Jumping and Escaping

The Abstract Path order

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Notation

let R be a binary relation over a set V, we define the following:

- R^+ – the transitive closure.
- R^* – the transitive reflexive closure.
- R^ϵ – the reflexive closure.
- the immortal elements for a relation R over V are does points $v \in V$ that initiate an infinite R-chain of (not necessarily distinct) points in V: $vRv'Rv''R\dots$
- $R^\infty = \{\langle u, v\rangle : u, v \in V, \text{u is immortal for R}\}$

Jumping and Escaping

let A,B be binary relations, and $E = A \cup B$.

Jumping

relation A jumps over relation B if

$$BA \subseteq AE^* + B.$$

Escaping

relation A escapes from relation B if there is some point in every infinite B-chain from which an A-step leads to a point that is immortal in E.

Constriction I

an infinite sequence $s_0 E s_1 E \ldots$ is constricting in B if whenever there is a B-step $s_i B s_{i+1}$ in the sequence, it is the case that all the neighbors t, such that $s_i A t$, are mortal in E.

Proposition 1

if s is immortal in E, then there is an infinite B-constricting sequence in E originating in S

Proof.

simply take a B-step only when all possible A-steps leads to mortality. □

Constriction II

$$B_{\#} = B \setminus AE^{\infty}$$

$B_{\#}$ steps are the only kind of B-steps in a constricting sequence, so we get: if relation A escapes from relation B then ${B_{\#}}^{\infty} = \emptyset$.

Theorem 1

if A and B are well founded and A jumps over B, then E is also well founded.

proof I

assume by way of contradiction that there is an infinite E-chain.then by proposition 1 there is also an infinite E'-chain: C where $E' = (B_{\#} + A)$.

$$C = v_1 E' v_2 E' \ldots$$

if C contains a finite number of $B_{\#}$-steps then it contains an infinite A-Chain and we are done.
if C does not contain an infinite number of $B_{\#}$-steps, then by the jumping property each $B_{\#}A$-step can be replaced with a $B_{\#}$-step(it cannot be replaced with a AE^* step since we assume all A steps leads to mortality.) by preforming this replacement repeatably we get :

proof II

$$
\begin{aligned}
C &= v_1 A^* v_i \underline{B_\# A^n} v_k B_\# \dots \\
&= v_1 A^* v_i B_\# A^{n-1} v_k B_\# \dots \\
&= v_1 A^* v_i B_\# A^{n-2} v_k B_\# \dots \\
\dots &= v_1 A^* v_i B_\# v_k \underline{B_\# A^*} B_\# v_l \dots \\
\dots &= v_1 A^* \underbrace{v_i B_\# v_k B_\# v_l \dots}_{\text{an infinite } B_\#\text{-chain}}
\end{aligned}
$$

and thus we get an infinite $B_\#$-chain in contradiction to the well-foundedness of B.

Theorem 2

if relation A jumps over relation B, escapes from B and is well founded then E is also well founded.

Proof.

the proof is identical to the one presented for Theorem 1 except when we get an infinite $B_{\#}$-chain it implies that ${B_{\#}}^{\infty} \neq \emptyset$ which is a contradiction to the fact that A escapes from B. $\square$

Abstract path ordering

$$t \succ u \quad \text{if} \quad \begin{cases} t \rhd u \quad \text{and} \quad t \rhd^{+} \succ^{*} u, \quad \text{or} & \text{(a)} \\ t \gg u \quad \text{and} \quad t(\rhd^{+} \succ^{*} + \succ)/ \rhd u & \text{(b)} \end{cases}$$

$$R/S = \{\langle x, y\rangle : \forall z. ySz \Rightarrow xRz\}$$

The abstract path ordering is not necessarily an ordering, as it can be non transitive.

Lemma 1

for the path ordering , relation $\rhd$ jumps over $\sqsupset$ where
$\sqsupset := \gg \cap (\rhd^{+}\succ^{*} + \succ)/\rhd$ (case b of the path ordering definition).

Proof.

by the division in (b), $\sqsupset\rhd \subseteq \rhd^{+}\succ^{*} + \succ$. by the definition of $\succ$ we get $\succ \subseteq \rhd^{+}\succ^{*} + \sqsupset$, giving $\sqsupset\rhd \subseteq \rhd^{+}\succ^{*} + \sqsupset$ as required. □

Theorem

the path ordering $\succ$ is transitive if $\gg$ is transitive and $\rhd$ is universal.

Proof.

let $\sqsupset$ be short for $\gg \cap \succ / \rhd$. we proceed by induction with respect to to $\rhd$ in any any of the three positions s,t or u in $s \succ t \succ u$.

1. if $s \rhd s' \succeq t$ then $s' \succ u$ by induction in the first position and $s \succ u$ by definition.
2. if $s \sqsupset t \rhd t' \succeq u$, then $s \succ t' \succeq u$ on account of the division clause and $s \succ u$ by induction in the second position.
3. if $s \sqsupset t \sqsupset u$, then we have $s \gg u$ and $s \succ t' \succ v$ for all $v \lhd u$. by induction in the third position, $s \succ u$ for all $v \lhd u$ from which it follows that $s \sqsupset u$, hence, $s \succ u$.

□

if $\succ$ is transitive then due to sub-term property ($\rhd \subseteq \succ$) we get a much simpler definition to $\succ$.

$$\succ := \rhd \succeq + [\gg \cap \succ / \rhd]$$

Well-Definedness

the following is an alternative mutually-recursive definition of $\succ$, which together with its transitive closure $\succ^*$, can be implemented "bottom-up":

$$\succ \;:=\; (\rhd \cap \rhd^{+}\succ^{*}) + \sqsupset \tag{a'}$$

$$\sqsupset \;:= \gg \cap(\rhd^{+}\succ^{*} + \sqsupset)/\rhd \tag{b'}$$

we can have $\sqsupset$ on the right side of the second line instead of $\succ$ as appears in case (b) of the original definition of $\succ$,since case (a) of $\succ$ is subsumed by the first by the first alternative , $\rhd^{+}\succ^{*}$.

the abstract path ordering may be viewed in the following stratified fashion, with the empty relation serving for the base case:

$$\succ_n \;:=\; (\rhd \cap \rhd^+ \succ_{n-1}^*) + \sqsupset_n + \succ_{n-1} \tag{a''}$$

$$\sqsupset_n \;:= \gg_n \cap (\rhd^+ \succ_{n-1}^* + \sqsupset_{n-1}) / \rhd + \sqsupset_{n-1} \tag{b''}$$

$$\gg_n \;:= \succ_{n-1}^{lex} + \gg_{n-1} \tag{c''}$$

where $\succ_{n-1}^{lex}$ looks at certain $\succ_{n-1}$ relations between $\rhd$-neighbors of the points in question.

Well-Foundedness I

Theorem

A Path ordering $\succ$ is well-founded if $\sqsupset$ is.

Proof.

since $\succ \subseteq (\rhd + \sqsupset)^+$, then by Theorem 1,Lemma 1 and the assumption that $\rhd$ is well founded, we get that $\succ$ is well founded. $\square$

Well-Foundedness II

Theorem

A Path ordering $\succ$ is well-founded if $\triangleright$ escapes from $\sqsupset$.

Proof.

since $\succ \subseteq (\triangleright + \sqsupset)^{+}$, then by Theorem 2,Lemma 1 and the assumption that $\triangleright$ is well founded, we get that $\succ$ is well founded. □