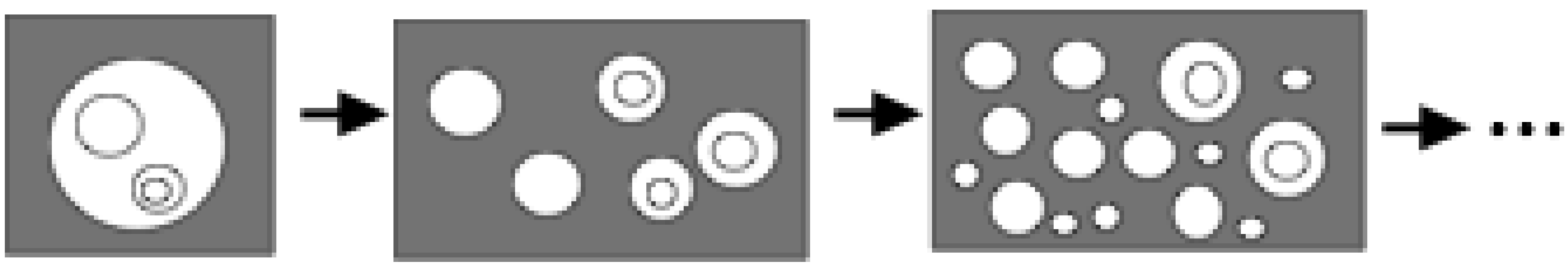
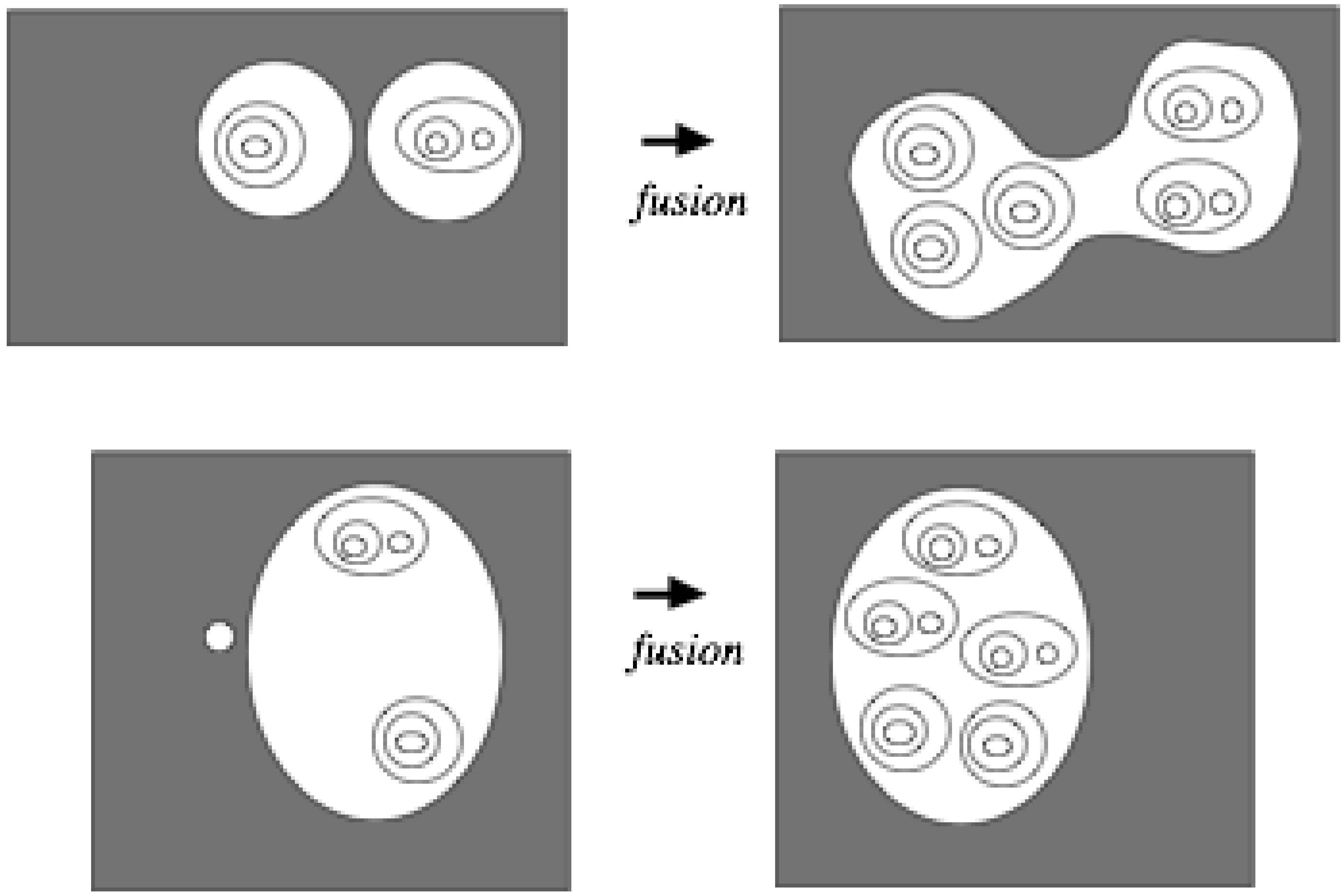
Termination

7. Rewriting

Fission




fusion
fusion

Better

- d(a) = depth(a)
- { {d(a) : a in A} : colony A }
- fission: depth decreases
- fusion: one deep item removed

DNF0

- $\neg\neg x \Rightarrow x$
- $\neg(x \vee y) \Rightarrow (\neg x) \wedge (\neg y)$
- $\neg(x \wedge y) \Rightarrow (\neg x) \vee (\neg y)$
- $x \wedge (y \vee z) \Rightarrow (x \wedge y) \vee (x \wedge z)$
- $(y \vee z) \wedge x \Rightarrow (y \wedge x) \vee (z \wedge x)$

DNF1

- $\neg \neg x \Rightarrow x$
- $\neg(x \lor y) \Rightarrow (\neg x) \land (\neg y)$
- $\neg(x \land y) \Rightarrow (\neg x) \lor (\neg y)$
- $x \land (y \land z) \Rightarrow (x \land y) \land z$
- $x \lor (y \lor z) \Rightarrow (x \lor y) \lor z$
- $x \land (y \lor z) \Rightarrow (x \land y) \lor (x \land z)$
- $(y \lor z) \land x \Rightarrow (y \land x) \lor (z \land x)$

DNF2

- $\neg \neg x \Rightarrow x$
- $\neg(x \lor y) \Rightarrow (\neg x) \land (\neg y)$
- $\neg(x \land y) \Rightarrow (\neg x) \lor (\neg y)$
- $(x \land y) \land z \Rightarrow x \land (y \land z)$
- $x \lor (y \lor z) \Rightarrow (x \lor y) \lor z$
- $x \land (y \lor z) \Rightarrow (x \land y) \lor (x \land z)$
- $(y \lor z) \land x \Rightarrow (y \land x) \lor (z \land x)$

DNF3

- $\neg\neg x \Rightarrow x$
- $\neg(x \vee y) \Rightarrow (\neg\neg\neg x) \wedge (\neg\neg\neg y)$
- $\neg(x \wedge y) \Rightarrow (\neg\neg\neg x) \vee (\neg\neg\neg y)$
- $x \wedge (y \vee z) \Rightarrow (x \wedge y) \vee (x \wedge z)$
- $(y \vee z) \wedge x \Rightarrow (y \wedge x) \vee (z \wedge x)$

DNF3

- $\neg \neg x \Rightarrow x$
- $\neg(x \vee y) \Rightarrow (\neg \neg \neg x) \wedge (\neg \neg \neg y)$
- $\neg(x \wedge y) \Rightarrow (\neg \neg \neg x) \vee (\neg \neg \neg y)$
- $x \wedge (y \vee z) \Rightarrow (x \wedge y) \vee (x \wedge z)$
- $(y \vee z) \wedge x \Rightarrow (y \wedge x) \vee (z \wedge x)$

$\neg \neg(a \wedge (b \vee c))$

DNF4

- $\neg \neg x \Rightarrow x$
- $\neg(x \vee y) \Rightarrow (\neg x) \wedge (\neg y)$
- $\neg(x \wedge y) \Rightarrow (\neg x) \vee (\neg y)$
- $x \wedge (y \vee z) \Rightarrow (x \wedge y) \vee (x \wedge z) \vee (x \wedge y) \vee (x \wedge z)$
- $(y \vee z) \wedge x \Rightarrow (x \wedge y) \vee (x \wedge z) \vee (x \wedge y) \vee (x \wedge z)$
- $x \vee x \Rightarrow x$

DNF5

- $\neg \neg x \Rightarrow x$
- $\neg(x \vee y) \Rightarrow (\neg x) \wedge (\neg y) \wedge (\neg x) \wedge (\neg y)$
- $\neg(x \wedge y) \Rightarrow (\neg x) \vee (\neg y) \vee (\neg x) \vee (\neg y)$
- $x \wedge (y \vee z) \Rightarrow (x \wedge y) \vee (x \wedge z)$
- $(y \vee z) \wedge x \Rightarrow (x \wedge y) \vee (x \wedge z)$
- $x \vee x \Rightarrow x$
- $x \wedge x \Rightarrow x$

DNF6

- $\neg \neg x \Rightarrow x$
- $\neg(x \vee y) \Rightarrow (\neg \neg \neg x) \wedge (\neg \neg \neg y) \wedge (\neg \neg \neg x) \wedge (\neg \neg \neg y)$
- $\neg(x \wedge y) \Rightarrow (\neg \neg \neg x) \vee (\neg \neg \neg y) \vee (\neg \neg \neg x) \vee (\neg \neg \neg y)$
- $x \vee x \Rightarrow x$
- $x \wedge x \Rightarrow x$

DNF7

- $\neg \neg x \Rightarrow x$
- $\neg(x \vee y) \Rightarrow (\neg x) \wedge (\neg y) \wedge (\neg x) \wedge (\neg y)$
- $\neg(x \wedge y) \Rightarrow (\neg x) \vee (\neg y) \vee (\neg x) \vee (\neg y)$
- $x \vee x \Rightarrow x$
- $x \wedge x \Rightarrow x$

Termination

- A rewrite system over a set of terms *Y* is terminating if, and only if, there exists a well-founded ordering '>' over *Y* such that, **t** ⇨ **u** implies **t > u**, for all terms **t** and **u** in *Y*.

Termination

- If s[x] ⇨ t[x] is a rule
- then c[s[v]] ⇨ c[t[v]] is a rewrite
- Want c[s[v]] > c[t[v]] in some wfo
- Want monotonicity
  - $s > t \Rightarrow f(\ldots,s,\ldots) > f(\ldots,t,\ldots)$

Symbolic Computation

- $Dt = 1$
- $Dc = 0$
- $D(x+y) = Dx + Dy$
- $D(xy) = xDy + yDx$
- ...

Rewriting

- $Dt \Rightarrow 1$
- $Dc \Rightarrow 0$
- $D(x+y) \Rightarrow Dx + Dy$
- $D(xy) \Rightarrow xDy + yDx$
- ...

Interpretations

- It is frequently convenient to separate a well-founded ordering of terms Y into two parts:
  - a **termination function [ ]**, that maps terms in Y to a set W
  - and a "standard" well-founded ordering '>' on that W

Polynomial Interpretation

- $[Dx] = [x]^2$
- $[x+y] = [x] + [y]$
- $[xy] = [x] + [y]$

Each rule must be shown to be reducing; that is, for each rule **l ⇨ r**, the polynomial **[l] – [r]** must be positive for non-negative interpretations of rule variables.

Polynomial Interpretation

- Prove rule reduction: $D(x+y) \Rightarrow Dx + Dy$
  - $(x^2 + 2xy + y^2) - (x^2 + y^2) = 2xy$
  - $2xy > 0$ for $x,y>0$
  - Consequently: $[D(x+y)] - [Dx + Dy] > 0$

Exponential Interpretation

- $[Dx] = 3^{[x]}$
- $[x] = 3$
- $[c] = 3$
- $[x+y] = [x] + [y]$
- $[xy] = [x] + [y]$

Exponential Interpretation

$$D(xy) \Rightarrow xDy + yDx$$

Prove reduction: $[D(xy)] > [xDy + yDx]$

- $[D(xy)] = 3^{[xy]} = 3^{[x]+[y]} = 3^{3+3} = 729$
- $[xDy + yDx] = [xDy] + [yDx] =$
- $= [x]+[Dy] + [y] + [Dx] = 3 + 3^{[y]} + 3 + 3^{[x]}$
- $= 3+3^3+3+3^3=60$
- $729>60$

Factorial

- $x+0 \Rightarrow x$
- $x+s(y) \Rightarrow s(x+y)$
- $x*0 \Rightarrow 0$
- $x*s(y) \Rightarrow y + x*y$
- $f(0) \Rightarrow s(0)$
- $f(s(x)) \Rightarrow s(x)*f(x)$

Factorial

- $x+0 \Rightarrow x$
- $x+s(y) \Rightarrow s(x+y)$
- $x*0 \Rightarrow 0$
- $x*s(y) \Rightarrow y + x*y$
- $f(0) \Rightarrow s(0)$
- $f(s(x)) \Rightarrow s(x)*f(p(s(x)))$
- $p(s(x)) \Rightarrow x$

Multiset Path Order

- $s = f(s_1,\ldots,s_m) \quad t = g(t_1,\ldots,t_n)$
- $s > t$ if $s_i \gtrsim t$ for some i
- $s > t$ if
  - $(f,\{s_1,\ldots,s_m\}) >_{lex} (g,\{t_1,\ldots,t_n\})$
  - and $s > t_j$ for all j

Lexicographic Path Order

- $s = f(s_1,\ldots,s_m) \quad t = g(t_1,\ldots,t_n)$
- $s > t$ if $s_i \gtrsim t$ for some i
- $s > t$ if
  - $(f, s_1,\ldots,s_m) >_{lex} (g, t_1,\ldots,t_n)$
  - and $s > t_j$ for all j

Boyer & Moore

- $if(if(x,y,z),u,v) \Rightarrow if(x,if(y,u,v),if(z,u,v))$

Recursive Path Order

- $s = f(s_1,...,s_m) \quad t = g(t_1,...,t_n)$
- $s > t$ if $s_i \gtrsim t$ for some i
- $s > t$ if
  - $(f, s_1, ..., \{s_i, ..., s_m\}) >_{lex} (g, t_1, ..., \{t_i, ..., t_n\})$
  - and $s > t_j$ for all j

Simplification Order

- Suppose finite vocabulary
- Subterm: $f(\ldots,s,\ldots) > s$
- Monotonic: $s > t \Rightarrow f(\ldots,s,\ldots) > f(\ldots,t,\ldots)$
- Must be well-founded

Weak Simplification Order

- Weak subterm: $f(\ldots,s_i,\ldots) \gtrsim s_i$
- Weak monotonicity:
  $s_i \gtrsim t_i \Rightarrow f(\ldots,s_i,\ldots) \gtrsim f(\ldots,t_i,\ldots)$
- Well-quasi-order by Kruskal
- Enough for termination of rewriting
  - Why?

Total Order

- Suppose finite vocabulary
- Monotonic: $s > t \Rightarrow f(\ldots,s,\ldots) > f(\ldots,t,\ldots)$
- Well-founded iff subterm

Semantic Path Order

- $s = f(s_1,\ldots,s_m) \quad t = g(t_1,\ldots,t_n) \quad >$
- $s > t$ if $s_i \gtrsim t$ for some i
- $s > t$ if
  - $(s, s_1,\ldots,s_m) >_{lex} (t, t_1,\ldots,t_n)$
  - and $s > t_j$ for all j
- require $s \Rightarrow t \Rightarrow f(\ldots s \ldots) \geq f(\ldots t \ldots)$

Proof

- Extend base order to a total w.f. order
- Consider minimal bad sequence
- Subterms are well-founded
- No use of $s_i \gtrsim t$ case
- So base order decreases and stabilizes