing syntactic and semantic criteria for cut-elimination (and analyticity) in various other proof-theoretic domains. For example, [14, 15] consider a family of two-sided single-conclusion calculi obtained by extending the full Lambek calculus with structural and logical rules. The systems of this family all include the usual identity axiom and the usual cut rule. Thus, in [14, 15] the primitive rules remain fixed, while the structural and logical rules change. This is in contrast to the current paper, where a large variety of primitive rules is allowed, but the structural rules are predetermined (as we discuss only fully-structural calculi). Note also that as opposed to the multiple-conclusion systems used in this paper, an investigation of cut-admissibility and analyticity for two-sided single-conclusion canonical calculi was carried out in [16].

Another interesting connection is provided by [17], where a general framework for specifying proof systems using linear logic is proposed (further extended in [18] using subexponentials), providing also a necessary condition that guarantees that the specified systems admit cut-elimination. Many systems of our framework (at least all the two-sided ones) can be specified using these methods. Due to the general nature of our framework, an algorithmic approach to such specification (instead of doing it manually for each concrete calculus) is a promising direction for further research.

9 Conclusions and Further Research

In this paper we have defined an abstract framework of canonical labelled calculi, which includes many calculi previously studied in the literature. To provide semantics for all the calculi in our framework, we have introduced *PNmatrices*, an *effective* generalization of Nmatrices, in which empty entries in logical truth tables are allowed. We have provided a method to construct a characteristic PNmatrix for every canonical labelled calculus, which in turn implies its decidability. We have also shown that this PNmatrix has no empty entries (i.e. is proper) iff the corresponding calculus is strongly analytic. For cut-saturated canonical calculi, these properties are also equivalent to strong cut-admissibility. Thus we obtain a simple and decidable semantic characterization of these crucial proof-theoretical properties for canonical labelled calculi.

Canonical labelled calculi are a general family of calculi which are decidable, have simple semantics and exhibit nice proof-theoretical properties. As demonstrated in this paper, they can be used as a formalism for "encoding" other types of calculi. It is thus interesting to characterize the type of calculi which can be translated into canonical labelled calculi, and investigate the properties preserved by such translations. Another direction for further research is generalizing the results of this paper to more complex classes of labelled calculi, e.g., those defined in [19] for inquisitive logic. Finally, to make the framework useful for more practical applications, extending the results to the first-order case is required.

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A Proof of Theorem 7.11

Lemma A.1. Let $\mathbf{G}$ be a quasi-canonical system. For every n-quasi-clause q, substitution σ, and sequent s: $T(\sigma(q)) \cup s \vdash_{\mathbf{G}^L} \sigma(T(q)) \cup s$.

Proof. We prove it for n-quasi-clauses with exactly one labelled formula. For larger n-quasi-clauses, the claim follows by repeatedly applying the following argument. Let q be an n-quasi-clause with exactly one labelled formula, σ be an arbitrary substitution, and s be an arbitrary sequent. By definition, $q = \{l : \varphi\}$, where $l \in \{t, f\}$, and φ is either p_i or $\star p_i$ for $1 \leq i \leq n$ and $\star \in \mathcal{U}$. Suppose first that $\varphi = \star p_i$. Then $T(\sigma(q)) = T(\{l : \star\sigma(p_i)\}) = \{l_\star : \sigma(p_i)\} = \sigma(\{l_\star : p_i\}) = \sigma(T(q))$. Obviously, $T(\sigma(q)) \cup s \vdash_{\mathbf{G}^L} \sigma(T(q)) \cup s$. Suppose now that $\varphi = p_i$. If $\sigma(p_i)$ is not a formula of the form $\star\psi$ for $\star \in \mathcal{U}$, then again $T(\sigma(q)) = T(\{l : \sigma(p_i)\}) = \{l : \sigma(p_i)\} = \sigma(\{l : p_i\}) = \sigma(T(q))$. Otherwise, $\sigma(p_i) = \star\psi$ for some $\star \in \mathcal{U}$. Then: $T(\sigma(q)) = T(\{l : \star\psi\}) = \{l_\star : \psi\}$. By definition, $\mathbf{G}^L$ includes the introduction rule $\{l_\star : p_1\}/\{l : \star p_1\}$. By applying this rule, we obtain that $T(\sigma(q)) \cup s \vdash_{\mathbf{G}^L} \{l : \star\psi\} \cup s$, where $\{l : \star\psi\} = \{l : \sigma(p_i)\} = \sigma(\{l : p_i\}) = \sigma(T(q))$. □

Proof of Theorem 7.11. ($\Rightarrow$) Suppose that $\mathcal{S} \vdash_{\mathbf{G}} s$. Thus there exists a sequence $P = s_1, \ldots, s_N$ such that $s_N = s$ and for every $1 \leq i \leq N$, s_i is either in $\mathcal{S}$, or derived from previous sequents by applying a rule of $\mathbf{G}$. We prove that for every $1 \leq i \leq N$: if $T(\mathcal{S}) \vdash_{\mathbf{G}^L} T(s_j)$ for every $1 \leq j < i$ then $T(\mathcal{S}) \vdash_{\mathbf{G}^L} T(s_i)$ (it then follows that $T(\mathcal{S}) \vdash_{\mathbf{G}^L} T(s)$). Let $1 \leq i \leq N$, and suppose that $T(\mathcal{S}) \vdash_{\mathbf{G}^L} T(s_j)$ for every $1 \leq j < i$. Consider the possible cases:

- $s_i \in \mathcal{S}$. By definition, $T(s_i) \in T(\mathcal{S})$, and so $T(\mathcal{S}) \vdash_{\mathbf{G}^L} T(s_i)$.
- $s_i = s_{i_1} \cup s$ is derived from a previous sequent s_{i_1} by applying the weakening rule. In this case, we can derive $T(s_i) = T(s_{i_1}) \cup T(s)$ from $T(s_{i_1})$ by applying weakening as well.
- s_i is obtained by applying the identity axiom. Thus $s_i = \{t : \psi, f : \psi\}$ for some formula ψ. $T(s_i)$ is either $\{t : \psi, f : \psi\}$ or $\{t_\star : \varphi, f_\star : \varphi\}$ (if $\psi = \star\varphi$). In any case, $T(s_i)$ is obtained by applying a canonical axiom of $\mathbf{G}^L$.
- s_i is derived from previous sequents s_{i_1} and s_{i_2} by applying the cut rule. Thus $s_i = c_1 \cup c_2$, $s_{i_1} = c_1 \cup \{t : \psi\}$, and $s_{i_2} = c_2 \cup \{f : \psi\}$ for some formula ψ and sequents c_1 and c_2. If ψ does not have the form $\star\varphi$ (for $\star \in \mathcal{U}$ and formula φ), then $T(s_{i_1}) = T(c_1) \cup \{t : \psi\}$ and $T(s_{i_2}) = T(c_2) \cup \{f : \psi\}$. By applying a canonical cut in $\mathbf{G}^L$, we obtain a derivation of $T(s_i)$. Otherwise, $\psi = \star\varphi$ for some $\star \in \mathcal{U}$ and formula φ, then $T(s_{i_1}) = T(c_1) \cup \{t_\star : \psi\}$ and $T(s_{i_2}) = T(c_2) \cup \{f_\star : \psi\}$. Again, by applying a canonical cut in $\mathbf{G}^L$, we obtain a derivation of $T(s_i)$.
- $s_i = \{l : \star\sigma(p_1)\} \cup c_1 \cup \ldots \cup c_m$ is derived from previous sequents $s_{i_1} = \sigma(q_1) \cup c_1, \ldots, s_{i_m} = \sigma(q_m) \cup c_m$ by applying a quasi-canonical rule of $\mathbf{G}$ of the form $\{q_1, \ldots, q_m\}/\{l : \star p_1\}$ where $l \in \{t, f\}$ and $\star \in \mathcal{U}$. By Lemma A.1, for every $1 \leq k \leq m$, $T(\sigma(q_k) \cup c_k) \vdash_{\mathbf{G}^L} \sigma(T(q_k)) \cup T(c_k)$. By definition, each q_k consists only of the formulas p_1 or $\triangleright p_1$ for $\triangleright \in \mathcal{U}$, and so $\sigma(T(q_k)) = (lab[T(q_k)] : \sigma(p_1))$. Thus, we obtain that $\mathcal{T}(\mathcal{S}) \vdash_{\mathbf{G}^L} (lab[T(q_k)] : \sigma(p_1)) \cup T(c_k)$ for every $1 \leq k \leq m$. By applying the corresponding primitive rule of $\mathbf{G}^L$ $\{lab[T(q_1)], \ldots, lab[T(q_m)]\}/\{l_\star\}$, we obtain a derivation in $\mathbf{G}^L$ of the sequent $T(s_i) = \{l_\star : \sigma(p_1)\} \cup T(c_1) \cup \ldots \cup T(c_m)$.
- $s_i = \{l : \sigma(\diamond(p_1, \ldots, p_n))\} \cup c_1 \cup \ldots \cup c_m$ is derived from previous sequents $s_{i_1} = \sigma(q_1) \cup c_1, \ldots, s_{i_m} = \sigma(q_m) \cup c_m$ by applying a quasi-canonical rule of $\mathbf{G}$ for $\diamond \notin \mathcal{U}$ of the form $\{q_1, \ldots, q_m\}/\{l : \diamond(p_1, \ldots, p_n)\}$ where $l \in \{t, f\}$. By Lemma A.1, for every $1 \leq k \leq m$, $T(\sigma(q_k) \cup c_k) \vdash_{\mathbf{G}^L} \sigma(T(q_k)) \cup T(c_k)$. By applying the corresponding canonical introduction rule of $\mathbf{G}^L$ $T(\{q_1, \ldots, q_m\})/\{l : \diamond(p_1, \ldots, p_n)\}$, we obtain a derivation in $\mathbf{G}^L$ of $T(s_i)$.
- $s_i = \{l : \sigma(\star \diamond (p_1, \ldots, p_n))\} \cup c_1 \cup \ldots \cup c_m$ is derived from previous sequents $s_{i_1} = \sigma(q_1) \cup c_1, \ldots, s_{i_m} = \sigma(q_m) \cup c_m$ by applying a quasi-canonical rule of $\mathbf{G}$ for $\diamond$ of the form $\{q_1, \ldots, q_m\}/\{l : \star \diamond (p_1, \ldots, p_n)\}$ where $l \in \{t, f\}$ and $\star \in \mathcal{U}$. The proof is similar to the previous case, using the rule $T(\{q_1, \ldots, q_m\})/\{l_\star : \diamond(p_1, \ldots, p_n)\}$ of $\mathbf{G}^L$.

($\Leftarrow$) For the converse, we define a translation T^{-1}, mapping labelled formulas over $\mathcal{L}_{\mathcal{U}}$ to labelled formulas over $\{t, f\}$: $T^{-1}(l : \psi) = l : \psi$ for $l \in \{t, f\}$, and $T^{-1}(l_\star : \psi) = l : \star\psi$ for $l_\star \in \mathcal{L}_{\mathcal{U}} \setminus \{t, f\}$. T^{-1} is extended to sequents for $\mathcal{L}_{\mathcal{U}}$ and sets of sequents for $\mathcal{L}_{\mathcal{U}}$ as follows:

$$T^{-1}(s) = \{T^{-1}(l : \psi) \mid l : \psi \in s\} \qquad T^{-1}(\mathcal{S}) = \{T^{-1}(s) \mid s \in \mathcal{S}\}$$

Since $T^{-1}(T(s)) = s$ for every (two-sided) sequent s, it suffices to show that $\mathcal{S} \vdash_{\mathbf{G}^L} s$ implies $T^{-1}(\mathcal{S}) \vdash_{\mathbf{G}} T^{-1}(s)$ (for every set $\mathcal{S} \cup \{s\}$ of sequents over $\mathcal{L}_{\mathcal{U}}$). Suppose that $\mathcal{S} \vdash_{\mathbf{G}^L} s$. Hence there exists a sequence $P = s_1, \ldots, s_N$ such that $s_N = s$ and for every $1 \leq i \leq N$, s_i is either in $\mathcal{S}$, or is derived from previous sequents by applying a rule of $\mathbf{G}^L$. We prove that for every $1 \leq i \leq N$: if $T^{-1}(\mathcal{S}) \vdash_{\mathbf{G}} T^{-1}(s_j)$ for every $1 \leq j < i$ then $T^{-1}(\mathcal{S}) \vdash_{\mathbf{G}} T^{-1}(s_i)$ (it then follows that $T^{-1}(\mathcal{S}) \vdash_{\mathbf{G}} T^{-1}(s)$). Let $1 \leq i \leq N$, and suppose that $T^{-1}(\mathcal{S}) \vdash_{\mathbf{G}} T^{-1}(s_j)$ for every $1 \leq j < i$. Consider the possible cases:

- $s_i \in \mathcal{S}$. By definition, $T^{-1}(s_i) \in T^{-1}(\mathcal{S})$, and so $T^{-1}(\mathcal{S}) \vdash_{\mathbf{G}^L} T^{-1}(s_i)$.
- $s_i = s_{i_1} \cup s$ is derived from a previous sequent s_{i_1} by applying the weakening rule. In this case, $T^{-1}(s_i) = T^{-1}(s_{i_1}) \cup T^{-1}(s)$ can be also derived from a previous sequent $T^{-1}(s_{i_1})$ by applying weakening as well.

- $s_i = (L : \varphi) \cup c_1 \cup \ldots \cup c_m$ is derived from previous sequents $s_{i_1} = (L_1 : \varphi) \cup c_1, \ldots, (L_m : \varphi) \cup c_m$ by applying a primitive rule r^L of $\mathbf{G}^L$ of the form $\{L_1, \ldots, L_m\}/L$. By the construction of $\mathbf{G}^L$, one of the following holds:
  - r^L is the canonical axiom $\emptyset/\{t, f\}$. Thus $s_i = \{t : \psi, f : \psi\}$ for some formula ψ. In this case, $T^{-1}(s_i) = s_i$ is obtained by applying the identity axiom of $\mathbf{G}$.
  - r^L is a canonical axiom of the form $\emptyset/\{t_\star, f_\star\}$ for some $\star \in \mathcal{U}$. Thus $s_i = \{t_\star : \psi, f_\star : \psi\}$ for some formula ψ. $T^{-1}(s_i) = \{t : \star\psi, f : \star\psi\}$ is obtained by applying the identity axiom of $\mathbf{G}$.
  - r^L is the canonical cut $\{t\}, \{f\}/\emptyset$. Thus $s_i = c_1 \cup c_2$, $s_{i_1} = c_1 \cup \{t : \psi\}$, and $s_{i_2} = c_2 \cup \{f : \psi\}$ for some formula ψ and sequents c_1 and c_2. Here, $T^{-1}(s_{i_1}) = T^{-1}(c_1) \cup \{t : \psi\}$ and $T^{-1}(s_{i_2}) = T^{-1}(c_2) \cup \{f : \psi\}$. $T^{-1}(s_i) = T^{-1}(c_1) \cup T^{-1}(c_2)$ is derived by applying the cut rule of $\mathbf{G}$.
  - r^L is a canonical cut of the form $\{t_\star\}, \{f_\star\}/\emptyset$ for some $\star \in \mathcal{U}$. Thus $s_i = c_1 \cup c_2$, $s_{i_1} = c_1 \cup \{t_\star : \psi\}$, and $s_{i_2} = c_2 \cup \{f_\star : \psi\}$ for some formula ψ and sequents c_1 and c_2. Here, $T^{-1}(s_{i_1}) = T^{-1}(c_1) \cup \{t : \star\psi\}$ and $T^{-1}(s_{i_2}) = T^{-1}(c_2) \cup \{f : \star\psi\}$. $T^{-1}(s_i) = T^{-1}(c_1) \cup T^{-1}(c_2)$ is derived by applying the cut rule of $\mathbf{G}$.
  - $L = \{l_\star\}$ for some $l \in \{t, f\}$ and $\star \in \mathcal{U}$, and there exists a quasi-canonical rule r in $\mathbf{G}$ of the form $\{q_1, \ldots, q_m\}/\{l : \star p_1\}$ in $\mathbf{G}$ such that $L_k = lab[T(q_k)]$ for every $1 \leq k \leq m$. In this case, let σ be any substitution such that $\sigma(p_1) = \varphi$. It is easy to see that for every $1 \leq k \leq m$, $T^{-1}(lab[T(q_k)] : \varphi) = \sigma(q_k)$. Therefore, for every $1 \leq k \leq m$, $T^{-1}(s_{i_k}) = \sigma(q_k) \cup T^{-1}(c_k)$. By applying the rule r on these provable sequents, we obtain $T^{-1}(s_i) = \{l : \star\varphi\} \cup T^{-1}(c_1) \cup \ldots \cup T^{-1}(c_m)$.

- $s_i = (L : \sigma(\diamond(p_1, \ldots, p_n))) \cup c_1 \cup \ldots \cup c_m$ is derived from previous sequents $s_{i_1} = \sigma(s'_1) \cup c_1, \ldots, \sigma(s'_m) \cup c_m$ by applying a canonical introduction rule r^L of $\mathbf{G}^L$ of the form $\{s'_1, \ldots, s'_m\}/(L : \diamond(p_1, \ldots, p_n))$. By the construction of $\mathbf{G}^L$, one of the following holds:
  - $r^L = \{l_\star : p_1\}/\{l : \star p_1\}$ for $l \in \{t, f\}$ and $\star \in \mathcal{U}$. Thus $s_i = \{l : \star\sigma(p_1)\} \cup c_1$, $m = 1$, and $s_{i_1} = \{l_\star : \sigma(p_1)\} \cup c_1$. In this case we have $T^{-1}(s_i) = T^{-1}(s_{i_1})$, and obviously we are done.
  - $L = \{l\}$ where $l \in \{t, f\}$, $\diamond \notin \mathcal{U}$, and there exists a quasi-canonical rule r of the form $\{q_1, \ldots, q_m\}/\{l : \diamond(p_1, \ldots, p_n)\}$ in $\mathbf{G}$ such that $s'_k = T(q_k)$ for every $1 \leq k \leq m$. Here, $T^{-1}(s_i) = \{l : \sigma(\diamond(p_1, \ldots, p_n))\} \cup T^{-1}(c_1) \cup \ldots \cup T^{-1}(c_m)$. It is easy to see that for every $1 \leq k \leq m$, $T^{-1}(\sigma(s'_k)) = T^{-1}(\sigma(T(q_k))) = \sigma(q_k)$. Thus we have $T^{-1}(s_{i_k}) = \sigma(q_k) \cup T^{-1}(c_k)$ for every $1 \leq k \leq m$. By applying the rule r to these m provable sequents, we obtain $T^{-1}(s_i)$.
  - $L = \{l_\star\}$ where $l \in \{t, f\}$ and $\star \in \mathcal{U}$, and there exists a quasi-canonical rule r of the form $\{q_1, \ldots, q_m\}/\{l : \star \diamond (p_1, \ldots, p_n)\}$ in $\mathbf{G}$ such that $s'_k = T(q_k)$ for every $1 \leq k \leq m$. The proof is similar to the previous case. □