Semantic Investigation of Canonical Gödel Hypersequent Systems

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Logic: Between Semantics and Proof Theory
A Workshop in Honor of Prof. Arnon Avron's 60th Birthday
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1. $\langle U, \leq \rangle$ is a linearly ordered *infinite* set of truth values, with a minimum value 0 and a maximum value 1.
2. A valuation is a function $v : \mathit{wff} \to U$ satisfying:

$$v(A \wedge B) = \min\{v(A), v(B)\} \qquad v(A \vee B) = \max\{v(A), v(B)\}$$

$$v(\bot) = 0 \qquad v(A \supset B) = v(A) \to v(B) = \begin{cases} 1 & v(A) \leq v(B) \\ v(B) & \mathit{otherwise} \end{cases}$$

Definition

$\Gamma \vdash A$ if for every valuation v: if $v(B) = 1$ for every $B \in \Gamma$ then $v(A) = 1$.

The Proof-Theory of Gödel Logic

$$(\textit{Linearity}) \qquad (A \supset B) \vee (B \supset A)$$

- "Syntactically", Gödel logic is obtained by adding (*Linearity*) to an axiomatization of intuitionistic logic.
- Various sequent systems have been introduced (e.g., [Sonobe '75], [Corsi '86], [Avellone et al. '99], [Dyckhoff '99], [Avron and Konikowska '01], [Dyckhoff and Negri '06]).
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- Each of them has some ad-hoc logical rules of a nonstandard form.
- In contrast, standard logical rules are used in **HG** [Avron '91], the system obtained by "lifting" **LJ** to the hypersequent level, and adding the communication rule.

Hypersequents

A *hypersequent* is a finite set of sequents denoted by:

$$\Gamma_1 \Rightarrow E_1 \mid \Gamma_2 \Rightarrow E_2 \mid \ldots \mid \Gamma_n \Rightarrow E_n$$

The Communication Rule

$$\frac{H \mid \Gamma, \Delta \Rightarrow E_1 \qquad H \mid \Gamma, \Delta \Rightarrow E_2}{H \mid \Gamma \Rightarrow E_1 \mid \Delta \Rightarrow E_2}$$

The System **HG**

Structural Rules:

$$(IW \Rightarrow)\ \frac{H \mid \Gamma \Rightarrow E}{H \mid \Gamma, A \Rightarrow E} \qquad (\Rightarrow IW)\ \frac{H \mid \Gamma \Rightarrow}{H \mid \Gamma \Rightarrow A} \qquad (EW)\ \frac{H}{H \mid \Gamma \Rightarrow E}$$

$$(com)\ \frac{H \mid \Gamma, \Delta \Rightarrow E_1 \quad H \mid \Gamma, \Delta \Rightarrow E_2}{H \mid \Gamma \Rightarrow E_1 \mid \Delta \Rightarrow E_2}$$

Identity Rules:

$$(id)\ \frac{}{A \Rightarrow A} \qquad (cut)\ \frac{H \mid \Gamma \Rightarrow A \quad H \mid \Gamma, A \Rightarrow E}{H \mid \Gamma \Rightarrow E}$$

Logical Rules:

$$(\Rightarrow\supset)\ \frac{H \mid \Gamma, A_1 \Rightarrow A_2}{H \mid \Gamma \Rightarrow A_1 \supset A_2} \qquad (\supset\Rightarrow)\ \frac{H \mid \Gamma \Rightarrow A_1 \quad H \mid \Gamma, A_2 \Rightarrow E}{H \mid \Gamma, A_1 \supset A_2 \Rightarrow E}$$

$$(\Rightarrow \wedge)\ \frac{H \mid \Gamma \Rightarrow A_1 \quad H \mid \Gamma \Rightarrow A_2}{H \mid \Gamma \Rightarrow A_1 \wedge A_2} \qquad (\wedge \Rightarrow)\ \frac{H \mid \Gamma, A_1, A_2 \Rightarrow E}{H \mid \Gamma, A_1 \wedge A_2 \Rightarrow E}$$

The System **HG**

Theorem

1. $\Gamma \vdash A$ iff $\{ \Rightarrow B \mid B \in \Gamma\} \vdash_{\mathbf{HG}} \Rightarrow A$.
2. (cut) is admissible in **HG**.

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Proof

By authority. Arnon says it's true. :)

Question

What happens if we “play” a bit with the logical rules of **HG**?

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Canonical Logical Rules

Right Rules: $\Pi_1, \Sigma_1, \ldots, \Pi_m, \Sigma_m \subseteq \{1, \ldots, n\}$ $\quad |\Sigma_1| = \ldots = |\Sigma_m| \leq 1$

$$\frac{H \mid \Gamma, \{A_j \mid j \in \Pi_1\} \Rightarrow \{A_j \mid j \in \Sigma_1\} \quad \ldots \quad H \mid \Gamma, \{A_j \mid j \in \Pi_m\} \Rightarrow \{A_j \mid j \in \Sigma_m\}}{H \mid \Gamma \Rightarrow \diamond(A_1, \ldots, A_n)}$$

Left Rules: $\Pi_1, \Sigma_1, \ldots, \Pi_m, \Sigma_m \subseteq \{1, \ldots, n\}$ $\quad |\Sigma_1| = \ldots = |\Sigma_m| \leq 1$
$\Theta_1, \ldots, \Theta_k \subseteq \{1, \ldots, n\}$

$$\frac{\begin{array}{c} H \mid \Gamma, \{A_j \mid j \in \Pi_1\} \Rightarrow \{A_j \mid j \in \Sigma_1\} \quad \ldots \quad H \mid \Gamma, \{A_j \mid j \in \Pi_m\} \Rightarrow \{A_j \mid j \in \Sigma_m\} \\ H \mid \Gamma, \{A_j \mid j \in \Theta_1\} \Rightarrow E \quad \ldots \quad H \mid \Gamma, \{A_j \mid j \in \Theta_k\} \Rightarrow E \end{array}}{H \mid \Gamma, \diamond(A_1, \ldots, A_n) \Rightarrow E}$$

Examples

- All logical rules of **HG** are canonical. E.g.,

$$\frac{H \mid \Gamma, A_1 \Rightarrow A_2}{H \mid \Gamma \Rightarrow A_1 \supset A_2} \qquad \frac{H \mid \Gamma \Rightarrow A_1 \quad H \mid \Gamma, A_2 \Rightarrow E}{H \mid \Gamma, A_1 \supset A_2 \Rightarrow E}$$

- And/Or Connective

$$\frac{H \mid \Gamma \Rightarrow A_1 \quad H \mid \Gamma \Rightarrow A_2}{H \mid \Gamma \Rightarrow A_1 \mathbin{\vee\!\!\!\!\wedge} A_2} \qquad \frac{H \mid \Gamma, A_1 \Rightarrow E \quad H \mid \Gamma, A_2 \Rightarrow E}{H \mid \Gamma, A_1 \mathbin{\vee\!\!\!\!\wedge} A_2 \Rightarrow E}$$

- Primal Implication [Gurevich, Neeman '09]

$$\frac{H \mid \Gamma \Rightarrow A_2}{H \mid \Gamma \Rightarrow A_1 \rightsquigarrow A_2} \qquad \frac{H \mid \Gamma \Rightarrow A_1 \quad H \mid \Gamma, A_2 \Rightarrow E}{H \mid \Gamma, A_1 \rightsquigarrow A_2 \Rightarrow E}$$

Canonical Gödel Systems

A Canonical Gödel System =

The structural rules of **HG**

$+$

The two identity rules

$+$

A (finite) set of canonical logical rules

Semantics of Canonical Gödel Systems

Let **G** be a canonical Gödel system.

- The rules in **G** for each connective $\diamond$ impose restrictions on the values assigned to $\diamond$-formulas.
- These restrictions are given by intervals whose lower and upper bounds are determined according to the right and left rules of **G** for $\diamond$ (resp.).

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$$v(\diamond(A_1,\ldots,A_n)) \in \left[\mathbf{G}^{\diamond}_{right}(v(A_1),\ldots,v(A_n))\ ,\ \mathbf{G}^{\diamond}_{left}(v(A_1),\ldots,v(A_n))\right]$$

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$$\mathbf{G}^{\diamond}_{right}(x_1,\ldots,x_n) = \max_{\substack{\Pi_1,\Sigma_1,\ldots,\Pi_m,\Sigma_m \\ \text{is a right rule} \\ \text{of } \mathbf{G} \text{ for } \diamond}} \left(\min_{1\leq i\leq m}\left(\min_{j\in\Pi_i} x_j \to \max_{j\in\Sigma_i} x_j\right)\right)$$

$$\mathbf{G}^{\diamond}_{left}(x_1,\ldots,x_n) = \min_{\substack{\Pi_1,\Sigma_1,\ldots,\Pi_m,\Sigma_m \quad \Theta_1,\ldots,\Theta_k \\ \text{is a left rule of } \mathbf{G} \text{ for } \diamond}} \left(\min_{1\leq i\leq m}\left(\min_{j\in\Pi_i} x_j \to \max_{j\in\Sigma_i} x_j\right) \to \max_{1\leq i\leq k}\left(\min_{j\in\Theta_i} x_j\right)\right)$$

Examples

- For all usual connectives, we obtain a degenerate interval. E.g.,

$$\frac{H \mid \Gamma, A_1 \Rightarrow A_2}{H \mid \Gamma \Rightarrow A_1 \supset A_2} \qquad \frac{H \mid \Gamma \Rightarrow A_1 \quad H \mid \Gamma, A_2 \Rightarrow E}{H \mid \Gamma, A_1 \supset A_2 \Rightarrow E}$$

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- And/Or

$$\frac{H \mid \Gamma \Rightarrow A_1 \quad H \mid \Gamma \Rightarrow A_2}{H \mid \Gamma \Rightarrow A_1 \between A_2} \qquad \frac{H \mid \Gamma, A_1 \Rightarrow E \quad H \mid \Gamma, A_2 \Rightarrow E}{H \mid \Gamma, A_1 \between A_2 \Rightarrow E}$$

$$v(A_1 \between A_2) \in [\min(v(A_1), v(A_2)), \max(v(A_1), v(A_2))]$$

- Primal Implication

$$\frac{H \mid \Gamma \Rightarrow A_2}{H \mid \Gamma \Rightarrow A_1 \leadsto A_2} \qquad \frac{H \mid \Gamma \Rightarrow A_1 \quad H \mid \Gamma, A_2 \Rightarrow E}{H \mid \Gamma, A_1 \leadsto A_2 \Rightarrow E}$$

$$v(A_1 \leadsto A_2) \in [v(A_2), v(A_1) \to v(A_2)]$$

Semantics of Identity Rules

Identity Rules:

$$(id)\ \frac{}{A \Rightarrow A} \qquad (cut)\ \frac{H \mid \Gamma \Rightarrow A \quad H \mid \Gamma, A \Rightarrow E}{H \mid \Gamma \Rightarrow E}$$

- **Question:** What is the semantic effect of the two identity rules?
- **Motivation:** Semantics for cut-free systems are useful in proofs of cut-admissibility.

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- **Intuition:** The identity rules bind together the two sides of the sequent. Without them each formula can have different values when it occurs on the left side, and on the right side.

$$(id)\quad \textit{left side} \leq \textit{right side} \qquad (cut)\quad \textit{right side} \leq \textit{left side}$$

Semantics of **HG** without Identity Rules

1. $\langle U, \leq \rangle$ is a linearly ordered infinite set of truth values, with a minimum value 0 and a maximum value 1.
2. A quasi-valuation is a function $q : wff \to U \times U$ satisfying:

$$q(A \wedge B) \in [0, \min(q^l(A), q^l(B))] \times [\min(q^r(A), q^r(B)), 1]$$

$$q(A \supset B) \in \left[0, \begin{cases} 1 & q^r(A) \leq q^l(B) \\ q^l(B) & otherwise \end{cases}\right] \times \left[\begin{cases} 1 & q^l(A) \leq q^r(B) \\ q^r(B) & otherwise \end{cases}, 1\right]$$

3. q is a *model* of a hypersequent H if

$$\min_{A \in \Gamma} q^l(A) \leq \max_{A \in E} q^r(A)$$

for some $\Gamma \Rightarrow E \in H$.

Semantics of **HG** without Identity Rules

Soundness and Completeness

$\Omega \vdash_{\mathbf{HG}-(id)-(cut)} H$ iff every quasi-valuation which is a model of Ω is also a model of H.

Variations

- For (id), use $q : wff \to \{\langle x, y\rangle \in U \times U \mid x \leq y\}$.
- For (cut), use $q : wff \to \{\langle x, y\rangle \in U \times U \mid y \leq x\}$.

Extension for Canonical Gödel Systems

$$q(\diamond(A_1, \ldots, A_n)) \in [0, \mathbf{G}^{\diamond}_{left}(q(A_1), \ldots, q(A_n))] \times \left[\mathbf{G}^{\diamond}_{right}(q(A_1), \ldots, q(A_n)), 1\right]$$

$$\mathbf{G}^{\diamond}_{right}(\langle x_1, y_1\rangle, \ldots, \langle x_n, y_n\rangle) = \max_{\substack{\Pi_1,\Sigma_1,\ldots,\Pi_m,\Sigma_m \\ \text{is a right rule} \\ \text{of } \mathbf{G} \text{ for } \diamond}} \left(\min_{1\leq i\leq m} \left(\min_{j\in\Pi_i} x_j \to \max_{j\in\Sigma_i} y_j \right)\right)$$

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Cut-Admissibility

Proving cut-admissibility reduces to proving that for every quasi-valuation which is not a model of some hypersequent H, there exists a valuation which is not a model of H.

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Definition

A valuation v is a *refinement* of a quasi-valuation q, if for every $A \in wff$: $q^l(A) \leq v(A) \leq q^r(A)$.

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A canonical Gödel system enjoys cut-admissibility if every quasi-valuation has a refinement.

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Corollary

A canonical Gödel system enjoys cut-admissibility if every quasi-valuation has a refinement.

For **HG**, this is straightforward. The refinement is obtained by recursion on the build-up of formulas.

Cut-Admissibility in Canonical Gödel Systems

Refinement is possible only in *coherent* canonical Gödel systems:

Definition

A canonical Gödel system $\mathbf{G}$ is called *coherent* if

$$\mathbf{G}^{\diamond}_{right}(x_1, \ldots, x_n) \leq \mathbf{G}^{\diamond}_{left}(x_1, \ldots, x_n)$$

for every n-ary connective $\diamond$ and $x_1, \ldots, x_n \in U$.

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Theorem

A canonical Gödel system enjoys cut-admissibility iff it is coherent.

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Syntactic Characterization of Coherence

A canonical Gödel system $\mathbf{G}$ is coherent iff for every right rule R_1 and left rule R_2 of $\mathbf{G}$ for some connective $\diamond$, the empty sequent is derivable from the premises of R_1 and R_2 using only cuts.

Further Work

- Extensions for higher-order logics.
  In particular, does the extension of **HG** with usual rules for first and second order quantifiers enjoy cut-admissibility?
- Is this approach applicable in substructural hypersequent calculi?

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Thank you!

"*The mediocre teacher tells.
The good teacher explains.
The superior teacher demonstrates.
The great teacher inspires.*"

(William Arthur Ward)