Non-deterministic Matrices for Semi-canonical Deduction Systems

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***Abstract*—We use non-deterministic finite-valued matrices to provide uniform effective semantics for a large family of logics, emerging from "well-behaved" sequent systems in which the cut rule and/or the identity-axiom are not present. We exploit this semantics to obtain important proof-theoretic properties of systems of this kind, such as cut-admissibility. Non-determinism is shown to be essential for these purposes, since the studied logics cannot be characterized by ordinary finite-valued matrices. Our results shed light on the dual semantic roles of the cut rule and the identity-axiom, showing that they are crucial for having deterministic (truth-functional) finite-valued semantics.**

I. Introduction

Non-deterministic multi-valued matrices (*Nmatrices*) (introduced in [2], surveyed in [3]) are a natural generalization of ordinary multi-valued matrices, in which the truth-value of a complex formula can be chosen non-deterministically out of some non-empty set of options. That is, the strict truth-functionality principle is relaxed, and non-deterministic semantics are used instead. Under these non-deterministic semantics, the meaning of a compound formula is restricted, rather than uniquely determined, by the meanings of its subformulas. Nmatrices have proved to be a powerful tool, the use of which preserves all the advantages of ordinary multi-valued matrices, but is applicable to a much wider range of logics. Indeed, there are many useful propositional logics, which have no finite-valued characteristic matrices, but do have finite-valued characteristic Nmatrices.

A particular important use of Nmatrices is to uniformly provide effective valuation semantics for various logics originally emerging from different proof systems. Indeed, it was shown in [2], that two-valued Nmatrices can be used to semantically characterize the family of *canonical systems*. The systems of this family are all "ideal" Gentzen-type sequent systems, which include the cut rule, the standard identity-axiom, the weakening rule, and each of their other rules is a "well-behaved" logical rule, where exactly one occurrence of a connective is introduced and no other connective is involved. The best-known system of this family is of-course Gentzen's $\mathbf{LK}$ for classical logic. Obviously, an ordinary matrix suffices for $\mathbf{LK}$. However, for other useful canonical systems (e.g. the system for primal logic in [6]), Nmatrices are necessary. In such systems, some sort of underspecification in the derivation rules can induce non truth-functional connectives, and the use of Nmatrices becomes inevitable.

In this paper, we provide a new application of Nmatrices in the same spirit. We investigate *semi-canonical* systems, which are sequent systems obtained from canonical systems by omitting the cut rule and/or the identity-axiom. We show that the semantics of logics induced by semi-canonical systems cannot be captured using ordinary finite-valued matrices. Importantly, unlike in canonical systems, the inability to capture these logics is not a result of having non truth-functional connectives. In fact, semantics induced by semi-canonical systems obtained from $\mathbf{LK}$ cannot be captured using ordinary finite-valued matrices as well. Next we prove that *three-valued* and *four-valued* Nmatrices do suffice for this matter. Moreover, these Nmatrices are algorithmically constructed from the derivation rules of a given semi-canonical system. In addition, we show that the Nmatrix semantics of semi-canonical systems is easily applicable to establish some properties of canonical systems and semi-canonical systems. This includes new general strong cut-admissibility for some families of semi-canonical systems without identity-axioms.

Semantics for sequent systems without cut or identity-axiom was studied in previous works. Following Schütte (see [8]), Girard (in [5]) studied the cut-free fragment of $\mathbf{LK}$, and provided semantics for this fragment using (non-deterministic) three-valued valuations.[1] Together with better understanding of the semantic role of the cut rule, this three-valued semantics was applied for proving several generalizations of the cut-elimination theorem (such as Takeuti's conjecture, see [5]). Later, the axiom-free fragment of $\mathbf{LK}$ was studied by Hösli and Jäger in [7]. As noted in [7], axiom-free systems play an important role in the proof-theoretic analysis of logic programming and in connection with the so called negation as failure. Hösli and Jäger provided a dual (non-deterministic) three-valued valuation semantics for axiom-free derivability in $\mathbf{LK}$. With respect to [5] and [7], the current work contributes in four main aspects:[2]

- Our results apply to the broad family of semi-canonical

[1]Note that cut-elimination implies that provability and cut-free provability coincide for $\mathbf{LK}$. However, it is well-known that cut-elimination fails in the presence of extra "non-logical" axioms (assumptions), and so the derivability relation induced by $\mathbf{LK}$ (which allows non-empty set of assumptions) is different from the one induced by its cut-free fragment.

[2]It should be mentioned, however, that [5] and [7] concerned also the usual quantifiers of $\mathbf{LK}$, while we only investigate *propositional* logics, leaving the more complicated first-order case (and beyond) to a future work.

systems, of which the cut-free and the axiom-free fragments of **LK** is just a particular example. For these systems, we obtain practically the same semantics that was suggested in [5] and [7].

- While the focus of [5] and [7] was on derivability between sequents, we also study the consequence relation between *formulas* induced by cut-free and axiom-free systems.
- We formulate the two kinds of three-valued valuation semantics inside the well-studied framework of (three-valued) Nmatrices, exploiting some known general properties of Nmatrices.
- In [7], it seems that the two dual kinds of three-valued valuation semantics cannot be combined. However, in this paper we show that a combination of them is obtained using *four-valued* Nmatrices. To the best of our knowledge, systems with neither cut nor identity-axiom were not studied before.

The structure of this paper is as follows: Section II provides the necessary background concerning Nmatrices and canonical systems. In Section III we define the family of semi-canonical systems and the logics induced by them. Our main results are given in Section IV, where we provide the general construction of Nmatrices for semi-canonical systems, and use it to prove some properties of these systems. Final remarks and further research topics are given in Section V.

II. Preliminaries

In what follows, $\mathcal{L}$ denotes an arbitrary propositional language, and $Frm_{\mathcal{L}}$ denotes its set of wffs. We assume that $p_1, p_2, \ldots$ are the atomic formulas of any propositional language. An $\mathcal{L}$-*substitution* is a function $\sigma : Frm_{\mathcal{L}} \to Frm_{\mathcal{L}}$, such that $\sigma(\diamond(\psi_1, \ldots, \psi_n)) = \diamond(\sigma(\psi_1), \ldots, \sigma(\psi_n))$ for every n-ary connective $\diamond$ of $\mathcal{L}$. Substitutions are extended to sets of formulas in the obvious way.

Definition 1: A *Tarskian consequence relation (tcr)* $\vdash$ for $\mathcal{L}$ is a binary relation between sets of $\mathcal{L}$-formulas and $\mathcal{L}$-formulas, satisfying the following conditions:

Reflexivity: if $\psi \in \mathcal{T}$ then $\mathcal{T} \vdash \psi$.
Monotonicity: if $\mathcal{T} \vdash \psi$ and $\mathcal{T} \subseteq \mathcal{T}'$, then $\mathcal{T}' \vdash \psi$.
Transitivity: if $\mathcal{T} \vdash \psi$ and $\mathcal{T}', \psi \vdash \varphi$, then $\mathcal{T}, \mathcal{T}' \vdash \varphi$.

Definition 2: A binary relation $\vdash$ between sets of $\mathcal{L}$-formulas and $\mathcal{L}$-formulas is *structural* if for every $\mathcal{L}$-substitution σ and every $\mathcal{T}$ and ψ, if $\mathcal{T} \vdash \psi$ then $\sigma(\mathcal{T}) \vdash \sigma(\psi)$. $\vdash$ is *finitary* if whenever $\mathcal{T} \vdash \psi$, there exists some finite $\Gamma \subseteq \mathcal{T}$, such that $\Gamma \vdash \psi$. $\vdash$ is *consistent* (or *non-trivial*) if there exist some non-empty $\mathcal{T}$ and some ψ such that $\mathcal{T} \not\vdash \psi$.

Definition 3: A (Tarskian propositional) logic is a pair $\langle \mathcal{L}, \vdash \rangle$, where $\mathcal{L}$ is a propositional language, and $\vdash$ is a structural, finitary, consistent tcr for $\mathcal{L}$.

A. Non-Deterministic Matrices

The most standard method for defining propositional logics is by using multi-valued (deterministic) matrices (see e.g. [9]). The following natural generalization was introduced in [2]:

Definition 4: A *non-deterministic matrix (Nmatrix)* $\mathbf{M}$ for $\mathcal{L}$ consists of: (i) a non-empty set $\mathcal{V}_{\mathbf{M}}$ of *truth-values*, (ii) a non-empty proper subset $\mathcal{D}_{\mathbf{M}} \subseteq \mathcal{V}_{\mathbf{M}}$ of *designated truth-values*, and (iii) a function $\diamond_{\mathbf{M}} : {\mathcal{V}_{\mathbf{M}}}^n \to P(\mathcal{V}_{\mathbf{M}}) \setminus \{\emptyset\}$ for every n-ary connective $\diamond$ of $\mathcal{L}$.

Ordinary (deterministic) *matrices* correspond to the case when each $\diamond_{\mathbf{M}}$ is a function taking singleton values only (then it can be treated as a function $\diamond_{\mathbf{M}} : {\mathcal{V}_{\mathbf{M}}}^n \to \mathcal{V}_{\mathbf{M}}$). An Nmatrix $\mathbf{M}$ is *finite-valued* if $\mathcal{V}_{\mathbf{M}}$ is finite. By an nNmatrix ($n \in \mathbb{N}$) we shall mean an Nmatrix for which $|\mathcal{V}_{\mathbf{M}}| = n$.

Definition 5: A *valuation* in an Nmatrix $\mathbf{M}$ (for $\mathcal{L}$) is a function v from $Frm_{\mathcal{L}}$ to $\mathcal{V}_{\mathbf{M}}$, such that for every compound formula $\diamond(\psi_1, \ldots, \psi_n) \in \mathcal{L}$, $v(\diamond(\psi_1, \ldots, \psi_n)) \in \diamond_{\mathbf{M}}(v(\psi_1), \ldots, v(\psi_n))$. v in $\mathbf{M}$ is a *model* of a *formula* ψ if $v(\psi) \in \mathcal{D}_{\mathbf{M}}$. v is a *model* of a *set* $\mathcal{T}$ of formulas if v is a model of every $\psi \in \mathcal{T}$. In addition, $\vdash_{\mathbf{M}}$, *the tcr induced by* $\mathbf{M}$, is defined by: $\mathcal{T} \vdash_{\mathbf{M}} \psi$, if every valuation v in $\mathbf{M}$ which is a model of $\mathcal{T}$ is also a model of ψ.

Following [2], we have that $\langle \mathcal{L}, \vdash_{\mathbf{M}} \rangle$ is a logic for every finite-valued Nmatrix $\mathbf{M}$ for $\mathcal{L}$.

In general, in order for a denotational semantics of a propositional logic to be useful and effective, it should be *analytic*. This means that to determine whether a formula φ follows from a theory $\mathcal{T}$, it suffices to consider *partial* valuations, defined on the set of all subformulas of the formulas in $\mathcal{T} \cup \{\varphi\}$. The semantics of Nmatrices is *analytic* in this sense:

Definition 6: A *partial valuation* in an Nmatrix $\mathbf{M}$ (for $\mathcal{L}$) is a function v from some subset $\mathcal{E}$ of $Frm_{\mathcal{L}}$, which is closed under subformulas, to $\mathcal{V}_{\mathbf{M}}$, such that for every compound formula $\diamond(\psi_1, \ldots, \psi_n) \in \mathcal{E}$, $v(\diamond(\psi_1, \ldots, \psi_n)) \in \diamond_{\mathbf{M}}(v(\psi_1), \ldots, v(\psi_n))$. The notion of a *model* is defined for partial valuations exactly like it is defined for valuations.

Proposition 1 ([3]): Every partial valuation in some Nmatrix $\mathbf{M}$ can be extended to a (full) valuation in $\mathbf{M}$.

Corollary 1: Let $\mathbf{M}$ be an Nmatrix for $\mathcal{L}$, and $\mathcal{T} \cup \{\psi\}$ be a set of $\mathcal{L}$-formulas. Denote by $\mathcal{E}$ the set of subformulas of $\mathcal{T} \cup \{\psi\}$. Then, $\mathcal{T} \vdash_{\mathbf{M}} \psi$ iff every partial valuation v in $\mathbf{M}$, defined on $\mathcal{E}$, which is a model of $\mathcal{T}$ is also a model of ψ.

As a result of the last corollary, we obtain that logics characterized by finite-valued Nmatrices are decidable (see Theorem 28 in [3]).

B. Canonical Systems

The proof-theoretical common way to define logics is based on the notion of a proof in some formal deduction system. In this paper we study *sequent systems*. A sequent system (for $\mathcal{L}$) is an axiomatic system that manipulates higher-level constructs, called *sequents*, rather than the formulas themselves. There are several variants of what exactly constitutes a sequent. Here it is convenient to take it to be an expression of the form $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite *sets* of formulas, and $\Rightarrow$ is a new symbol, not occurring in $\mathcal{L}$. Each sequent system induces a derivability relation between sets of sequents and sequents. We shall write $\mathcal{S} \vdash^{seq}_{\mathbf{G}} s$ if the sequent s is derivable in a sequent system $\mathbf{G}$ from the set of sequents $\mathcal{S}$. The sequents of $\mathcal{S}$ are called *assumptions*. If $\vdash^{seq}_{\mathbf{G}} s$ (or formally, $\emptyset \vdash^{seq}_{\mathbf{G}} s$), we say that s is *provable* in $\mathbf{G}$. There are two natural ways to obtain a relation between sets of formulas and formulas from a given sequent system (see [1]):

Definition 7: Let $\mathbf{G}$ be a sequent system.

1) $\mathcal{T} \vdash^{v}_{\mathbf{G}} \varphi$ if $\{\Rightarrow \psi \mid \psi \in \mathcal{T}\} \vdash^{seq}_{\mathbf{G}} \Rightarrow \varphi$.
2) $\mathcal{T} \vdash^{t}_{\mathbf{G}} \varphi$ if $\vdash^{seq}_{\mathbf{G}} \Gamma \Rightarrow \varphi$ for some finite $\Gamma \subseteq \mathcal{T}$.

For many natural sequent systems, we have $\vdash^{v}_{\mathbf{G}} = \vdash^{t}_{\mathbf{G}}$ (see [1]). However, this usually does not hold for the cut-free and axiom-free systems investigated in this paper, where the difference between $\vdash^{v}_{\mathbf{G}}$ and $\vdash^{t}_{\mathbf{G}}$ becomes crucial. While it is evident that $\vdash^{v}_{\mathbf{G}}$ is a tcr for every sequent system $\mathbf{G}$ (without any assumptions on the derivation rules of $\mathbf{G}$), the status of $\vdash^{t}_{\mathbf{G}}$ is different. Obviously, $\vdash^{t}_{\mathbf{G}}$ is always monotone. Its reflexivity and transitivity, however, are not guaranteed. For example, if $\mathbf{G}$ does not include any axiomatic derivation rule (a rule with an empty set of premises) then clearly $p_1 \not\vdash^{t}_{\mathbf{G}} p_1$, and $\vdash^{t}_{\mathbf{G}}$ is not reflexive. Thus we will focus on the relation $\vdash^{v}_{\mathbf{G}}$, leaving for a future work the task of identifying the exact conditions under which $\vdash^{t}_{\mathbf{G}}$ is a tcr, and providing a similar study of it.

Henceforth, we shall write $\vdash_{\mathbf{G}}$ instead of $\vdash^{v}_{\mathbf{G}}$.

The general framework of sequent systems is perhaps too broad to obtain any interesting general results. Thus we study narrower families of sequent systems with "well-behaved" derivation rules. In fact, we consider three variants of *canonical systems*, a family of sequent systems that was introduced in [2]. The rest of this subsection is devoted to review relevant definitions and results concerning canonical systems.

The derivation rules of canonical systems are divided to *structural* rules and *logical* rules. The structural rules are fixed, and they consist of the following three rules:

- The *cut rule* (CUT): allows to infer sequents of the form $\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2$ from $\Gamma_1 \Rightarrow \varphi, \Delta_1$ and $\Gamma_2, \varphi \Rightarrow \Delta_2$ (φ is called the *cut-formula*).
- The *identity-axiom* (ID): allows to infer sequents of the form $\varphi \Rightarrow \varphi$ (without any premises).
- The *weakening rule* (Weak): allows to infer sequents of the form $\Gamma, \Gamma' \Rightarrow \Delta, \Delta'$ from $\Gamma \Rightarrow \Delta$.

On the other hand, the logical rules of canonical systems are not pre-determined. All of them should be "well-behaved" rules, as precisely defined below:

Definition 8: An *n-clause* is a sequent consisting only of formulas from $\{p_1, \dots, p_n\}$.

Definition 9: A *canonical rule* for an n-ary connective $\diamond$ (of $\mathcal{L}$) is an expression of the form $\mathcal{S}/C$, where $\mathcal{S}$ is a finite set of n-clauses (called *premises*), and C is either $\Rightarrow \diamond(p_1, \dots, p_n)$ or $\diamond(p_1, \dots, p_n) \Rightarrow$. When C is $\Rightarrow \diamond(p_1, \dots, p_n)$ the rule is called a *right* rule, and when C is $\diamond(p_1, \dots, p_n) \Rightarrow$ the rule is called a *left* rule. An *application* of a canonical rule $\{\Pi_1 \Rightarrow \Sigma_1, \dots, \Pi_m \Rightarrow \Sigma_m\} / \Rightarrow \diamond(p_1, \dots, p_n)$ is any inference step inferring a sequent of the form $\Gamma_1, \dots, \Gamma_m \Rightarrow \sigma(\diamond(p_1, \dots, p_n)), \Delta_1, \dots, \Delta_m$ from $\Gamma_i, \sigma(\Pi_i) \Rightarrow \sigma(\Sigma_i), \Delta_i$ for every $1 \leq i \leq m$, where σ is an $\mathcal{L}$-substitution. Applications of left rules are defined similarly.

Example 1: The usual derivation rules for the classical connectives can all be presented as canonical rules. For example, the usual rules for $\supset$, $\wedge$ and $\neg$ are the following:

$\{\Rightarrow p_1; p_2 \Rightarrow\}/p_1 \supset p_2 \Rightarrow$ $\qquad$ $\{p_1 \Rightarrow p_2\}/\Rightarrow p_1 \supset p_2$
$\{p_1, p_2 \Rightarrow\}/p_1 \wedge p_2 \Rightarrow$ $\qquad$ $\{\Rightarrow p_1; \Rightarrow p_2\}/\Rightarrow p_1 \wedge p_2$
$\{\Rightarrow p_1\}/\neg p_1 \Rightarrow$ $\qquad$ $\{p_1 \Rightarrow\}/\Rightarrow \neg p_1$

Applications of the rules for $\supset$ have the form:

$$\frac{\Gamma_1 \Rightarrow \psi, \Delta_1 \qquad \Gamma_2, \varphi \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2, \psi \supset \varphi \Rightarrow \Delta_1, \Delta_2} \qquad \frac{\Gamma, \psi \Rightarrow \varphi, \Delta}{\Gamma \Rightarrow \psi \supset \varphi, \Delta}$$

Example 2: An alternative implication connective (denoted here by $\leadsto$), was studied in [6], and shown there to be useful for access-control applications. It is defined by the two canonical rules: $\{\Rightarrow p_1; p_2 \Rightarrow\}/p_1 \leadsto p_2 \Rightarrow$ and $\{\Rightarrow p_2\}/\Rightarrow p_1 \leadsto p_2$ Applications of the left rule have the same form as application of the left rule for $\supset$. On the other hand, applications of the right rule allow to infer $\Gamma \Rightarrow \psi \leadsto \varphi, \Delta$ from $\Gamma \Rightarrow \varphi, \Delta$.

A *canonical system* is in turn a sequent system that includes (CUT), (ID), and (Weak), and each of its other rules is a canonical rule. Clearly, the propositional fragment of Gentzen's $\mathbf{LK}$ can be presented as a canonical system. This is the most important example of a canonical system, but many more canonical systems can be introduced with various new connectives. Not all combinations of canonical rules, however, are meaningful. A natural demand is that the premises of a right rule and a left rule for the same connective are contradictory. This is captured by the *coherence* criterion:

Definition 10: A canonical system $\mathbf{G}$ is called *coherent* if $\mathcal{S}_1 \cup \mathcal{S}_2$ is classically unsatisfiable whenever $\mathbf{G}$ includes two rules of the form $\mathcal{S}_1/\Rightarrow \diamond(p_1, \dots, p_n)$ and $\mathcal{S}_2/\diamond(p_1, \dots, p_n) \Rightarrow$, for some connective $\diamond$.

Finally, the following result (see [2]) shows that coherence is a minimal demand from every canonical system:

Theorem 1: A canonical system $\mathbf{G}$ for $\mathcal{L}$ is coherent iff $\langle \mathcal{L}, \vdash_{\mathbf{G}} \rangle$ is a logic.

III. Semi-Canonical Sequent Systems

In this paper, we study *semi-canonical* sequent systems, defined as follows:

Definition 11: A *semi-canonical* system is a system obtained from a canonical system by omitting (CUT) and/or (ID).

Notation: We shall refer to sequent systems that include (CUT) as (+C)-systems. Similarly, (+A)-systems are sequent systems that include (ID). We also use (−C), (−A), and their combinations (for example, (−C−A)-systems include neither (CUT) nor (ID)). Given a (+C)-system $\mathbf{G}$, we denote by $\mathbf{G}$−C the system obtained from $\mathbf{G}$ by omitting (CUT). Similarly, $\mathbf{G}$−A stands for omitting (ID), and $\mathbf{G}$−C−A stands for omitting both (CUT) and (ID).

Remark 1: Semi-canonical (−A)-systems may look strange at first sight. Indeed, (−A)-systems, in which each canonical rule has a non-empty set of premises, have no provable sequents ($\not\vdash^{seq} s$ for every sequent s). However, the interest in such systems arises when we consider derivations from non-empty sets of assumptions.

It is easy to verify that $\vdash_{\mathbf{G}}$ is structural and finitary for every semi-canonical system $\mathbf{G}$. Semi-canonical (−C)-systems trivially have the global subformula property (i.e. if $\mathcal{S} \vdash^{seq}_{\mathbf{G}} s$, then there exists a derivation of s from $\mathcal{S}$ that contains only subformulas of formulas occurring in $\mathcal{S}$ and s), and so clearly $p_1 \not\vdash_{\mathbf{G}} p_2$. Thus for every semi-canonical (−C)-system $\mathbf{G}$ for $\mathcal{L}$, $\vdash_{\mathbf{G}}$ is consistent, and $\langle \mathcal{L}, \vdash_{\mathbf{G}} \rangle$ is a logic. The task of the next section will be to provide an adequate semantics for logics of this sort. The next proposition shows that finite-valued matrices do not suffice for this purpose:

Theorem 2: Suppose that $\mathcal{L}$ contains an unary connective denoted by $\neg$. Let $\mathbf{G}$ be a semi-canonical (−C)-system, whose rules for $\neg$ are the usual rules. There is no finite-valued (ordinary) matrix $\mathbf{M}$ such that $\vdash_{\mathbf{G}} = \vdash_{\mathbf{M}}$.

Proof Outline: Let $\mathbf{M}$ be an (ordinary) matrix, such that $\vdash_{\mathbf{G}} = \vdash_{\mathbf{M}}$. Note that for every $n \geq 0$, $\{\neg^i p_1 \mid i > n\} \not\vdash_{\mathbf{G}} \neg^n p_1$ (this can be easily verified using the Nmatrix semantics of the next section). Consequently, $\{\neg^i p_1 \mid i > n\} \not\vdash_{\mathbf{M}} \neg^n p_1$. For every $n \geq 0$, let v_n be a valuation in $\mathbf{M}$, which is a model of $\{\neg^i p_1 \mid i > n\}$, and not a model of $\neg^n p_1$. Next, one shows that $v_n(p_1) \neq v_m(p_1)$ for every $n > m \geq 0$. It follows that $\mathbf{M}$ has infinite number of truth-values. ■

On the other hand, in semi-canonical (+C−A)-systems the consistency of $\vdash_{\mathbf{G}}$ is not guaranteed. The coherence criterion (defined exactly as for canonical systems) is naturally required here also. However, there are non-coherent semi-canonical (+C−A)-systems that induce logics. These systems can be excluded by the next proposition:

Proposition 2: Let $\mathbf{G}$ be a semi-canonical (−A)-system, in which every left rule has at least one premise of the form $\Pi \Rightarrow$. Let $\mathbf{G}'$ be the semi-canonical (−A)-system obtained from $\mathbf{G}$ by omitting all left rules. Then, $\vdash_{\mathbf{G}'} = \vdash_{\mathbf{G}}$.

Assumption: Henceforth, we assume that every semi-canonical (−A)-system either has no left rules, or has at least one left rule with no premises of the form $\Pi \Rightarrow$.

With this assumption, we can have the following:

Proposition 3: Let $\mathbf{G}$ be a semi-canonical (+C−A)-system. $\vdash_{\mathbf{G}}$ is consistent iff $\mathbf{G}$ is coherent.

Proof Outline: Obviously $\vdash_{\mathbf{G}} \subseteq \vdash_{\mathbf{G}+\mathtt{A}}$ ($\mathbf{G}$+A is the canonical system obtained from $\mathbf{G}$ by adding (ID)). The consistency of $\vdash_{\mathbf{G}}$ then follows from Theorem 1. Based on the assumption above, the proof of the converse is an adaptation of the corresponding proof concerning canonical systems in [2]. ■

Thus, we now have that for every semi-canonical (+C−A)-system $\mathbf{G}$ for $\mathcal{L}$, $\mathbf{G}$ is coherent iff $\langle \mathcal{L}, \vdash_{\mathbf{G}} \rangle$ is a logic. Nmatrix semantics for logics induced by coherent semi-canonical (+C−A)-systems will be developed in the next section. Again, it is possible to show that even in simple cases there does not exist an adequate finite-valued matrix:

Theorem 3: Suppose that $\mathcal{L}$ contains a unary connective denoted by $\neg$. Let $\mathbf{G}$ be a coherent semi-canonical (−A)-system, whose rules for $\neg$ are the usual rules. There is no finite-valued (ordinary) matrix $\mathbf{M}$ such that $\vdash_{\mathbf{G}} = \vdash_{\mathbf{M}}$.

IV. Non-deterministic Matrices for Semi-canonical Systems

In this section, which is the heart of this paper, we construct Nmatrices for all logics induced by semi-canonical systems. These Nmatrices are based on the four truth-values $t, f, \top$, and $\bot$, used in Dunn-Belnap matrix (see [4]). We also use Dunn-Belnap "knowledge" partial order on $\{t, f, \top, \bot\}$, denoted here by $\leq$. According to $\leq$, $\bot$ is the minimal element, $\top$ – the maximal one, and t, f are intermediate incomparable values ($\leq$ is the transitive reflexive closure of $\{\langle \bot, t \rangle, \langle \bot, f \rangle, \langle t, \top \rangle, \langle f, \top \rangle\}$). In addition, the following definitions and propositions are used:

Definition 12: Let $\mathbf{M}$ be an Nmatrix, such that $\mathcal{V}_{\mathbf{M}} \subseteq \{t, f, \top, \bot\}$. A valuation v in $\mathbf{M}$ is a *model of a sequent* $\Gamma \Rightarrow \Delta$ if either $v(\psi) \geq f$ for some $\psi \in \Gamma$, or $v(\psi) \geq t$ for some $\psi \in \Delta$. v is a model of a set $\mathcal{S}$ of sequents if it is a model of every $s \in \mathcal{S}$. In addition, $\vdash^{seq}_{\mathbf{M}}$, *the relation induced by* $\mathbf{M}$ *between sets of sequents and sequents*, is defined as follows: $\mathcal{S} \vdash^{seq}_{\mathbf{M}} s$, if every valuation v in $\mathbf{M}$ which is a model of $\mathcal{S}$ is also a model of s.

Proposition 4: Let $\mathbf{M}$ be an Nmatrix, such that $\mathcal{V}_{\mathbf{M}} \subseteq \{t, f, \top, \bot\}$, and $\mathcal{D}_{\mathbf{M}} = \mathcal{V}_{\mathbf{M}} \cap \{t, \top\}$. Let $\mathbf{G}$ be a sequent system, such that $\vdash^{seq}_{\mathbf{M}} = \vdash^{seq}_{\mathbf{G}}$. Then, $\vdash_{\mathbf{M}} = \vdash_{\mathbf{G}}$.

Definition 13: Let $x_1, ..., x_n \in \{t, f, \top, \bot\}$. $\langle x_1, \ldots, x_n \rangle$ *satisfies* an n-clause $\Pi \Rightarrow \Sigma$ if there exists either some $p_i \in \Pi$ such that $x_i \geq f$, or some $p_i \in \Sigma$ such that $x_i \geq t$. $\langle x_1, \ldots, x_n \rangle$ *fulfils* a canonical rule r for an n-ary connective, if it satisfies every premise of r.

The structure of the rest of this section is as follows. We begin with the known construction of 2Nmatrices for coherent canonical systems. Next, we separately investigate (−C+A)-systems, coherent (+C−A)-systems, and (−C−A)-systems.

A. Canonical Systems

A general construction of 2Nmatrices to characterize logics induced by coherent canonical systems was given in [2]. For the sake of completeness, we review this construction.[3]

Definition 14: Let $\mathbf{G}$ be a coherent canonical system for $\mathcal{L}$. $\mathbf{M_G}$ is the 2Nmatrix for $\mathcal{L}$ defined by $\mathcal{V}_{\mathbf{M_G}} = \{t, f\}$, $\mathcal{D}_{\mathbf{M_G}} = \{t\}$, and for every n-ary connective $\diamond$ of $\mathcal{L}$, $\diamond_{\mathbf{M_G}}(x_1, \ldots, x_n) = \{t\}$ iff $\langle x_1, \ldots, x_n \rangle$ fulfils some right rule of $\mathbf{G}$ for $\diamond$, $\diamond_{\mathbf{M_G}}(x_1, \ldots, x_n) = \{f\}$ iff $\langle x_1, \ldots, x_n \rangle$ fulfils some left rule of $\mathbf{G}$ for $\diamond$, and otherwise $\diamond_{\mathbf{M_G}}(x_1, \ldots, x_n) = \{t, f\}$.

The following proposition ensures that $\mathbf{M_G}$ is well-defined.

Proposition 5: Let $\mathbf{G}$ be a coherent semi-canonical system, and let $x_1, ..., x_n \in \{t, f, \bot\}$. $\langle x_1, ..., x_n \rangle$ cannot fulfil *both* a right rule and a left rule of $\mathbf{G}$ for some n-ary connective $\diamond$.

Example 3: Suppose that the rules for $\supset$ and $\leadsto$ in some coherent canonical system $\mathbf{G}$ are those given in Examples 1 and 2. The following truth-tables represent $\supset_{\mathbf{M_G}}$ and $\leadsto_{\mathbf{M_G}}$:

[3] While this is not the exact formulation used in [2], it is easy to see that the same Nmatrix is obtained.

$\supset_{\mathbf{M_G}}$	t	f
t	$\{t\}$	$\{f\}$
f	$\{t\}$	$\{t\}$

$\leadsto_{\mathbf{M_G}}$	t	f
t	$\{t\}$	$\{f\}$
f	$\{t\}$	$\{t,f\}$

Theorem 4 ([2]): For every coherent canonical system $\mathbf{G}$, $\vdash^{seq}_{\mathbf{G}}=\vdash^{seq}_{\mathbf{M_G}}$ and $\vdash_{\mathbf{M_G}}=\vdash_{\mathbf{G}}$.

B. *(−C+A)-Systems*

Semantics for semi-canonical (−C+A)-systems is given in the form of 3Nmatrices, using the truth-values t, f and $\top$:

Definition 15: Let $\mathbf{G}$ be a semi-canonical (−C+A)-system for $\mathcal{L}$. $\mathbf{M_G}$ is the 3Nmatrix defined by $\mathcal{V}_{\mathbf{M_G}} = \{t, f, \top\}$, $\mathcal{D}_{\mathbf{M_G}} = \{t, \top\}$, and for every n-ary connective $\diamond$ of $\mathcal{L}$, $\diamond_{\mathbf{M_G}}(x_1, \ldots, x_n) = \{t, \top\}$ iff $\langle x_1, \ldots, x_n\rangle$ fulfils some right rule of $\mathbf{G}$ for $\diamond$ and does not fulfil any left rule of $\mathbf{G}$ for $\diamond$, $\diamond_{\mathbf{M_G}}(x_1, \ldots, x_n) = \{f, \top\}$ iff $\langle x_1, \ldots, x_n\rangle$ fulfils some left rule of $\mathbf{G}$ for $\diamond$ and does not fulfil any right rule of $\mathbf{G}$ for $\diamond$, $\diamond_{\mathbf{M_G}}(x_1, \ldots, x_n) = \{\top\}$ iff $\langle x_1, \ldots, x_n\rangle$ fulfils both some left rule and some right rule of $\mathbf{G}$ for $\diamond$, and otherwise $\diamond_{\mathbf{M_G}}(x_1, \ldots, x_n) = \{t, f, \top\}$.

Example 4: Suppose that the rules for $\supset$ and $\wedge$ in some semi-canonical (−C+A)-system $\mathbf{G}$ are the usual rules. The following truth-tables represent $\supset_{\mathbf{M_G}}$ and $\wedge_{\mathbf{M_G}}$:

$\supset_{\mathbf{M_G}}$	t	f	$\top$
t	$\{t,\top\}$	$\{f,\top\}$	$\{\top\}$
f	$\{t,\top\}$	$\{t,\top\}$	$\{t,\top\}$
$\top$	$\{t,\top\}$	$\{\top\}$	$\{\top\}$

$\wedge_{\mathbf{M_G}}$	t	f	$\top$
t	$\{t,\top\}$	$\{f,\top\}$	$\{\top\}$
f	$\{f,\top\}$	$\{f,\top\}$	$\{f,\top\}$
$\top$	$\{\top\}$	$\{f,\top\}$	$\{\top\}$

More intuition about this construction can be gained from the field of logical circuits. Consider a noisy circuit in which the logical gates (or the wires themselves) might produce borderline output values ($\top$ represents a borderline value). Having this in mind, the gates are designed to recognize borderline inputs, calculate the output using this information, and in some cases intentionally produce a borderline output. An AND gate of this type is described by the right table above. For example, when both inputs are t the output should be t, but noise might make it $\top$. On the other hand, when one of the inputs is t and the other is $\top$, the output cannot be calculated, so a borderline output is produced by the gate.

Theorem 5: For every semi-canonical (−C+A)-system $\mathbf{G}$, $\vdash^{seq}_{\mathbf{G}}=\vdash^{seq}_{\mathbf{M_G}}$ and $\vdash_{\mathbf{M_G}}=\vdash_{\mathbf{G}}$.

Proof Outline: It suffices to prove that $\vdash^{seq}_{\mathbf{G}}=\vdash^{seq}_{\mathbf{M_G}}$ ($\vdash_{\mathbf{M_G}}=\vdash_{\mathbf{G}}$ then follows from Proposition 4). Soundness is proved by usual induction on the length of derivations in $\mathbf{G}$ (note that every valuation over $\{t, f, \top\}$ is a model of every sequent derived by (ID)). Now, suppose that $\mathcal{S} \not\vdash^{seq}_{\mathbf{G}} \Gamma_0 \Rightarrow \Delta_0$. It is a routine matter to obtain *maximal* sets of formulas $\mathcal{T},\mathcal{U}$ such that $\Gamma_0 \subseteq \mathcal{T}$ and $\Delta_0 \subseteq \mathcal{U}$, and $\mathcal{S} \not\vdash^{seq}_{\mathbf{G}} \Gamma \Rightarrow \Delta$ for every finite $\Gamma \subseteq \mathcal{T}$ and $\Delta \subseteq \mathcal{U}$. Let $v : Frm_{\mathcal{L}} \to \{t, f, \top\}$ defined by $v(\psi) = t$ iff $\psi \in \mathcal{T}$, $v(\psi) = f$ iff $\psi \in \mathcal{U}$, and $v(\psi) = \top$ otherwise. The availability of (ID) ensures that v is well-defined. Next, one shows that v is a valuation in $\mathbf{M_G}$, which is a model of $\mathcal{S}$ but not of $\Gamma_0 \Rightarrow \Delta_0$. ■

Next, we use the previous results to provide a new proof of the fact that (CUT) is admissible in every coherent canonical system (originally proved in [2]).

Theorem 6: Let $\mathbf{G}$ be a coherent canonical system, and let s be a sequent. If every valuation in $\mathbf{M_G}$ is a model of s, then so does every valuation in $\mathbf{M_{G\text{-}C}}$.

Proof Outline: Suppose that there exists a valuation v' in $\mathbf{M_{G\text{-}C}}$ which is not a model of s. We recursively construct a valuation v in $\mathbf{M_G}$, such that $v(\psi) \le v'(\psi)$ for every $\psi \in Frm_{\mathcal{L}}$. For atomic formulas, we (arbitrarily) choose $v(p)$ to be either t or f, so that $v(p) \le v'(p)$ would hold. Now, let $\diamond$ be an n-ary connective of $\mathcal{L}$, and suppose $v(\psi_i)$ was defined for every $1 \le i \le n$. We choose $v(\diamond(\psi_1, \ldots, \psi_n))$ to be equal to $v'(\diamond(\psi_1, \ldots, \psi_n))$, if the latter is either t or f. Otherwise, we choose $v(\diamond(\psi_1, \ldots, \psi_n))$ to be some element of $\diamond_{\mathbf{M_G}}(v(\psi_1), \ldots, v(\psi_n))$. Using the definitions of $\mathbf{M_G}$ and of $\mathbf{M_{G\text{-}C}}$, it is possible to prove that v is indeed a valuation in $\mathbf{M_G}$. Clearly, v is not be a model of s. ■

The following is an easy corollary of Theorems 4, 5 and 6:

Corollary 2: Every coherent canonical system enjoys cut-admissibility, i.e. $\vdash^{seq}_{\mathbf{G}} s$ implies $\vdash^{seq}_{\mathbf{G\text{-}C}} s$.

C. *(+C−A)-Systems*

Semantics for coherent semi-canonical (+C−A)-systems is given in the form of 3Nmatrices:

Definition 16: Let $\mathbf{G}$ be a coherent semi-canonical (+C−A)-system for $\mathcal{L}$. $\mathbf{M_G}$ is the 3Nmatrix obtained from $\mathbf{M_{G+A}}$ (see Definition 14) by adding the truth-value $\perp$ to $\mathcal{V}_{\mathbf{M_G}}$, and replacing $\{t, f\}$ by $\{t, f, \perp\}$ in all truth-tables.

Example 5: Suppose that the rules for $\supset$ and $\wedge$ in some coherent semi-canonical (+C−A)-system $\mathbf{G}$ are the usual rules. The following truth-tables represent $\supset_{\mathbf{M_G}}$ and $\wedge_{\mathbf{M_G}}$ ($\mathcal{V}$ stands for $\{t, f, \perp\}$):

$\supset_{\mathbf{M_G}}$	t	f	$\perp$
t	$\{t\}$	$\{f\}$	$\mathcal{V}$
f	$\{t\}$	$\{t\}$	$\{t\}$
$\perp$	$\{t\}$	$\mathcal{V}$	$\mathcal{V}$

$\wedge_{\mathbf{M_G}}$	t	f	$\perp$
t	$\{t\}$	$\{f\}$	$\mathcal{V}$
f	$\{f\}$	$\{f\}$	$\{f\}$
$\perp$	$\mathcal{V}$	$\{f\}$	$\mathcal{V}$

Again, the truth-tables above can be interpreted as specifications of logical gates (compare with Example 4). Consider a circuit with noisy inputs, that can either be t, f or $\perp$ ($\perp$ represents a noisy input). The logical gates cannot always recognize the noisy inputs. In these cases they treat them either as t or as f. An AND gate of this type is described by the right table above. For example, when one input is f and the other is noisy, the output is f regardless of the value that was recognized for the noisy input. On the other hand, when both inputs are noisy the output can be t (if both inputs are recognized as t), f (if at least one input is recognized as f) or $\perp$ (otherwise, i.e. when one of the outputs is recognized as noisy, and the other one is not recognized as f).

Theorem 7: For every coherent semi-canonical (+C−A)-system $\mathbf{G}$, $\vdash^{seq}_{\mathbf{G}}=\vdash^{seq}_{\mathbf{M_G}}$, and $\vdash_{\mathbf{M_G}}=\vdash_{\mathbf{G}}$.

Proof Outline: It suffices to prove that $\vdash^{seq}_{\mathbf{G}}=\vdash^{seq}_{\mathbf{M_G}}$ ($\vdash_{\mathbf{M_G}}=\vdash_{\mathbf{G}}$ then follows from Proposition 4). Soundness is left to the reader. For completeness, suppose that $\mathcal{S} \not\vdash^{seq}_{\mathbf{G}} \Gamma_0 \Rightarrow \Delta_0$.

It is a routine matter to obtain sets of formulas $\mathcal{T},\mathcal{U}$, as in the proof of Theorem 5. We construct a valuation v in $\mathbf{M_G}$ as follows. For atomic formulas $v(p) = \bot$ iff $p \in \mathcal{T} \cap \mathcal{U}$, $v(p) = t$ iff $p \in \mathcal{T}$ and $p \notin \mathcal{U}$, and otherwise $v(p) = f$. Now let $\diamond$ be an n-ary connective of $\mathcal{L}$, and suppose that $v(\psi_1), \ldots, v(\psi_n)$ were defined. $v(\diamond(\psi_1, \ldots, \psi_n)) = x$ if $\diamond_{\mathbf{M_G}}(v(\psi_1), \ldots, v(\psi_n)) = \{x\}$ (for $x \in \{t, f\}$), and otherwise $v(\diamond(\psi_1, \ldots, \psi_n))$ is defined like $v(p)$ is defined above. Now, it is possible to prove the following properties: (a) if $\varphi \in \mathcal{T}$ then $v(\varphi) \neq f$; and (b) if $\varphi \in \mathcal{U}$ then $v(\varphi) \neq t$. Using these properties, we show that v is a model of $\mathcal{S}$, but not of $\Gamma_0 \Rightarrow \Delta_0$. For the latter, note that since $\Gamma_0 \subseteq \mathcal{T}$ and $\Delta_0 \subseteq \mathcal{U}$, (a) and (b) imply that v is not a model of $\Gamma_0 \Rightarrow \Delta_0$. Now let $\Gamma \Rightarrow \Delta$ be a sequent of $\mathcal{S}$. If $\Gamma \Rightarrow \Delta$ is a sequent of the form $\varphi \Rightarrow \varphi$, then $\varphi \notin \mathcal{T} \cap \mathcal{U}$, and so $v(\varphi) \neq \bot$. It follows that v satisfies $\Gamma \Rightarrow \Delta$. Otherwise, the availability of (CUT) entails that for every $\psi \in \Gamma \cup \Delta$ either $\psi \in \mathcal{T}$ or $\psi \in \mathcal{U}$. Since $\mathcal{S}_0 \vdash^{seq}_{\mathbf{G}} \Gamma \Rightarrow \Delta$, there exists either some $\psi \in \Gamma$ such that $\psi \notin \mathcal{T}$ and $\psi \in \mathcal{U}$, or some $\psi \in \Delta$ such that $\psi \in \mathcal{T}$ and $\psi \notin \mathcal{U}$. From (a) and (b) (and since $v(\psi) = \bot$ only if $\psi \in \mathcal{T} \cap \mathcal{U}$), we obtain that v satisfies $\Gamma \Rightarrow \Delta$. ■

Since only cuts on formulas occurring in $\mathcal{S} \setminus \{\psi \Rightarrow \psi \mid \psi \in \mathit{Frm}_{\mathcal{L}}\}$ are needed in the last proof, it follows that coherent semi-canonical (−A)-systems enjoy the following strong form of cut-admissibility:

Corollary 3: Let $\mathbf{G}$ be a coherent semi-canonical (+C−A)-system. If $\mathcal{S} \vdash^{seq}_{\mathbf{G}} s$, then there exists a derivation of s from $\mathcal{S}$ in $\mathbf{G}$, in which only formulas occurring in $\mathcal{S} \setminus \{\psi \Rightarrow \psi \mid \psi \in \mathit{Frm}_{\mathcal{L}}\}$ serve as cut-formulas.

D. (−C−A)-Systems

Finally, semantics for semi-canonical (−C−A)-systems is given in the form of 4Nmatrices:

Definition 17: Let $\mathbf{G}$ be a semi-canonical (−C−A)-system for $\mathcal{L}$. $\mathbf{M_G}$ is the 4Nmatrix obtained from $\mathbf{M_{G+A}}$ (see Definition 15) by adding the truth-value $\bot$ to $\mathcal{V}_{\mathbf{M_G}}$, and replacing $\{t, f, \top\}$ by $\{t, f, \top, \bot\}$ in all truth-tables.

Example 6: Suppose that the rules for $\supset$ in some semi-canonical (−C−A)-system $\mathbf{G}$ are the usual rules. The following truth-table represents $\supset_{\mathbf{M_G}}$:

$\supset_{\mathbf{M_G}}$	t	f	$\top$	$\bot$
t	$\{t,\top\}$	$\{f,\top\}$	$\{\top\}$	$\{t,f,\top,\bot\}$
f	$\{t,\top\}$	$\{t,\top\}$	$\{t,\top\}$	$\{t,\top\}$
$\top$	$\{t,\top\}$	$\{\top\}$	$\{\top\}$	$\{t,\top\}$
$\bot$	$\{t,\top\}$	$\{t,f,\top,\bot\}$	$\{t,\top,\bot\}$	$\{t,f,\top,\bot\}$

Theorem 8: For every semi-canonical (−C−A)-system $\mathbf{G}$, $\vdash^{seq}_{\mathbf{G}} = \vdash^{seq}_{\mathbf{M_G}}$ and $\vdash_{\mathbf{M_G}} = \vdash_{\mathbf{G}}$.

The proof goes along the lines of the previous cases, and is omitted here.

V. Conclusions and Further Research

The results of this paper shed light on the semantic roles of (CUT) and (ID) in sequent systems. The availability of each of these two components rules out one truth-value ($\top$ or $\bot$), and reduces the level of non-determinism in the induced Nmatrix. The presence of both of them (as happens in $\mathbf{LK}$) is crucial for having deterministic semantics.

Besides possible proof-theoretic applications of semi-canonical systems and their semantics (e.g. Corollary 3), Examples 4 and 5 roughly suggest other possible applications of semi-canonical systems, e.g. as a tool for reasoning about uncertain logical circuits. Combining different sorts of uncertainties in logical circuits naturally requires extensions of the current framework (e.g. to handle cases in which (CUT) and/or (ID) may apply only on some formulas).

From a proof-theoretic perspective, it would be interesting and useful to extend the results of this paper to broader families of sequent (and higher-level objects) systems without (CUT) and/or (ID). This can include first-order canonical systems, many-sided sequent systems, and systems allowing a larger variety of logical rules. Finally, a natural question, arising from the definition of semi-canonical systems, concerns the necessity of the weakening rule. It would be interesting to understand the semantic effect of (Weak) in canonical systems, and develop semantics for systems lacking this rule. However, following the next final proposition, finite-valued Nmatrices semantics does not suffice for this purpose:

Proposition 6: Suppose that $\mathcal{L}$ contains two binary connectives denoted by $\vee_1$ and $\vee_2$. Let $\mathbf{G}$ be a sequent system consisting only on (CUT), (ID) and the following three canonical rules: $\{\Rightarrow p_1, p_2\} / \Rightarrow p_1 \vee_1 p_2$, $\{\Rightarrow p_1\} / \Rightarrow p_1 \vee_2 p_2$, and $\{\Rightarrow p_2\} / \Rightarrow p_1 \vee_2 p_2$. There is no finite-valued Nmatrix $\mathbf{M}$ such that $\vdash_{\mathbf{G}} = \vdash_{\mathbf{M}}$.

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