Non-deterministic Connectives in Propositional Gödel Logic

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Background and Motivation

- Current fuzzy logics follow the principle of truth-functionality, and fuzziness is limited to the level of the atomic formulas.
- Non-deterministic semantics [Avron,Lev '01] relaxes the truth-functionality principle, and allows uncertainty also on the level of the connectives.
- However, non-deterministic semantics has not yet been applied for fuzzy logics.
- We provide a first step towards a theory of non-deterministic semantics for fuzzy logics, by identifying a family of non-deterministic connectives that can be added to Gödel logic.

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		$\tilde{\wedge}$
T	T	T
T	F	F
F	T	F
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		$\tilde{\land}$
T	T	T
T	F	F
F	T	F
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$$\frac{\Gamma, \varphi \Rightarrow \Delta \quad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \lor \psi \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \varphi, \psi, \Delta}{\Gamma \Rightarrow \varphi \lor \psi, \Delta}$$

		$\tilde{\lor}$
T	T	T
T	F	T
F	T	T
F	F	F

Simple Non-deterministic Semantics

This naturally leads to non-deterministic semantics.

$$\frac{\Gamma, \varphi \Rightarrow \Delta \quad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \circ \psi \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \varphi, \Delta \quad \Gamma \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \circ \psi, \Delta}$$

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T	T	{T}
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$$v(\varphi \circ \psi) \in \tilde{\circ}\,(v(\varphi), v(\psi))$$

Proof-Theory of Gödel Logic

- The only known "ideal" (in the above sense) system for Gödel logic is the single-conclusion hypersequent system **HG** [Avron '91].
- A single-conclusion hypersequent is a set of single-conclusion sequents denoted by:

$$\Gamma_1 \Rightarrow E_1 \mid \Gamma_2 \Rightarrow E_2 \mid \ldots \mid \Gamma_n \Rightarrow E_n$$

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$$\frac{H \mid \Gamma, \Delta \Rightarrow E_1 \quad H \mid \Gamma, \Delta \Rightarrow E_2}{H \mid \Gamma \Rightarrow E_1 \mid \Delta \Rightarrow E_2}\ (com)$$

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- All logical rules are the single-version hypersequent version of classical rules. E.g.

$$\frac{H \mid \Gamma, \varphi, \psi \Rightarrow E}{H \mid \Gamma, \varphi \wedge \psi \Rightarrow E} \qquad \frac{H \mid \Gamma \Rightarrow \varphi \quad H \mid \Gamma \Rightarrow \psi}{H \mid \Gamma \Rightarrow \varphi \wedge \psi}$$

New Connectives in Gödel Logic

- Example:

$$\frac{H \mid \Gamma, \varphi \Rightarrow E \quad H \mid \Gamma, \psi \Rightarrow E}{H \mid \Gamma, \varphi \circ \psi \Rightarrow E} \qquad \frac{H \mid \Gamma \Rightarrow \varphi \quad H \mid \Gamma \Rightarrow \psi}{H \mid \Gamma \Rightarrow \varphi \circ \psi}$$

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- In general, new connectives can be added to Gödel logic by adding to **HG** rules of the following forms:

$$\frac{\{H \mid \Gamma, \Pi_i \Rightarrow E_i\}_{1 \leq i \leq m} \quad \{H \mid \Gamma, \Sigma_i \Rightarrow E\}_{1 \leq i \leq k}}{H \mid \Gamma, \diamond(\psi_1, \ldots, \psi_n) \Rightarrow E} \qquad \frac{\{H \mid \Gamma, \Pi_i \Rightarrow E_i\}_{1 \leq i \leq m}}{H \mid \Gamma \Rightarrow \diamond(\psi_1, \ldots, \psi_n)}$$

where $\Pi_i, E_i, \Sigma_i \subseteq \{\psi_1, \ldots, \psi_n\}$

Many-valued Semantics

Gödel valuation

- Non-empty linearly ordered set $\langle V, \leq \rangle$ with a maximal element 1 and a minimal element 0
- A valuation function $v : Frm_{\mathcal{L}} \to V$

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For example:

$$\frac{H \mid \Gamma, \varphi \Rightarrow E \quad H \mid \Gamma, \psi \Rightarrow E}{H \mid \Gamma, \varphi \circ \psi \Rightarrow E} \qquad \frac{H \mid \Gamma \Rightarrow \varphi \quad H \mid \Gamma \Rightarrow \psi}{H \mid \Gamma \Rightarrow \varphi \circ \psi}$$

$$v(\varphi \circ \psi) \in [\min(v(\varphi), v(\psi)), \max(v(\varphi), v(\psi))]$$

Many-valued Semantics

In general:

$$v(\diamond(\psi_1, \ldots, \psi_n)) \in [F_\diamond(v(\psi_1), \ldots, v(\psi_n)), G_\diamond(v(\psi_1), \ldots, v(\psi_n))]$$

where $F_\diamond$ and $G_\diamond$ involve $\min, \max$, and $\rightarrow$

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$$\frac{\{H \mid \Gamma,\Pi_i \Rightarrow E_i\}_{1\leq i\leq m}}{H \mid \Gamma \Rightarrow \diamond(\psi_1,\ldots,\psi_n)} \qquad \frac{\{H \mid \Gamma,\Theta_i \Rightarrow F_i\}_{1\leq i\leq l} \quad \{H \mid \Gamma,\Sigma_i \Rightarrow E\}_{1\leq i\leq k}}{H \mid \Gamma,\diamond(\psi_1,\ldots,\psi_n) \Rightarrow E}$$

$$F_\diamond(x_1,\ldots,x_n) = \min_{1\leq i\leq m}(\min x(\Pi_i) \rightarrow \max x(E_i))$$

$$G_\diamond(x_1,\ldots,x_n) = \min_{1\leq i\leq l}(\min x(\Theta_i) \rightarrow \max x(F_i)) \rightarrow \max_{1\leq i\leq k}(\min x(\Sigma_i))$$

where for every set $\Delta \subseteq \{\psi_1,\ldots,\psi_n\}$, $x(\Delta) = \{x_i \mid \psi_i \in \Delta\}$

Example

- Usual implication:

$$\frac{H \mid \Gamma, \varphi \Rightarrow \psi}{H \mid \Gamma \Rightarrow \varphi \supset \psi} \qquad \frac{H \mid \Gamma \Rightarrow \varphi \quad H \mid \Gamma, \psi \Rightarrow E}{H \mid \Gamma, \varphi \supset \psi \Rightarrow E}$$

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$$v(\varphi \rightsquigarrow \psi) \in [v(\psi), v(\varphi) \to v(\psi)]$$

$$v(\varphi \rightsquigarrow \psi) \in \begin{cases} \{v(\psi)\} & v(\varphi) > v(\psi) \\ [v(\psi), 1] & \text{otherwise} \end{cases}$$

Summary

- We characterize proof-theoretically and semantically a family of (non-deterministic) connectives that can be added to propositional Gödel logic.
- The paper also provides:
  - General strong cut-admissibility results.
  - Decidability results.
  - Non-deterministic Kripke-style semantics.
- Further Research:
  - Provide an independent semantic characterization of this family of connectives.
  - Apply these methods for other fuzzy logics.

Thank you!