# A propagation process on Cayley graphs

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**Abstract**

A *propagation process* on a finite undirected graph $G = (V, E)$ with starting set $S \subset V$ is a sequence of sets $S_0 = S \subset S_1 \subset \ldots \subset S_r$, where $S_i \subset V$, and for each $i > 0$, $S_i$ is obtained from $S_{i-1}$ by adding to it a vertex $v \in V - S_{i-1}$ such that there exists a vertex $u \in S_{i-1}$ adjacent to $v$ so that all neighbors of $u$ besides $v$ are in $S_{i-1}$. $S$ is *covering* if there is such a sequence ending with $S_r = V$. The *propagation number* $pn(G)$ of $G$ is the minimum possible cardinality of a covering set $S \subset V$. The study of this parameter is motivated by the investigation of quantum networks, influence in social networks and power dominating sets. We establish an algebraic lower bound for the propagation number of a graph, and apply it to show that the propagation number of the $d$-cube is $2^{d-1}$. This settles a conjecture of Aazami. The propagation numbers of other Cayley graphs of the group $Z_2^d$ are determined as well.

## 1 Introduction

A *propagation process* on a finite undirected graph $G = (V, E)$ with starting set $S \subset V$ is a sequence of sets $S_0 = S \subset S_1 \subset \ldots \subset S_r$, where $S_i \subset V$, and for each $i > 0$, $S_i$ is obtained from $S_{i-1}$ by adding to it a vertex $v \in V - S_{i-1}$ such that there exists a vertex $u \in S_{i-1}$ adjacent to $v$ so that all neighbors of $u$ besides $v$ are in $S_{i-1}$. $S$ is *covering* if there is such a sequence ending with $S_r = V$. The *propagation number* $pn(G)$ of $G$ is the minimum possible cardinality of a covering set $S \subset V$.

The investigation of the propagation numbers of graphs is motivated by related questions in the study of quantum networks, power dominating sets and influence in social networks, see, for example, [2], [5], [4], [3], [1] and their references. Aazami (see [1]) conjectured that the propagation number of the $d$-cube $Q_d$ is $2^{d-1}$. The $d$ cube is the graph whose vertices are all binary vectors of length $d$, where two are adjacent iff they differ in exactly one coordinate. It is easy to see that $pn(Q_d) \leq 2^{d-1}$, indeed, the set $S$ consisting of all vertices whose first coordinate is 0 is covering. Aazami showed that equality holds for all $d \leq 5$ and that in general the propagation number of any graph is at least

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as large as its path-width, implying that for the $d$-cube, $pn(Q_d) \geq \Omega(\frac{2^d}{\sqrt{d}})$. Here we prove a general result that supplies a lower bound for the propagation number of a graph, and apply it to prove this conjecture. The proof illustrates nicely the power of tools from linear algebra in the study of extremal problems, and provides the propagation numbers of other graphs as well.

## 2 The main results

Given a graph $G = (V, E)$, we say that a matrix $A = (a_{uv})_{u,v \in V}$ over a field $F$ *represents* $G$ if for every $uv \in E$ $a_{uv} \neq 0$, for every distinct $u, v$ with $uv \notin E$, $a_{uv} = 0$, and the diagonal elements $a_{uu}$ are arbitrary. Let $r(A)$ denote the rank of $A$ over $F$. We prove the following.

**Theorem 2.1** *Let $G = (V, E)$ be a graph, let $F$ be a field, and suppose that $A$ represents $G$ over $F$. Then $pn(G) \geq |V| - r(A)$.*

**Theorem 2.2** *Let $A_d$ be the adjacency matrix of the d-cube $Q_d$, and let $I$ be the identity matrix of order $2^d$. Then, for every even d, the rank of $A_d$ over $GF(2)$ is $2^{d-1}$, and for every odd d, the rank of $I + A_d$ over $GF(2)$ is $2^{d-1}$. Moreover, both $A_d$ and $I + A_d$ represent $Q_d$ over any field, and therefore $pn(Q_d) = 2^{d-1}$ for every d.*

The last theorem is a special case of a more general result. For a subset $T$ of nonzero elements of the group $Z_2^d$, the *Cayley Graph* $C = C(Z_2^d, T)$ is the graph whose vertices are all elements of $Z_2^d$ where two such elements are adjacent iff their sum (which is also their difference) is in $T$. This is clearly a $|T|$-regular graph. In case $T$ is the set of all $d$ unit vectors, the corresponding Cayley graph is $Q_d$.

**Theorem 2.3** *Let $C = C(Z_2^d, T)$ be the Cayley graph of $Z_2^d$ with respect to $T \subset Z_2^d$, let $A$ be the adjacency matrix of $C$, and let $I$ be the identity matrix of order $2^d$. If $|T|$ is even, then the rank of $A$ over $GF(2)$ is at most $2^{d-1}$, and if $|T|$ is odd then the rank of $I + A$ over $GF(2)$ is at most $2^{d-1}$. Therefore, the propagation number of $C$ is at least $2^d - 2^{d-1} = 2^{d-1}$.*

As we briefly discuss at the end of the note, the last theorem can be used to identify a large class of Cayley graphs of $Z_2^d$, all having propagation number exactly $2^{d-1}$.

## 3 Proofs

**Proof of Theorem 2.1:** Let $S \subset V$ be covering, and let $S = S_0 \subset S_1 \subset \cdots \subset S_r = V$ be a propagation process with starting set $S$. We have to prove that $|S| \geq |V| - r(A)$. Consider the following system of homogeneous linear equations in the set $x = (x_v)_{v \in V}$ of $|V|$ variables over the field $F$:

$$Ax = 0 \quad \text{and} \quad x_s = 0 \ \text{ for every } \ s \in S$$

We claim that the only solution of this system is the trivial solution $x_v = 0$ for all $v$. Indeed, by the definition of the system, for each such solution $x$, $x_v = 0$ for all $v \in S = S_0$. Assume, by induction, that $x_u = 0$ for all $u \in S_{i-1}$. The propagation process then implies that if $v$ is the unique element of $S_i - S_{i-1}$ then there is a homogeneous equation in our system (given by the row indexed by $u$ in $Ax = 0$, where $u \in S_{i-1}$ is the neighbor of $v$ that caused its insertion to $S_i$) in which the coefficient of $x_v$ is nonzero and all other variables are zero, implying that $x_v = 0$ as well. Since $S$ is covering, the assertion of the claim follows. As the system has only the trivial solution it follows that the rank of its defining matrix is at least the number of variables, implying that $r(A) + |S| \geq |V|$ and completing the proof. $\Box$

**Proof of Theorem 2.2:** Let $U$ denote the set of all vertices of $Q_d$ represented by binary vectors whose first coordinate is zero, and let $U'$ be the set of all other vertices. For each $u \in U$, let $u'$ be the vertex obtained from $u$ by changing the first coordinate of $u$ from 0 to 1. Suppose, first, that $d$ is even. We show that the set of all rows of $A_d$ corresponding to vertices of $U$ forms a basis to the row-space of $A_d$. Indeed, the submatrix of $A_d$ whose rows are indexed by the elements of $U$ and whose columns are indexed by those of $U'$ is the identity matrix of order $2^{d-1}$. Thus these rows are linearly independent. We next show that they span all other rows. Let $u' \in U'$ be an arbitrary vertex of $U'$, and let $u, u'_2, u'_3, \ldots, u'_d$ be its neighbors in $Q_d$, where $u \in U$, and for $i \geq 2$, $u'_i \in U'$ is the vertex obtained from $u'$ by flipping its $i$th coordinate. It is not difficult to check that the row of $u'$ is equal (over $GF(2)$) to the sum of the $d-1$ rows of the vertices $u_2, u_3, \ldots, u_d$, where $u_i$ is the vertex obtained from $u'_i$ by changing the first coordinate from 1 to 0.

The case of odd $d$ is similar. Here, too, the set of rows of all vertices $u \in U$ forms a basis for $I + A_d$, where here the row corresponding to a vertex $u' \in U'$ is the sum (over $GF(2)$) of the rows corresponding to the vertices $u, u_2, u_3, \ldots, u_d$, using the same notation as before. By Theorem 2.1 it follows that $pn(Q_d) \geq 2^d - 2^{d-1} = 2^{d-1}$, and as we have already noted that $pn(Q_d) \leq 2^{d-1}$, the desired equality holds. This completes the proof. An alternative one is given in the proof of the next theorem. $\Box$

**Proof of Theorem 2.3:** Suppose, first, that $|T|$ is even. We show that in this case every two (not necessarily distinct) rows of $A$ are orthogonal (over $GF(2)$), that is, $A^2 = 0$. Thus, the row-space of $A$ is self-orthogonal, and its dimension is at most half its length, as needed. We proceed with the proof of this (simple) fact. Every row is orthogonal to itself, as it has $|T|$ one entries, and $|T|$ is even. The inner product between two distinct rows corresponding to the vertices $u$ and $v$ is the number of common neighbors of $u$ and $v$. This is exactly the number of ordered pairs $(t, t')$ of distinct elements of $T$ so that $u + t = v + t'$ (as each such pair corresponds to the common neighbor $u + t = v + t'$.) However, the number of such pairs is even, since if $(t, t')$ is such a pair, so is $(t', t)$. This completes the proof for even $|T|$. For odd $|T|$ the proof is similar, where in this case the rows of $I + A$ are orthogonal. $\square$

**Remarks:**

- Since the adjacency matrix $A_d$ of $Q_d$ has zeros on the diagonal, and its rank over $GF(2)$ for even $d$ is $2^{d-1}$, our proof actually supplies the tight $2^{d-1}$ lower bound for even $d$, even if we consider a more general propagation process in which an uncovered vertex $v \in V - S_{i-1}$ can join $S_{i-1}$ to form $S_i$ if there is a vertex $u \in V$ (not necessarily in $S_{i-1}$) which is adjacent to $v$ and all its neighbors but $v$ are in $S_{i-1}$. A similar comment applies to any Cayley graph of $Z_2^d$ whose degree is even.

- Theorem 2.3 can be used to produce many examples of graphs on $2^d$ vertices with propagation number exactly $2^{d-1}$. In particular, if $T$ is a subset of nonzero elements of $Z_2^d$, and there is a nonzero element $x \in Z_2^d$ so that there is a unique element $t \in T$ whose inner product with $x$ (over $GF(2)$) is 1, then the corresponding Cayley graph $G$ satisfies $pn(G) = 2^{d-1}$. Indeed, the set $S$ of all vectors $y \in Z_2^d$ whose inner product with $x$ is 0, is covering.

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