Contents

Part I

Introduction

Orders

Definition. *A* ***Quasi Order*** *is a reflexive and transitive relation.*

Definition. *A set A is Well Quasi Ordered under $\precsim$ if for all infinite sequences from A:*

$$a_1, a_2, a_3, \ldots$$

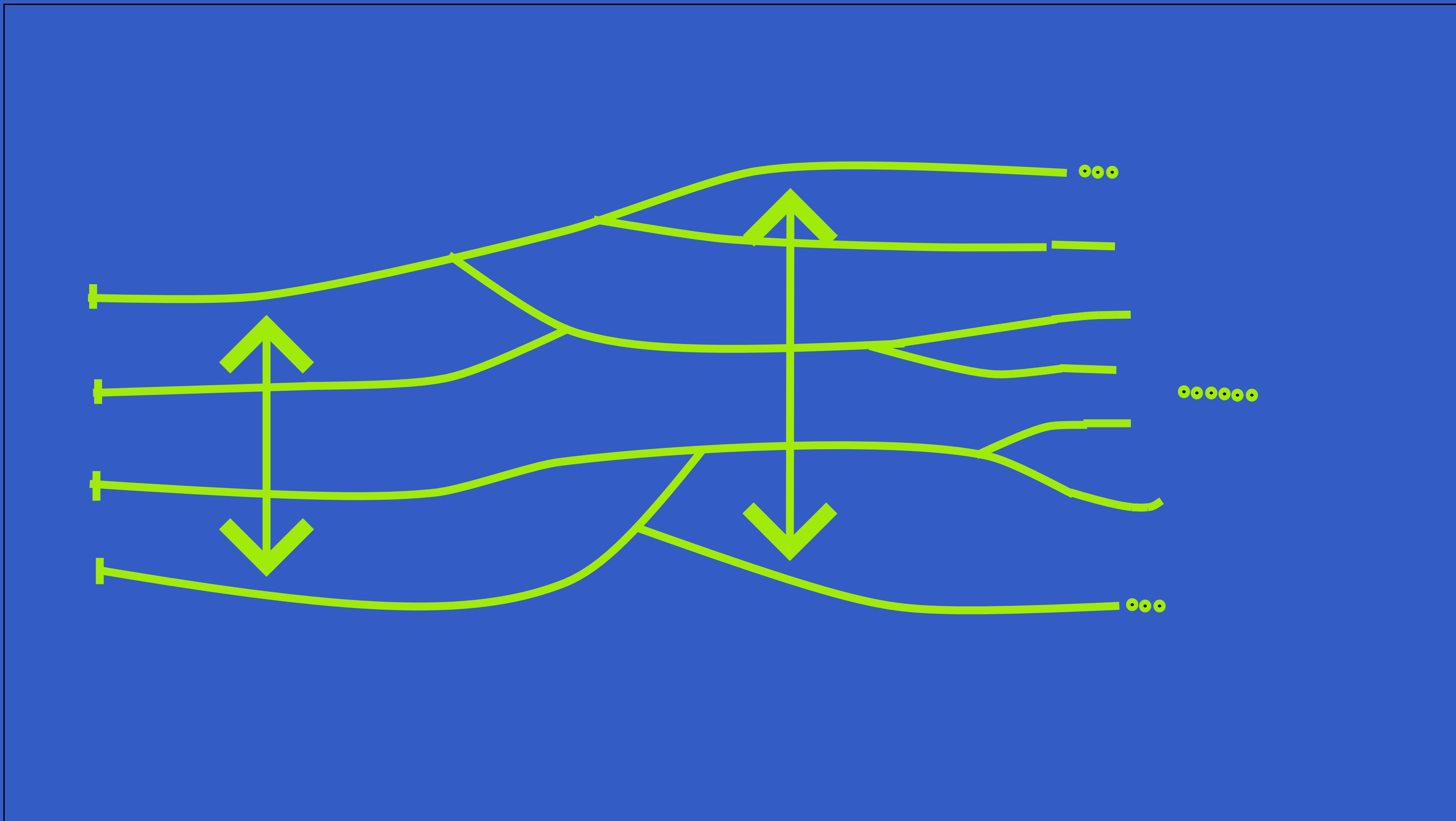
there exists some $i < j$ such that $a_i \precsim a_j$.

Good/Bad Sequences

Definition. *A sequence $a_1, a_2, a_3, \ldots$ s.t. for every $i < j$, $a_i \not\precsim a_j$ holds is called a bad sequence;otherwise called good.*

a_i and a_j are comparable if either $a \precsim b$ or $b \precsim a$; otherwise they are incomparable. If a_i, a_j are incomparable for all i, j then the sequence is an antichain.

Illustration of a well-partial ordering



Equivalent Definitions for WQO

- Q is a WQO.

Equivalent Definitions for WQO

- Q is a WQO.
- Q has no infinite strictly decreasing chains and no infinite antichains (Ramsey).

Equivalent Definitions for WQO

- Q is a WQO.
- Q has no infinite strictly decreasing chains and no infinite antichains (Ramsey).
- Any infinite Q-sequence contains an increasing chain (Ramsey).

Equivalent Definitions for WQO

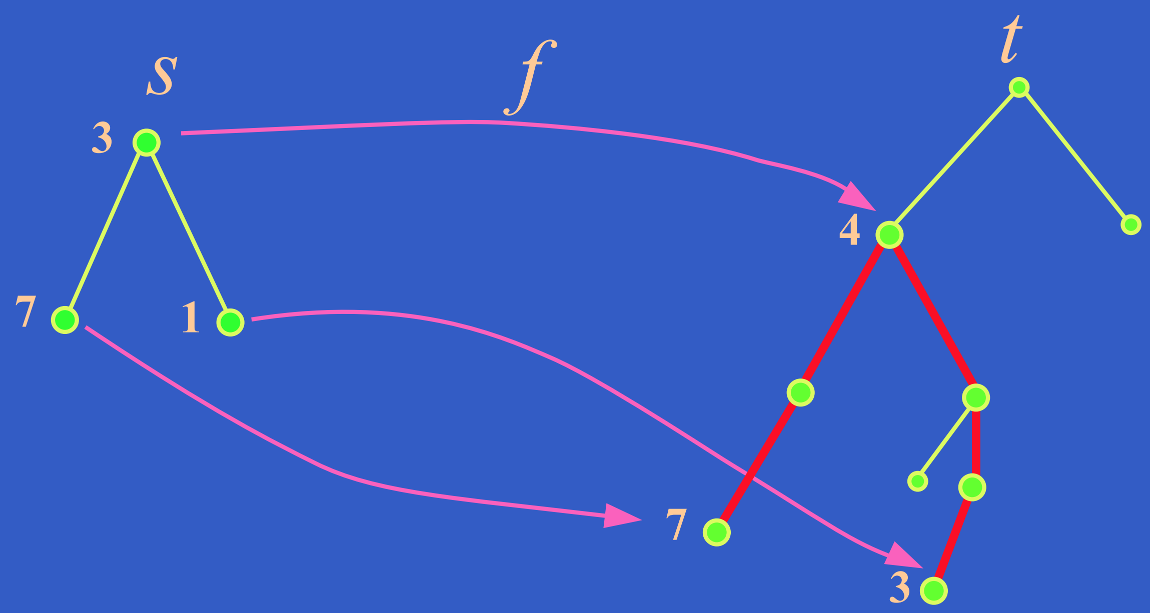
- Q is a WQO.
- Q has no infinite strictly decreasing chains and no infinite antichains (Ramsey).
- Any infinite Q-sequence contains an increasing chain (Ramsey).
- Every linear extension of $\succsim$ on $Q/_{\approx}$ is a well-order.

Tree embedding

Definition. *For two labelled trees s is embedded into t if there is a 1-1 function f, mapping vertices to $\precsim$ vertices and edges to unique disjoint paths.*

Tree embedding

Definition. *For two labelled trees s is embedded into t if there is a 1-1 function f, mapping vertices to $\precsim$ vertices and edges to unique disjoint paths.*



Tree embedding

Formally:

- for all nodes v, u in s,

$$f(v \wedge u) = f(v) \wedge f(u) \ ,$$

where $a \wedge b$ denotes the closest common ancestor of a, b

- $$v \precsim f(v)$$

Kruskal's Theorem

Theorem. (Kruskal 1960) *Finite trees are wqo under the embedding relation.*

Kruskal's Theorem

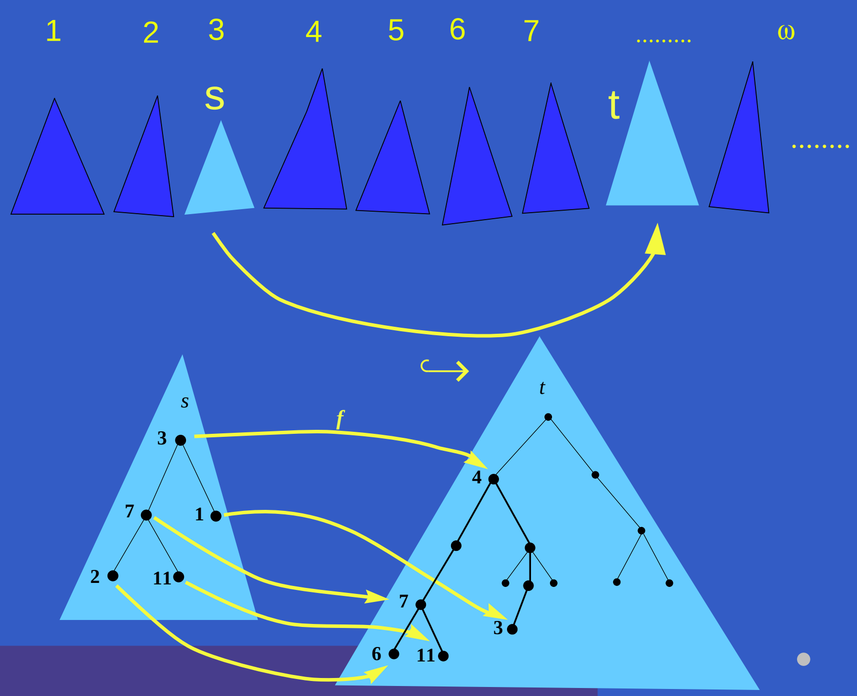
Theorem. (Kruskal 1960) *Finite trees are wqo under the embedding relation.*

Hence, every infinite sequence of trees is a good sequence.

Kruskal's Theorem

Theorem. (Kruskal 1960) *Finite trees are wqo under the embedding relation.*

Hence, every infinite sequence of trees is a good sequence.



Gap Embedding

Definition. (Gap embedding) s *is* embedded with gaps *in* t *(denoted* $s \hookrightarrow t$*) if there is an embedding* $f : s \to t$ *satisfying:*

Gap Embedding

Definition. (Gap embedding) s *is* embedded with gaps *in* t *(denoted* $s \hookrightarrow t$*) if there is an embedding* $f : s \to t$ *satisfying:*

Gap condition: For all edges (u, v) in s and for all w in the path from $f(u)$ to $f(v)$ in t, $w \gtrsim v$;

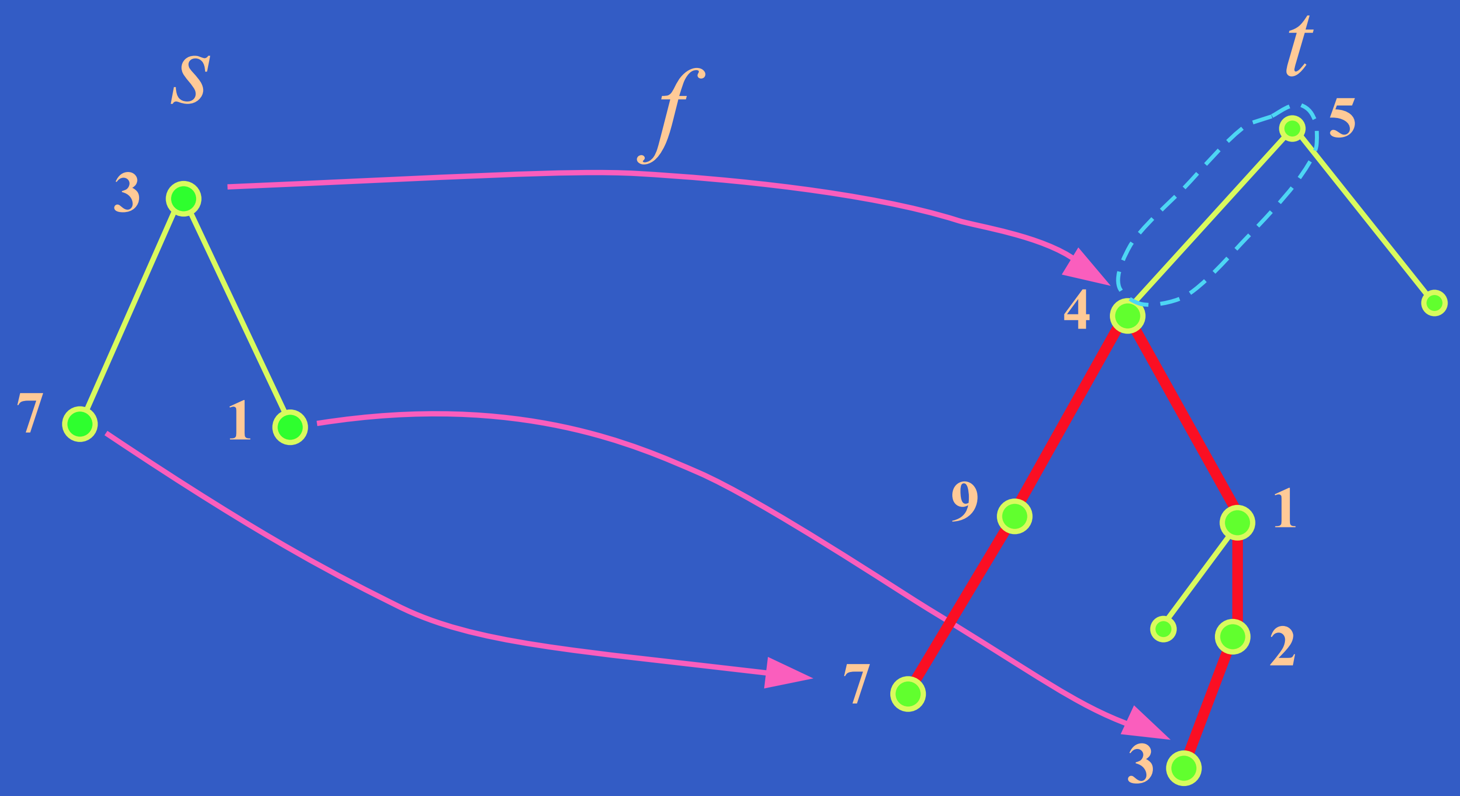
Gap Embedding

Definition. (Gap embedding) s *is* embedded with gaps *in* t *(denoted* $s \hookrightarrow t$*) if there is an embedding* $f : s \rightarrow t$ *satisfying:*

Gap condition: For all edges (u, v) in s and for all w in the path from $f(u)$ to $f(v)$ in t, $w \gtrsim v$;

Root gap condition: $u \gtrsim r(s)$ for all vertices u in the access path from $t^\bullet$ to $f(s^\bullet)$.

Gap Embedding



Known Results

Theorem ([Kříž '89]) *The set of finite trees with ordinal labels is a wqo under gap embedding.*

(Proved a similar result for infinite trees ['95 Kříž].)

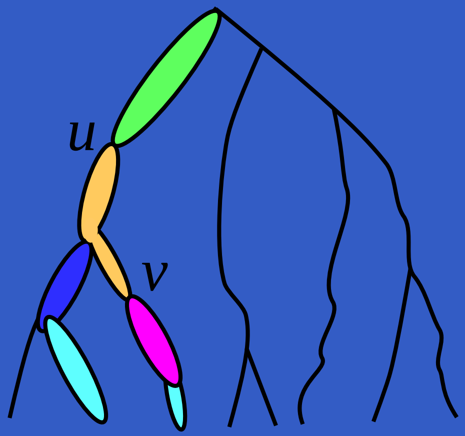
Our Result

Definition. *Given a tree path* $[u, v]$, *we say that the path is* comparable *if all the vertices in it have comparable labels, that is,*
$\forall x, y \in [u, v].\, x \precsim y \vee y \precsim x$.

Theorem. *The set of finite trees with well quasi ordered labels, with each node comparable to its ancestors, is a wqo under gap embedding.*

Our Result

Let Q be a wqo and let T_k be the set of all trees such that each path in a tree can be partitioned into some fixed $k \in \mathbb{N}$ or less comparable sub-paths.



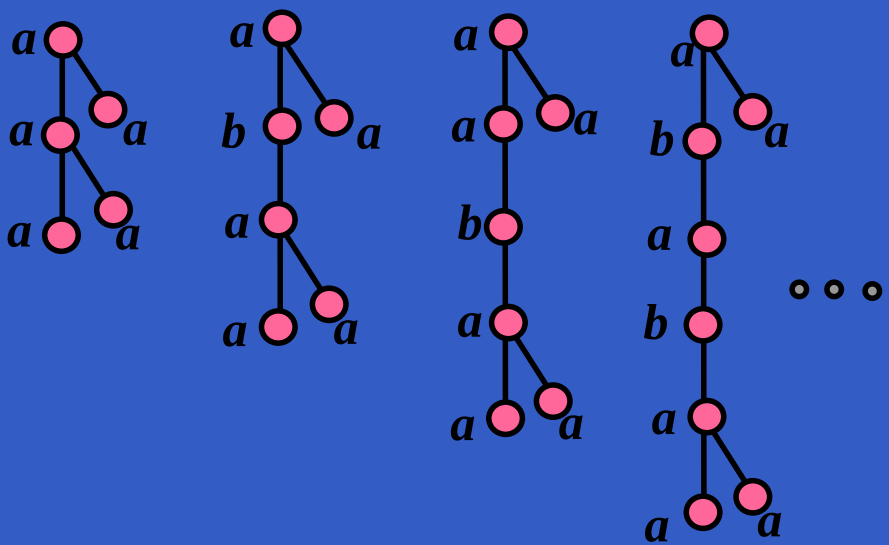
Our Result

Main Theorem *T_k is wqo under gap embedding.*

This is an optimal setting for partiality on nodes ordering:

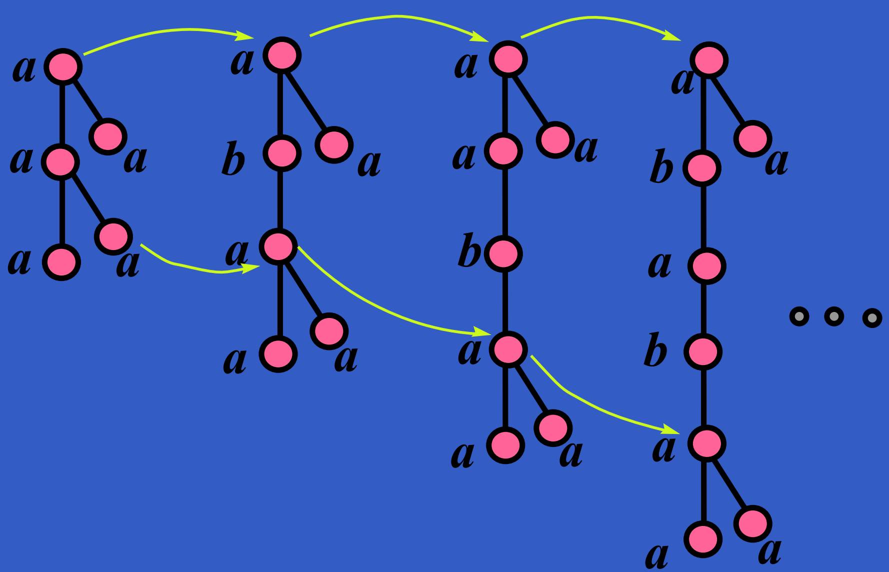
Our Result

Proposition. *If the node ordering is not total then finite trees are not wqo under gap embedding.*



Our Result

Proposition. *If the node ordering is not total then finite trees are not wqo under gap embedding.*



Our Result

Proposition. *If the node ordering is not total then finite trees are not wqo under gap embedding.*

Theorem proves that this is the canonic counter-example.

Part II

Kruskal's Theorem — Proof Idea

The Proof of Kruskal's Theorem

Theorem. (Kruskal 1960) *Finite trees are wqo under the embedding relation.*

The Proof of Kruskal's Theorem

Theorem. (Kruskal 1960) *Finite trees are wqo under the embedding relation.*

Kruskal's original proof ('60): Long, constructive

The Proof of Kruskal’s Theorem

Theorem. (Kruskal 1960) *Finite trees are wqo under the embedding relation.*

Kruskal’s original proof (’60): Long, constructive

We shall see:

Nash-Williams’s proof (’63): Short, simple, non-constructive

Higman's Lemma

Definition. *Let Q be a quasi order. We denote by $[Q]^{<\omega}$ the set of all finite sequences from Q.*

Higman's Lemma

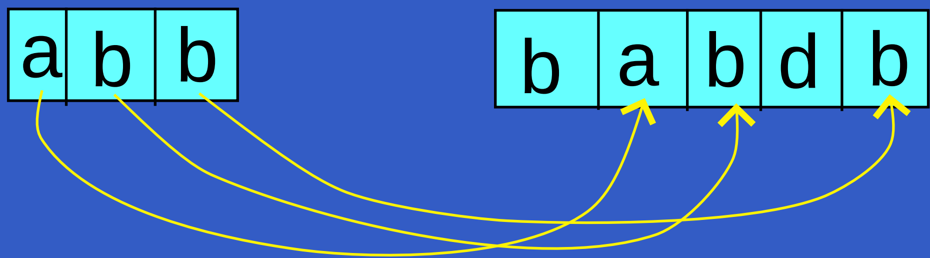
Definition. *Let Q be a quasi order. We denote by $[Q]^{<\omega}$ the set of all finite sequences from Q.*

For $s, t \in [Q]^{<\omega}$ we say that there is an embedding of s into t if there is a 1-1 mapping f from s into t such that for all $s_i \in s$, $s_i \precsim f(s_i)$.

Higman's Lemma

Definition. *Let Q be a quasi order. We denote by $[Q]^{<\omega}$ the set of all finite sequences from Q.*

For $s, t \in [Q]^{<\omega}$ we say that there is an embedding of s into t if there is a 1-1 mapping f from s into t such that for all $s_i \in s$, $s_i \precsim f(s_i)$.



Higman's Lemma

Theorem. (Higman '52) *: If Q is wqo then $[Q]^{<\omega}$ is wqo too.*

Higman's Lemma

Theorem. (Higman '52) *: If Q is wqo then $[Q]^{<\omega}$ is wqo too.*

Proof: By a Minimal Bad Sequence method.

Higman's Lemma: Proof

Assume there is a bad sequence.

Higman's Lemma: Proof

Assume there is a bad sequence.
Take a minimal bad sequence, w.r.t. the length of each string (finite sequence).

Higman's Lemma: Proof

Assume there is a bad sequence.
Take a minimal bad sequence, w.r.t. the length of each string (finite sequence).

T:= a , b

Higman's Lemma: Proof

Assume there is a bad sequence.
Take a minimal bad sequence, w.r.t. the length of each string (finite sequence).

T:= a , b a,c,d

Higman's Lemma: Proof

Assume there is a bad sequence.
Take a minimal bad sequence, w.r.t. the length of each string (finite sequence).

T:= a , b a,c,d e, e, f

Higman's Lemma: Proof

Assume there is a bad sequence.
Take a minimal bad sequence, w.r.t. the length of each string (finite sequence).

T:= [a , b] [a,c,d] [e, e, f] [a, f, d, a]

Higman's Lemma: Proof (cont.)

Take off the first element of each string (T is bad $\Rightarrow$ there is no empty string in T). Denote by S.

T:= a , b a,c,d e, e, f a, f, d, a

Higman's Lemma: Proof (cont.)

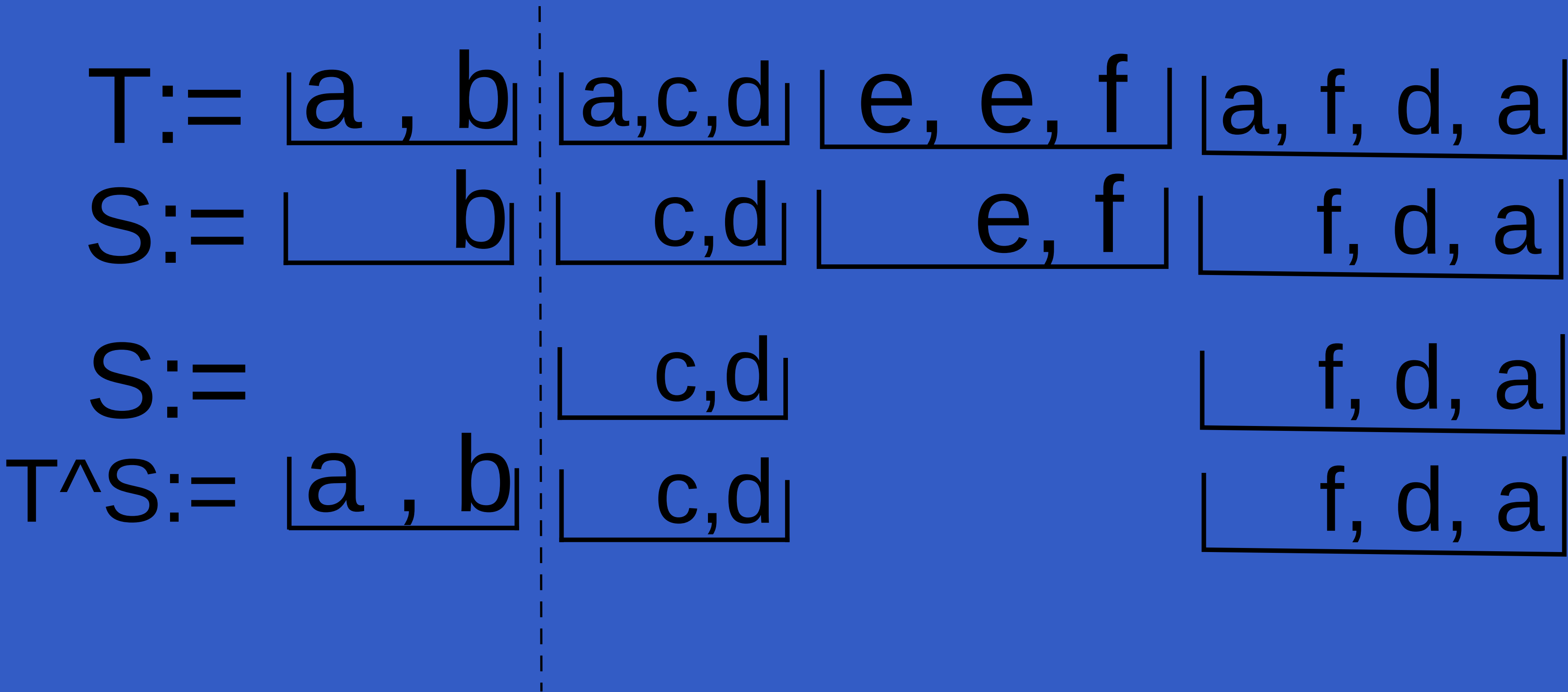
Take off the first element of each string (T is bad $\Rightarrow$ there is no empty string in T). Denote by S.

Higman's Lemma: Proof (cont.)

S contains no infinite bad subsequence S', or else contradict minimality of T:

Higman's Lemma: Proof (cont.)

S contains no infinite bad subsequence S', or else contradict minimality of T:



Higman's Lemma: Proof (cont.)

T contains an infinite subsequence with increasing heads.

Higman's Lemma: Proof (cont.)

T contains an infinite subsequence with increasing heads. Since S contains no infinite bad subsequence, the corresponding subsequence of S contains an embedded pair:

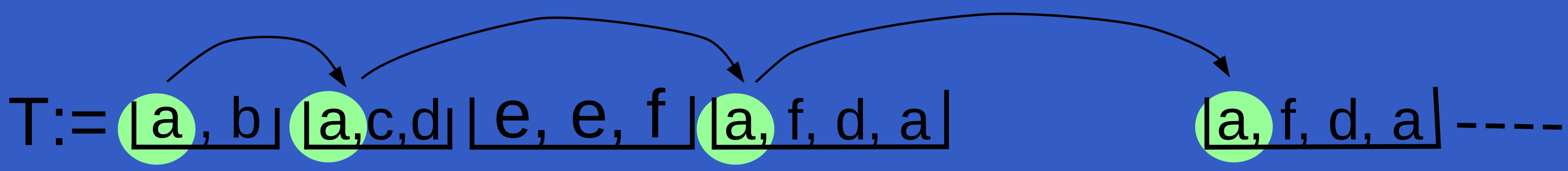
Higman's Lemma: Proof (cont.)

T contains an infinite subsequence with increasing heads. Since S contains no infinite bad subsequence, the corresponding subsequence of S contains an embedded pair:

T:= [a , b] [a,c,d] [e, e, f] [a, f, d, a] [a,e, f, g] [a, f, d, a] ----

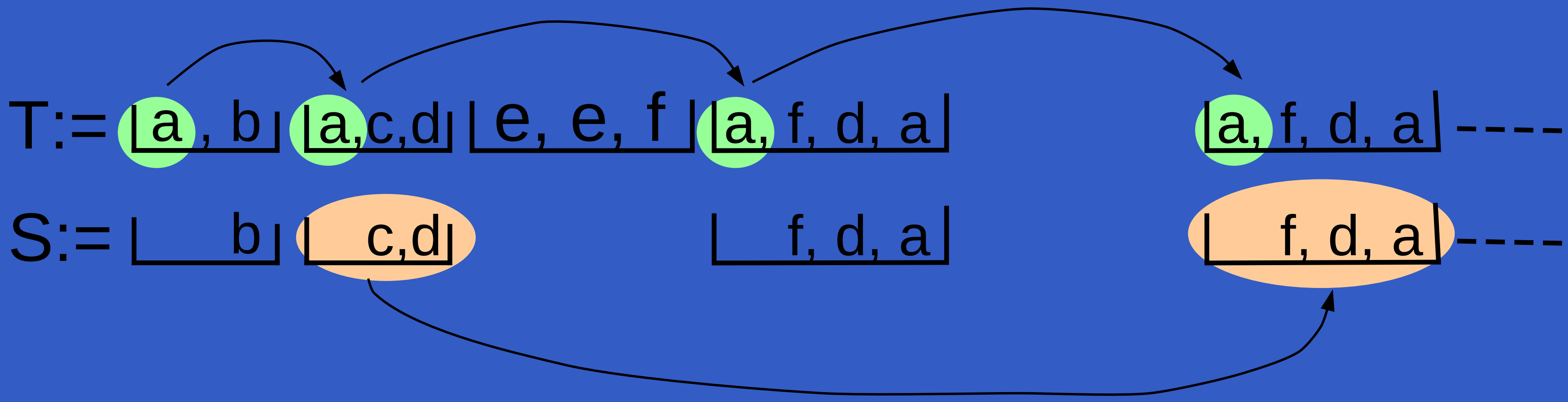
Higman's Lemma: Proof (cont.)

T contains an infinite subsequence with increasing heads. Since S contains no infinite bad subsequence, the corresponding subsequence of S contains an embedded pair:



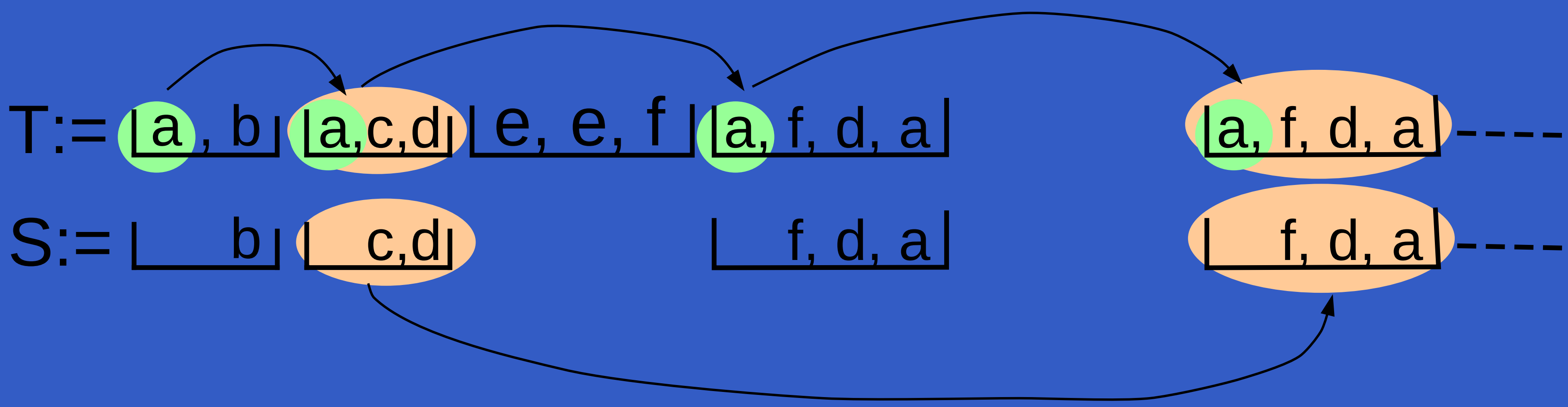
Higman's Lemma: Proof (cont.)

T contains an infinite subsequence with increasing heads. Since S contains no infinite bad subsequence, the corresponding subsequence of S contains an embedded pair:



Higman's Lemma: Proof (cont.)

T contains an infinite subsequence with increasing heads. Since S contains no infinite bad subsequence, the corresponding subsequence of S contains an embedded pair:



Higman's Lemma: Proof (cont.)

T contains an infinite subsequence with increasing heads. Since S contains no infinite bad subsequence, the corresponding subsequence of S contains an embedded pair:

contradiction to T's badness.

$\square$

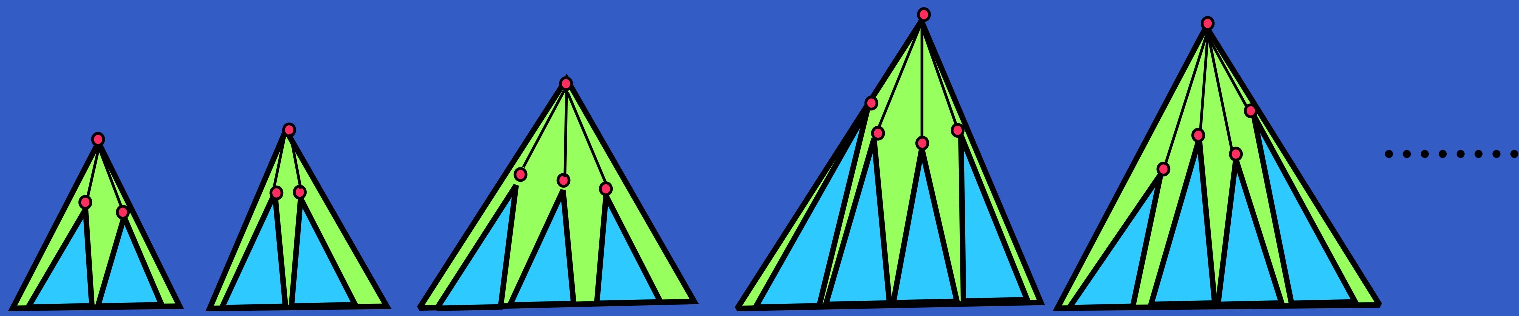
Kruskal’s Theorem: Proof Idea

Find a Minimal Bad Sequence: No subsequence of its children is bad!
Minimality w.r.t sizes of trees suffices.

Kruskal’s Theorem: Proof Idea

Find a Minimal Bad Sequence: No subsequence of its children is bad!
Minimality w.r.t sizes of trees suffices.

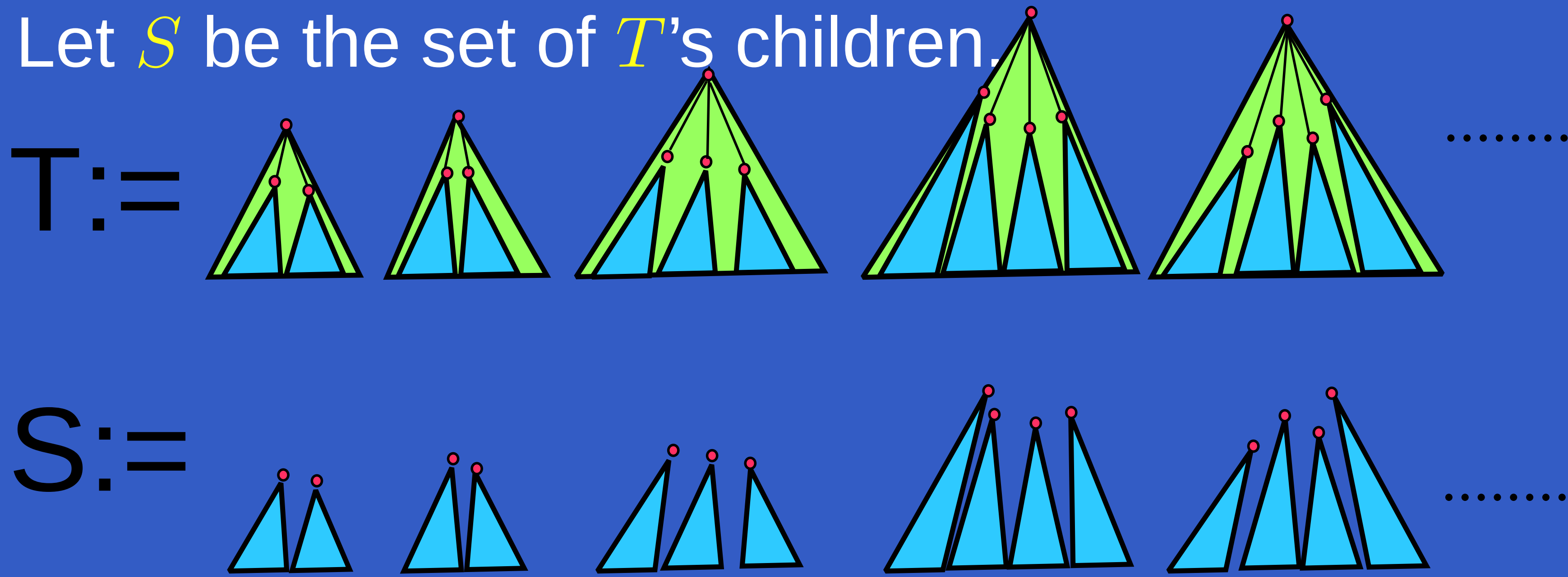
T:=



Kruskal's Theorem: Proof Idea

Find a Minimal Bad Sequence: No subsequence of its children is bad!
Minimality w.r.t sizes of trees suffices.

Let S be the set of T's children.

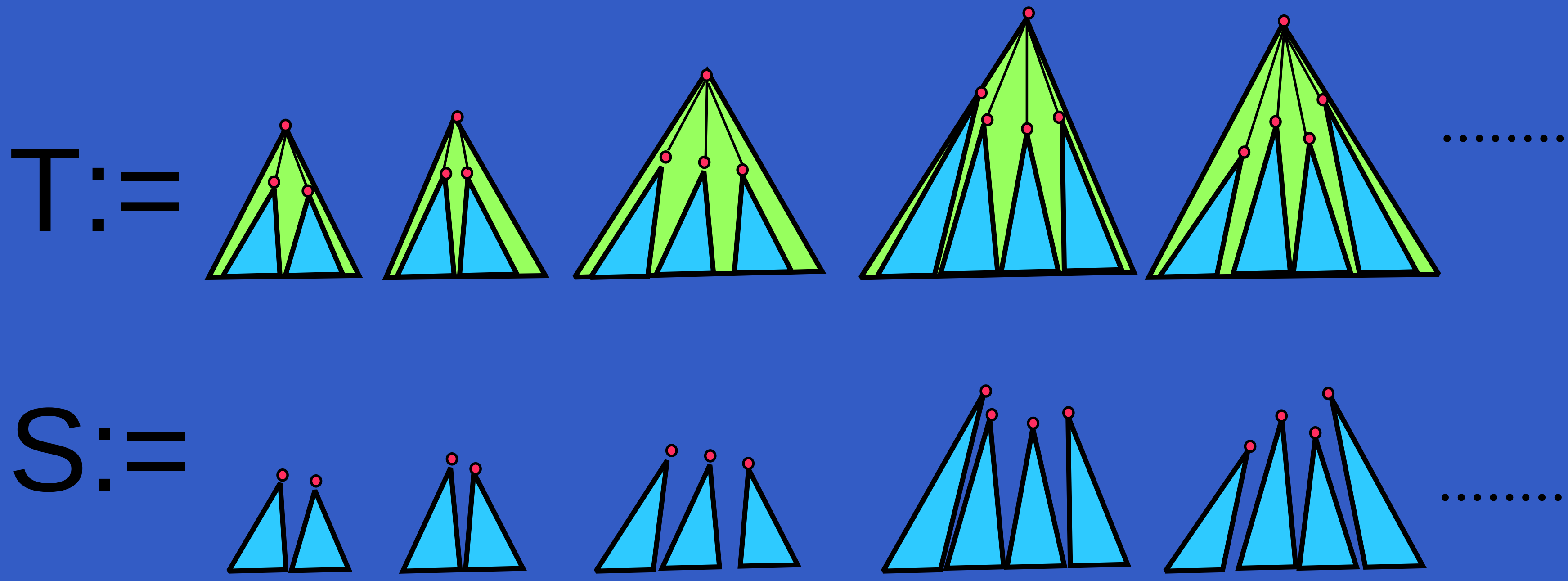


Kruskal's Theorem: Proof Idea (cont.)

No subsequence of children is bad: Otherwise there would be a bad sequence $S' \subseteq S$ appended with T's prefix: Contradict minimality of T.

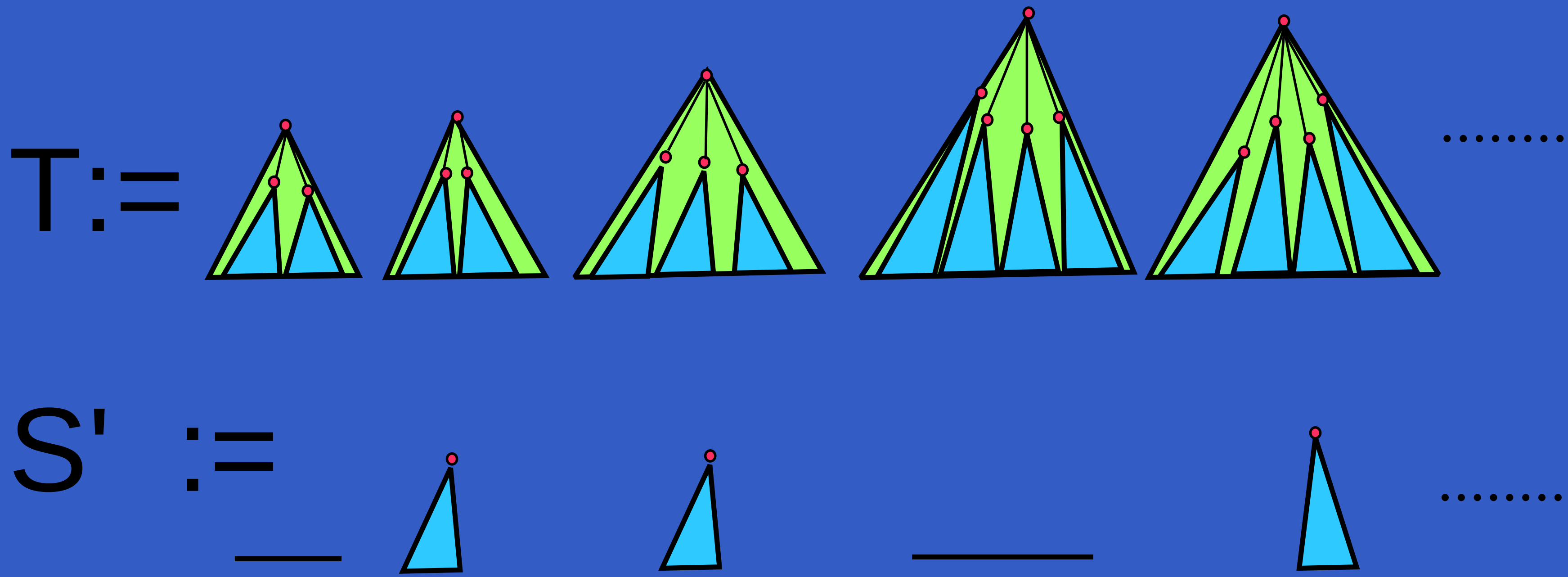
Kruskal's Theorem: Proof Idea (cont.)

No subsequence of children is bad: Otherwise there would be a bad sequence $S' \subseteq S$ appended with T's prefix: Contradict minimality of T.



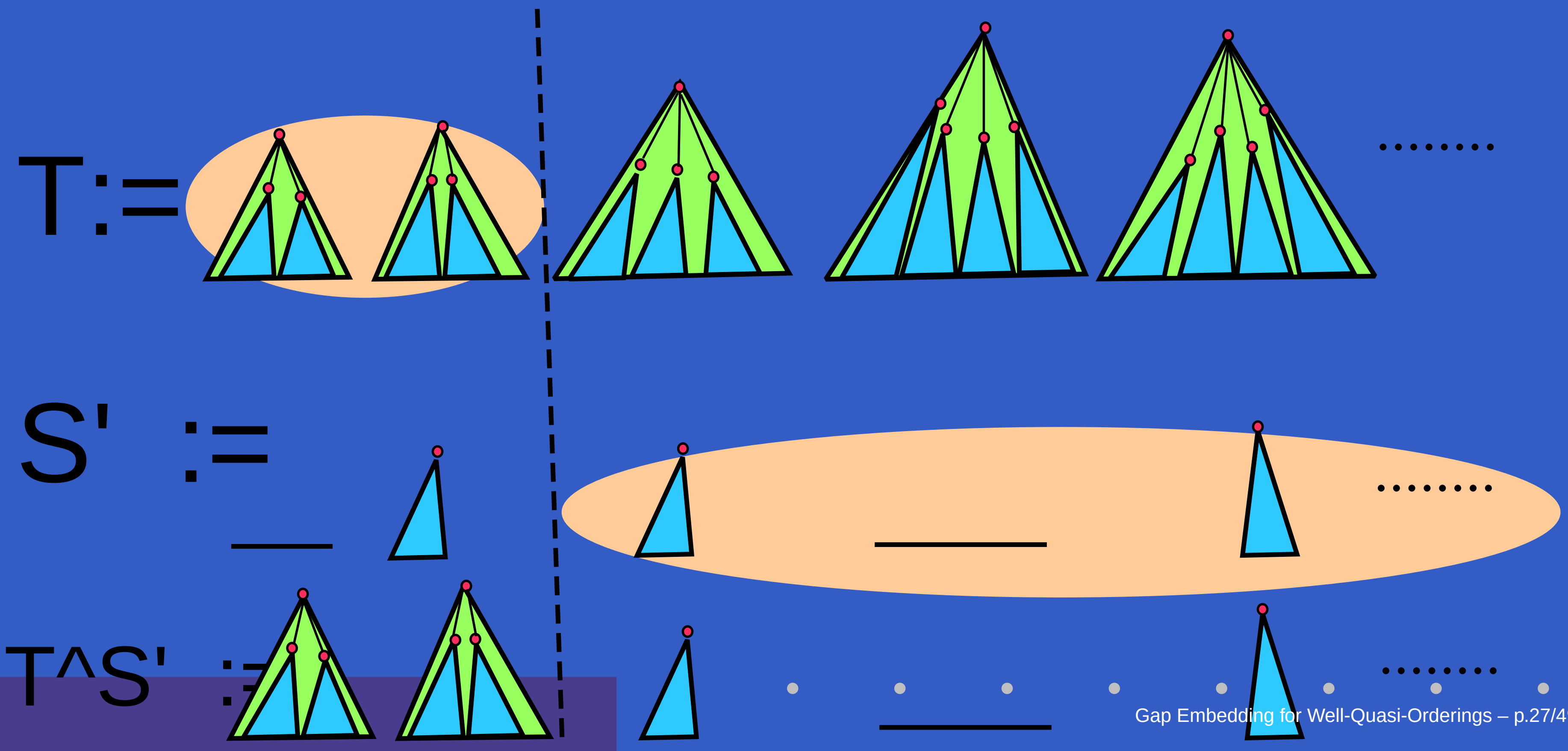
Kruskal's Theorem: Proof Idea (cont.)

No subsequence of children is bad: Otherwise there would be a bad sequence $S' \subseteq S$ appended with T's prefix: Contradict minimality of T.



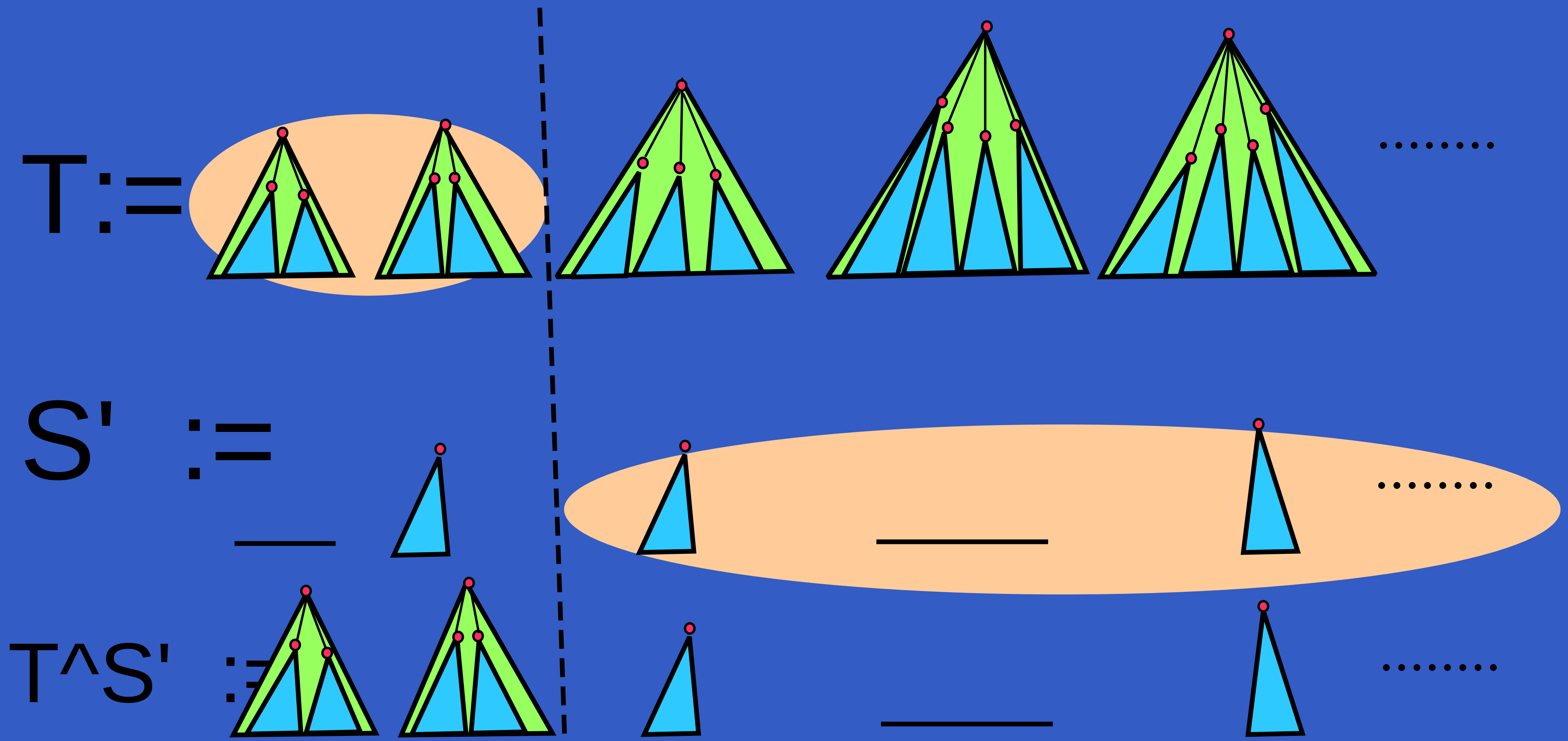
Kruskal's Theorem: Proof Idea (cont.)

No subsequence of children is bad: Otherwise there would be a bad sequence $S' \subseteq S$ appended with T's prefix: Contradict minimality of T.



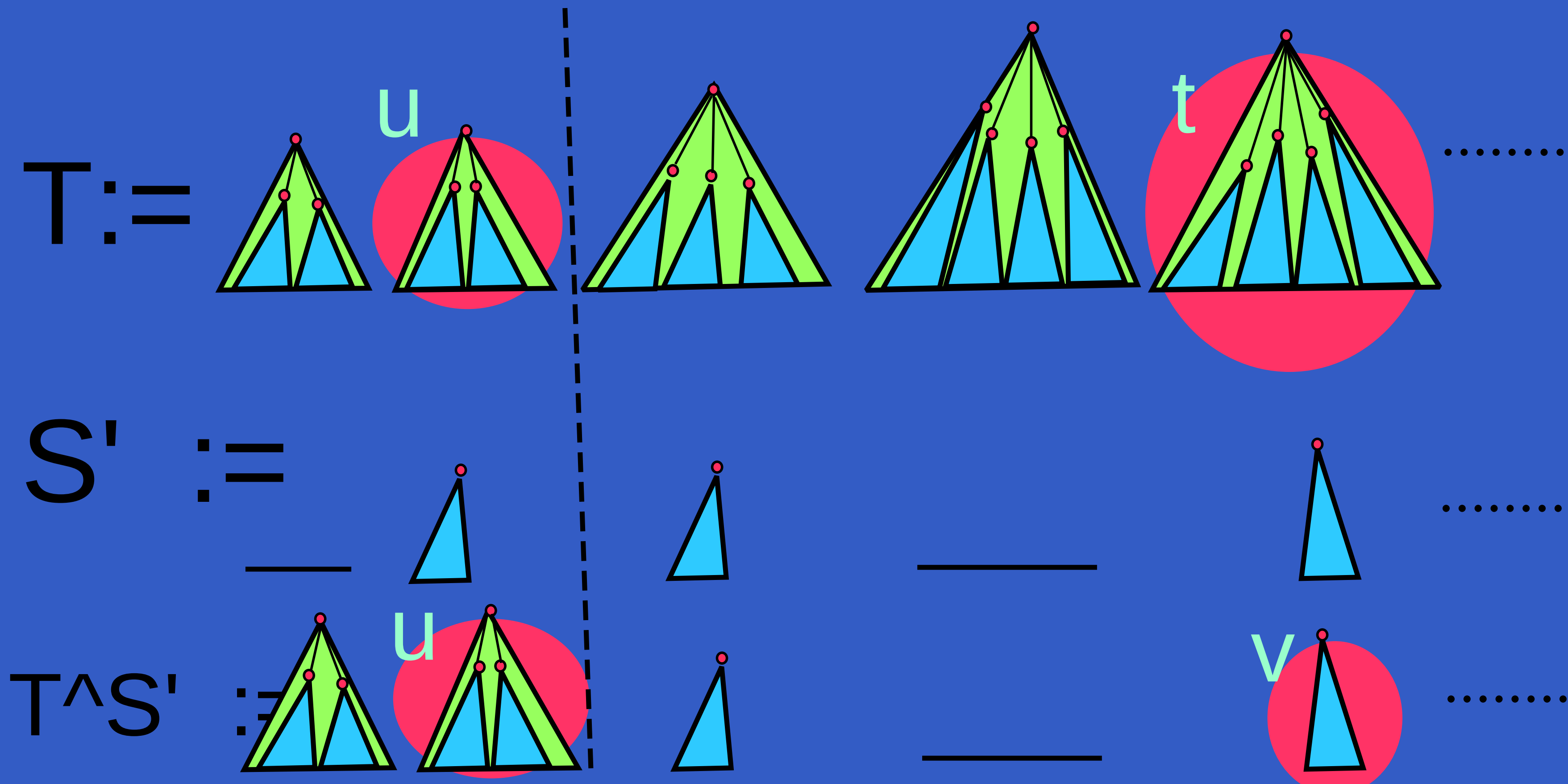
Kruskal's Theorem: Proof Idea (cont.)

Note that u embedded in $v \implies u$ embedded in t. Contradicts badness of T. Thus, S is WQO.



Kruskal's Theorem: Proof Idea (cont.)

Note that u embedded in v $\implies$ u embedded in t. Contradicts badness of T. Thus, S is WQO.



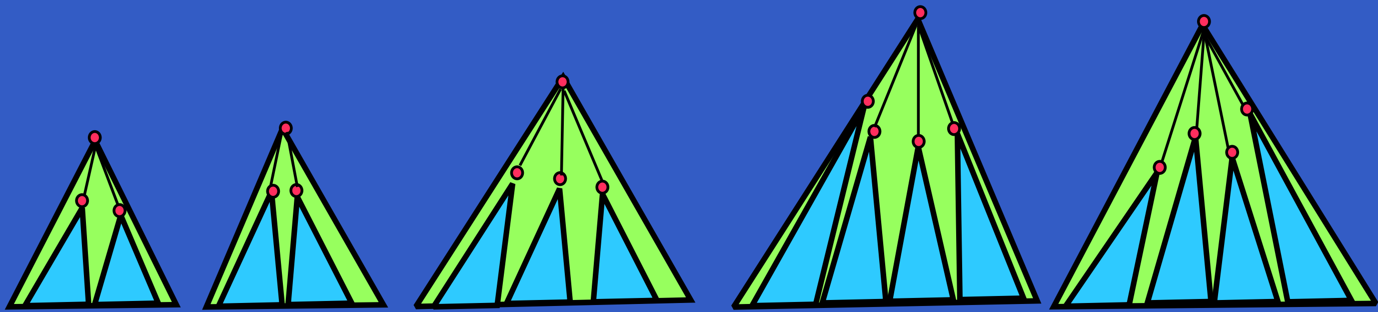
Kruskal's Theorem: Proof Idea (cont.)

Let $T' \subseteq T$ be root increasing and $S' \subseteq S$ the corresponding sequence of children.

Kruskal's Theorem: Proof Idea (cont.)

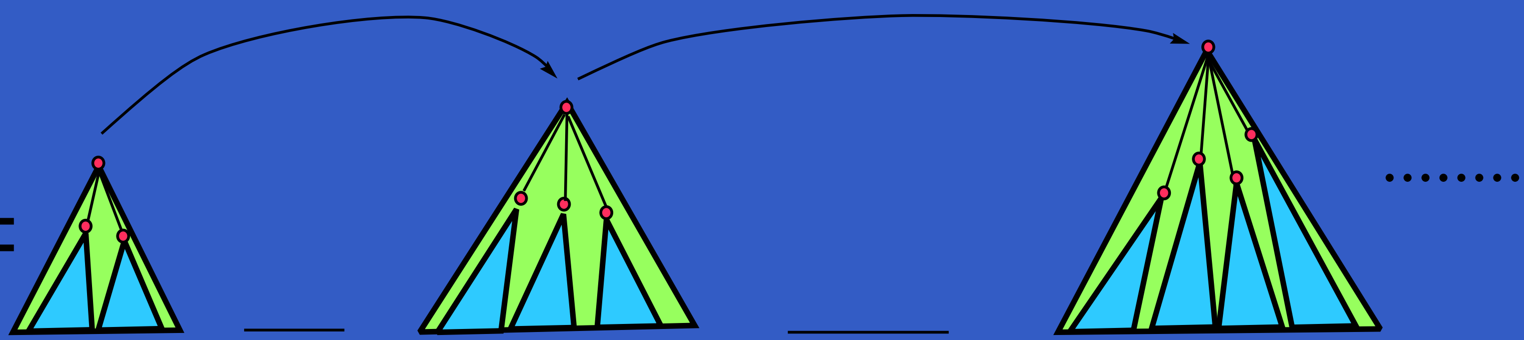
Let $T' \subseteq T$ be root increasing and $S' \subseteq S$ the corresponding sequence of children.

S is wqo $\Rightarrow$ Higman's Lemma yields S' contains an embedded pair.

$T :=$  ....

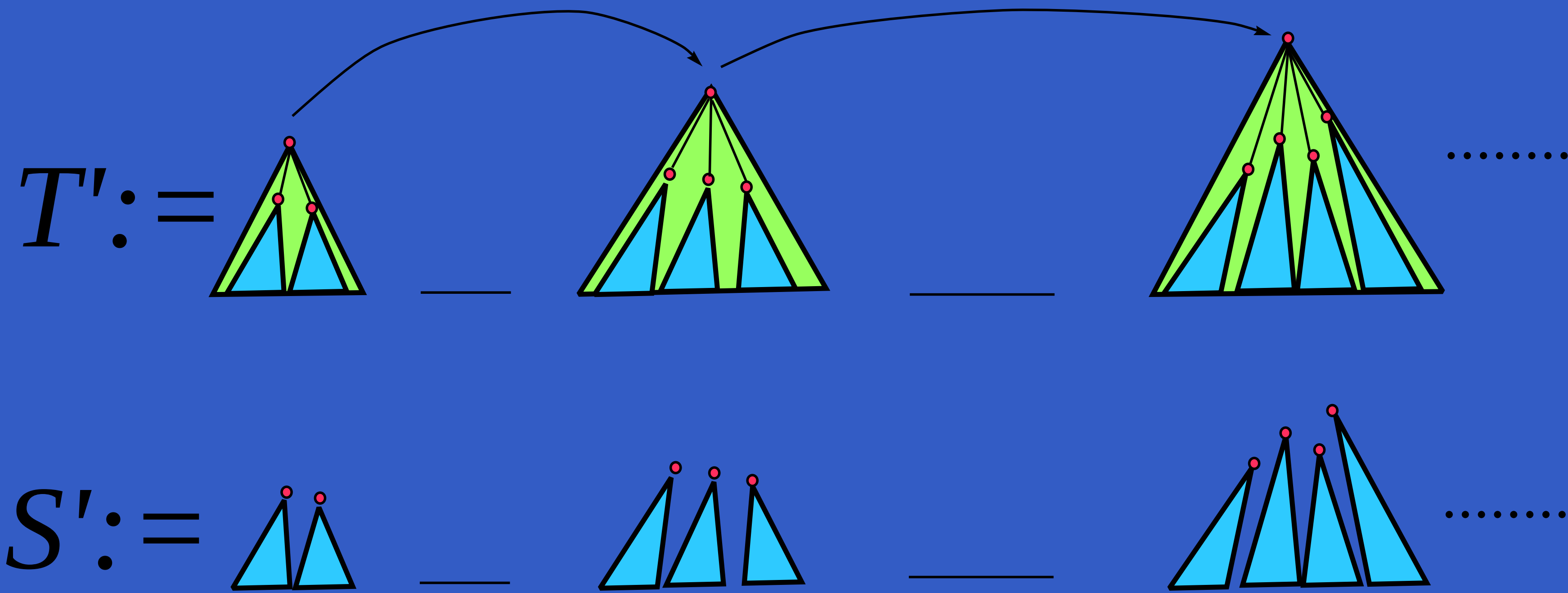
Kruskal's Theorem: Proof Idea (cont.)

Let $T' \subseteq T$ be root increasing and $S' \subseteq S$ the corresponding sequence of children.
S is wqo $\Rightarrow$ Higman's Lemma yields S' contains an embedded pair.

$T' :=$ 

Kruskal's Theorem: Proof Idea (cont.)

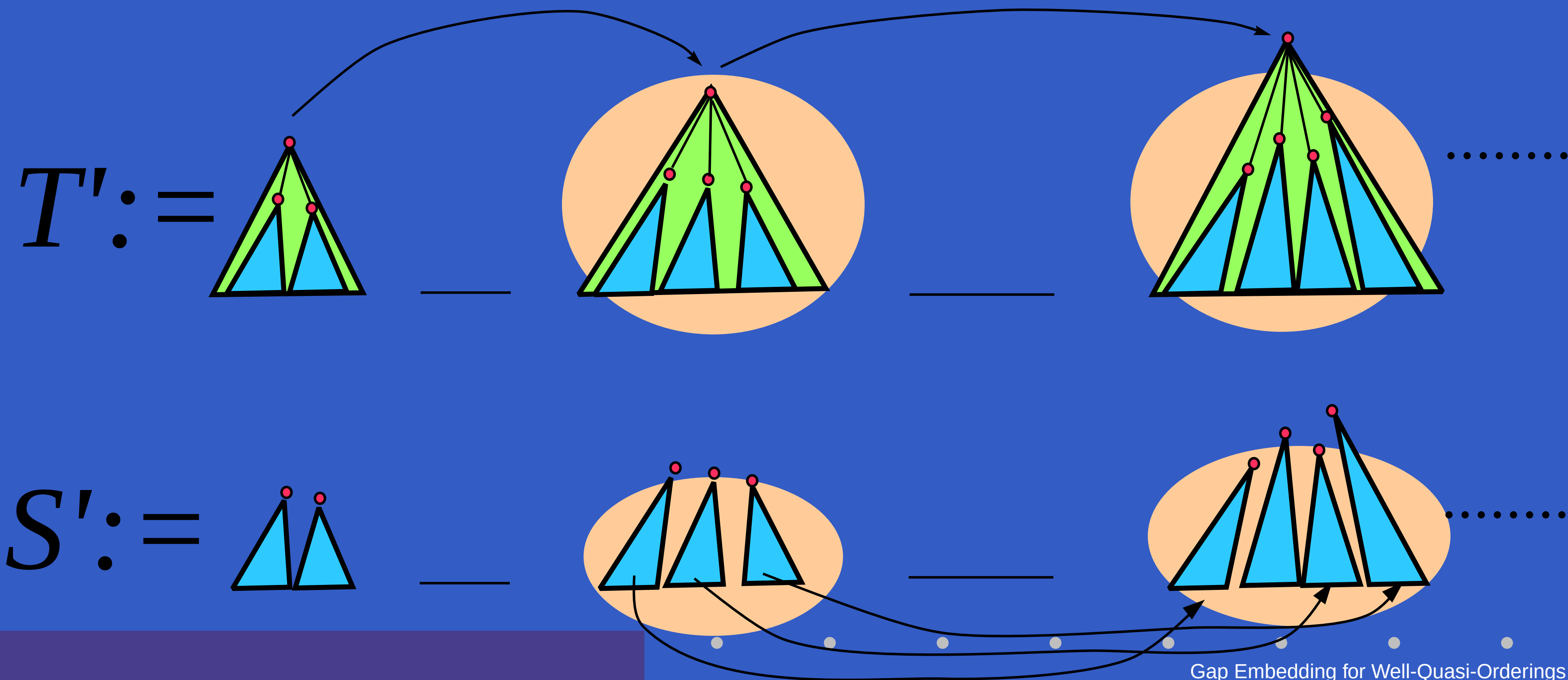
Let $T' \subseteq T$ be root increasing and $S' \subseteq S$ the corresponding sequence of children.
S is wqo $\Rightarrow$ Higman's Lemma yields S' contains an embedded pair.



Kruskal's Theorem: Proof Idea (cont.)

Let $T' \subseteq T$ be root increasing and $S' \subseteq S$ the corresponding sequence of children.

S is wqo $\Rightarrow$ Higman's Lemma yields S' contains an embedded pair.



Kruskal's Theorem: Proof Idea (cont.)

Let $T' \subseteq T$ be root increasing and $S' \subseteq S$ the corresponding sequence of children.
S is wqo $\Rightarrow$ Higman's Lemma yields S' contains an embedded pair.

Contradiction to $T's$ badness!

□

Part III

Gap Embedding

Gap Embedding

Definition. (Gap embedding) s *is* embedded with gaps *in* t *(*$s \hookrightarrow t$*) if there is an embedding* $f : s \rightarrow t$ *satisfying:*

Gap Embedding

Definition. (Gap embedding) s *is* embedded with gaps *in* t *(*$s \hookrightarrow t$*) if there is an embedding* $f : s \to t$ *satisfying:*

Gap condition: For all edges (u, v) in s and for all w in the path from $f(u)$ to $f(v)$ in t, $w \gtrsim v$;

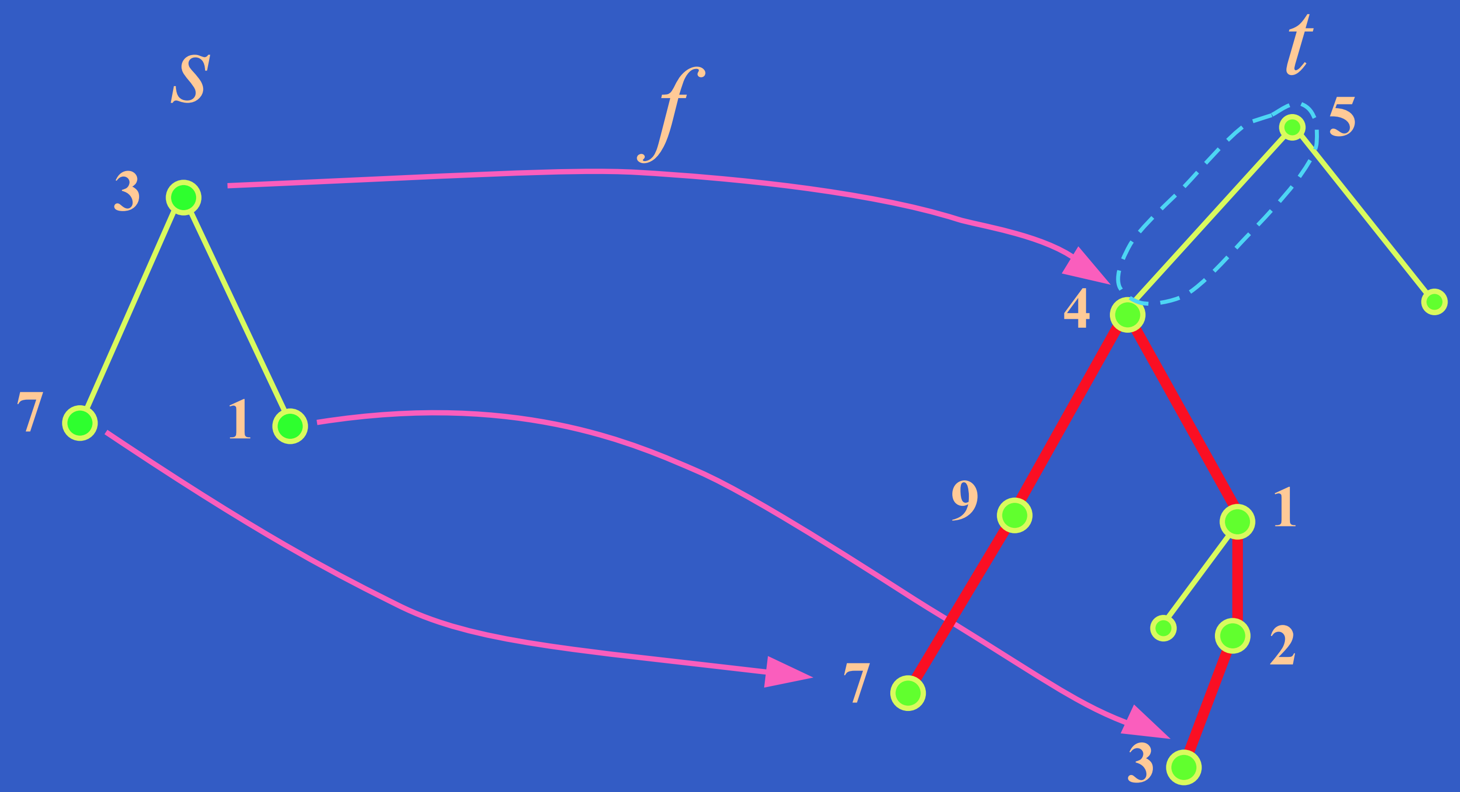
Gap Embedding

Definition. (Gap embedding) s *is* embedded with gaps *in* t *(*$s \hookrightarrow t$*) if there is an embedding* $f : s \to t$ *satisfying:*

Gap condition: For all edges (u, v) in s and for all w in the path from $f(u)$ to $f(v)$ in t, $w \gtrsim v$;

Root gap condition: $u \gtrsim r(s)$ for all vertices u in the access path from $t^\bullet$ to $f(s^\bullet)$.

Gap Embedding



Gap Embedding

Theorem. *The set of finite trees with ordinal labels are wqo under gap embedding.*

Proof method: again, Nash-Williams.

The problem: Construction of a minimal bad sequence.

Gap Embedding: Proof Idea

The problem: Construction of a minimal bad sequence.

Minimality w.r.t size of trees won't work:

Gap Embedding: Proof Idea

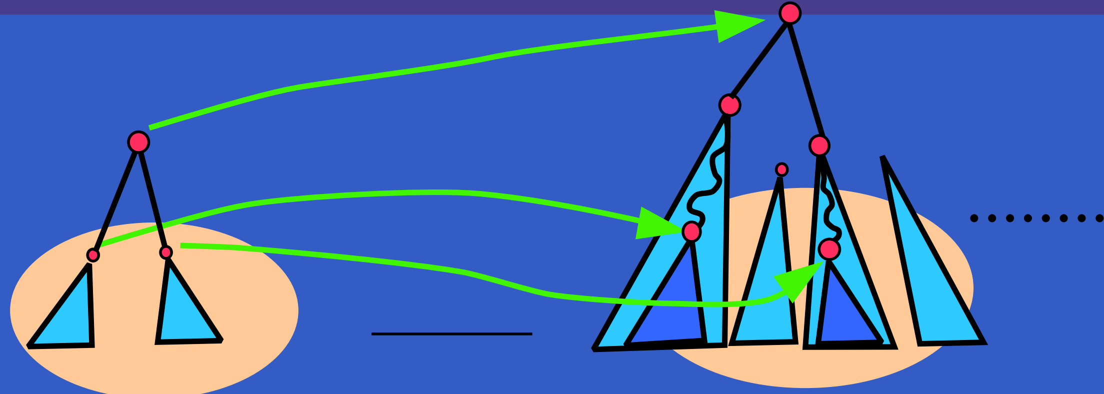
The problem: Construction of a minimal bad sequence.

Minimality w.r.t size of trees won't work:

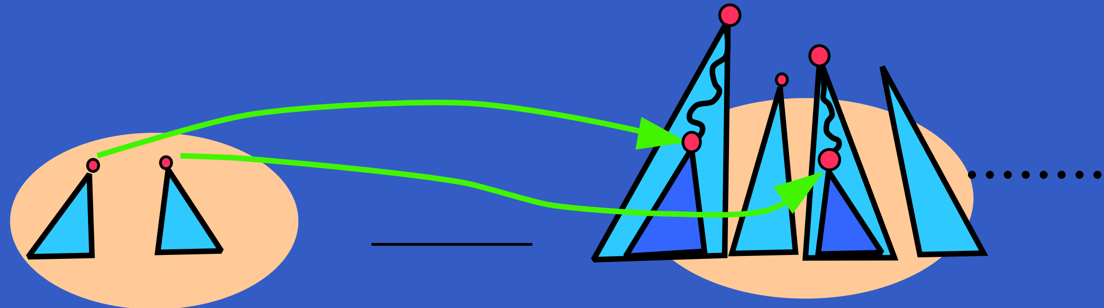
- We need the root gap condition for induction: "Higman's stage" maps root to root.
- Thus, we can't prove that S the set of children of the minimal sequence T is WQO: T^S' not necessarily bad.

Gap Embedding: Proof Idea

T' :=



S' :=



Gap Embedding: Proof Idea

Solution: Prove the existence of a **minimal** bad sequence h, i.e. with no bad children subsequences directly.

Gap Embedding: Proof Idea

Solution: Prove the existence of a **minimal** bad sequence h, i.e. with no bad children subsequences directly.

- By inductive construction of bad sequences of children, this process must end sometime via a cardinality argument $\Longrightarrow$

Gap Embedding: Proof Idea

Solution: Prove the existence of a **minimal** bad sequence h, i.e. with no bad children subsequences directly.

- By inductive construction of bad sequences of children, this process must end sometime via a cardinality argument $\Longrightarrow$
- There exists a bad sequence h with NO bad subsequence S.

Gap Embedding: The Construction

Assume the process doesn't end, i.e. no minimal bad sequence. Build by transfinite induction an infinite table H of (presumable) size $\omega_1 \times \omega$:

Gap Embedding: The Construction

Assume the process doesn't end, i.e. no minimal bad sequence. Build by transfinite induction an infinite table H of (presumable) size $\omega_1 \times \omega$:

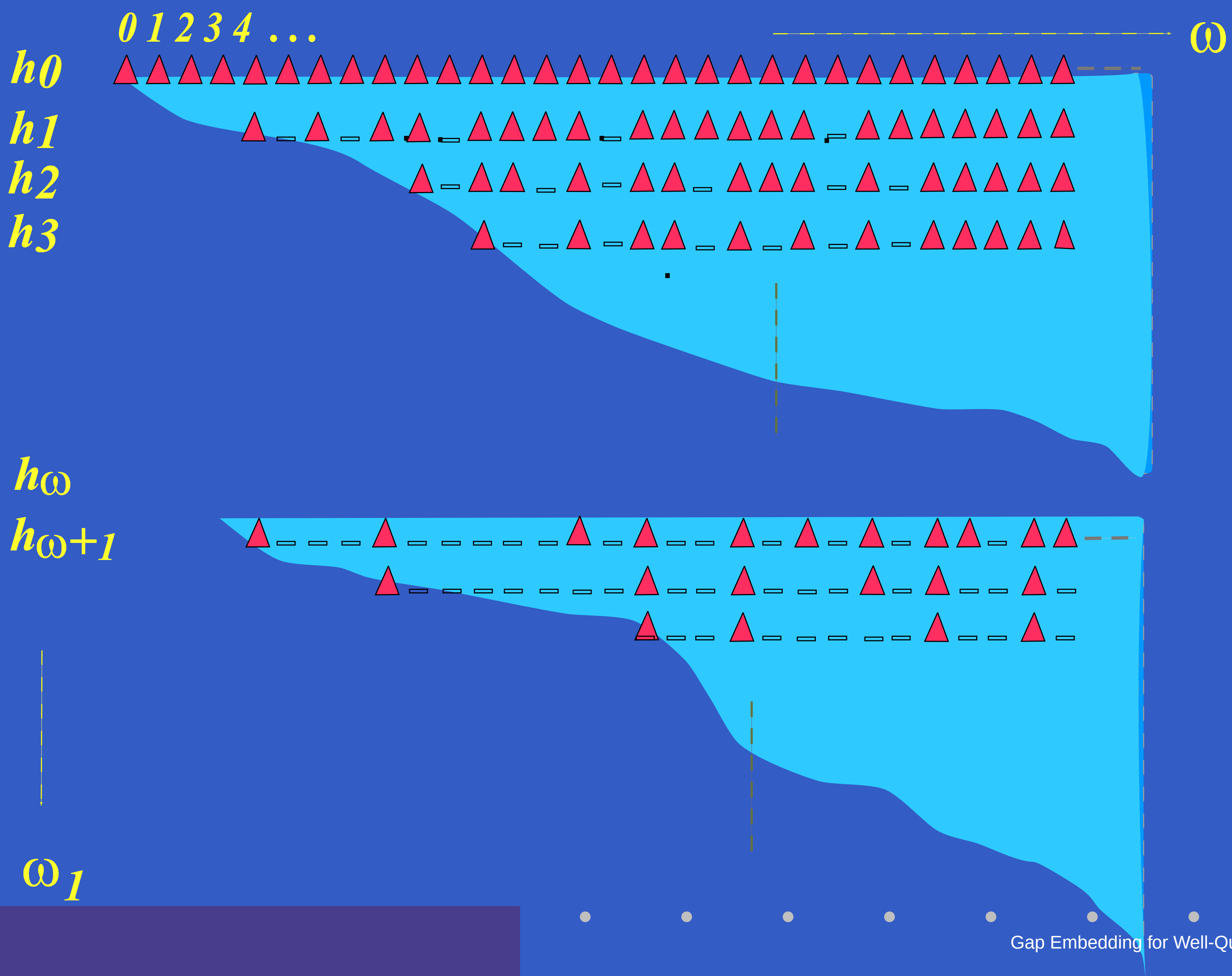
- Each row h_α is a bad sequence of length ω.

Gap Embedding: The Construction

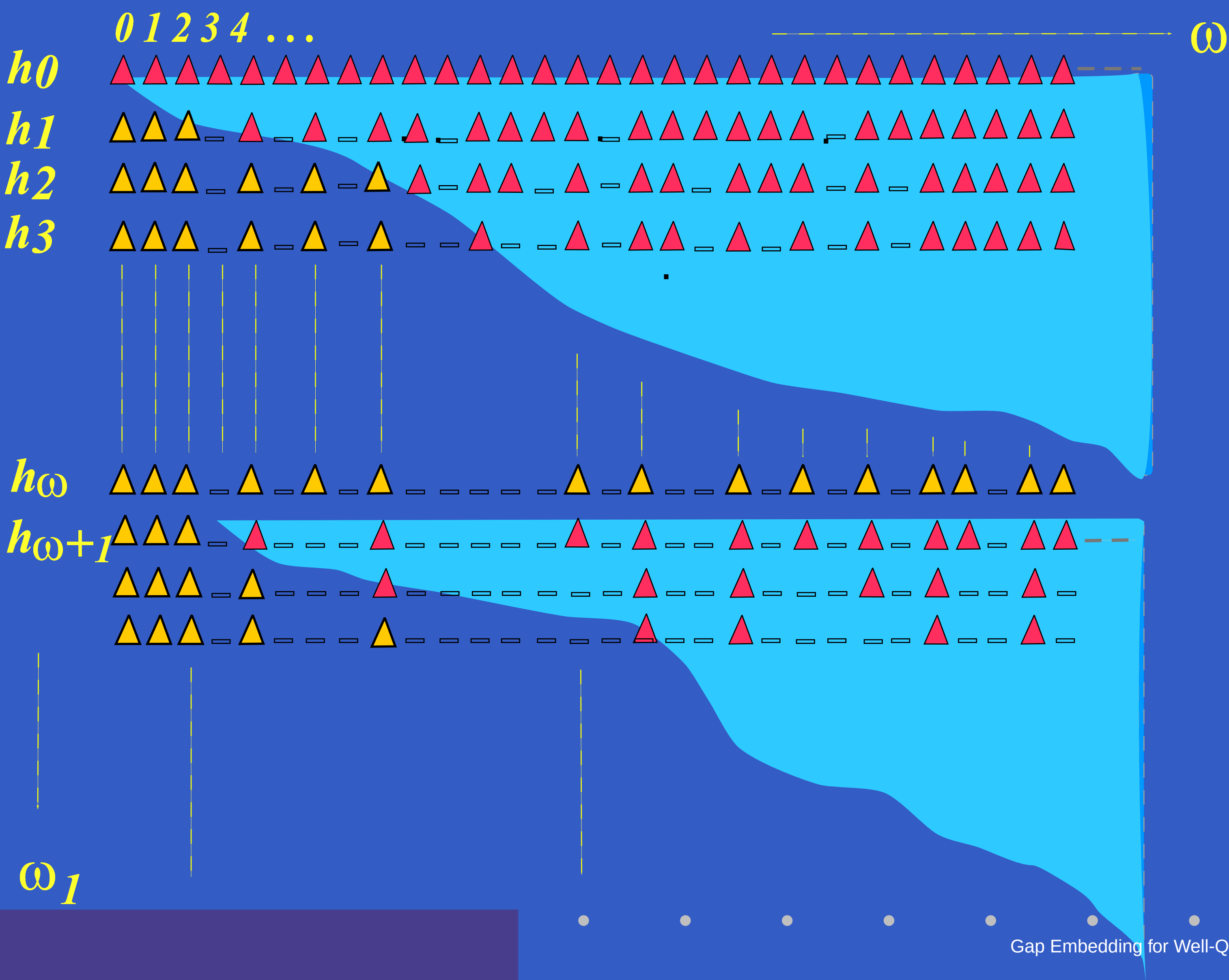
Assume the process doesn't end, i.e. no minimal bad sequence. Build by transfinite induction an infinite table H of (presumable) size $\omega_1 \times \omega$:

- Each row h_α is a bad sequence of length ω.
- By induction build next row $h_{\alpha+1}$ from the subtrees of h_α.

Gap Embedding: The Construction



Gap Embedding: The Construction



Gap Embedding: The Construction

- For a limit ordinal λ, row h_λ must converge since trees are finite.

Gap Embedding: The Construction

- For a limit ordinal λ, row h_λ must converge since trees are finite.
- The construction must terminate before ω_1 (first uncountable ordinal), since we cannot take more than $\aleph_0$ many subtrees.

Gap Embedding: The Construction

- **Difficulty**: The rows might **vanish** in the limit case! $\Rightarrow$

Gap Embedding: The Construction

- **Difficulty**: The rows might **vanish** in the limit case! $\Rightarrow$
- We need to append the preceding row to the new one to compensate for this (close to $T\,\hat{}\,S'$ in Nash-Williams' proof) $\Rightarrow$

Gap Embedding: The Construction

- For the appending to result in a **bad** sequence, need to maintain special invariants throughout induction:

Gap Embedding: The Construction

- For the appending to result in a **bad** sequence, need to maintain special invariants throughout induction:
- Root labels increasing in both directions: **Rows and columns**.

Gap Embedding: The Construction

- For the appending to result in a **bad** sequence, need to maintain special invariants throughout induction:
- Root labels increasing in both directions: **Rows and columns**.
- Row increase achieved by Ramsey — the question is at what stage take an increasing subsequence.

Gap Embedding: The Construction

- Reason for root increase in rows: So that $h_{\alpha+1} := h_\alpha\hat{\ }h_{\alpha+1}$ is **bad**.

Gap Embedding: The Construction

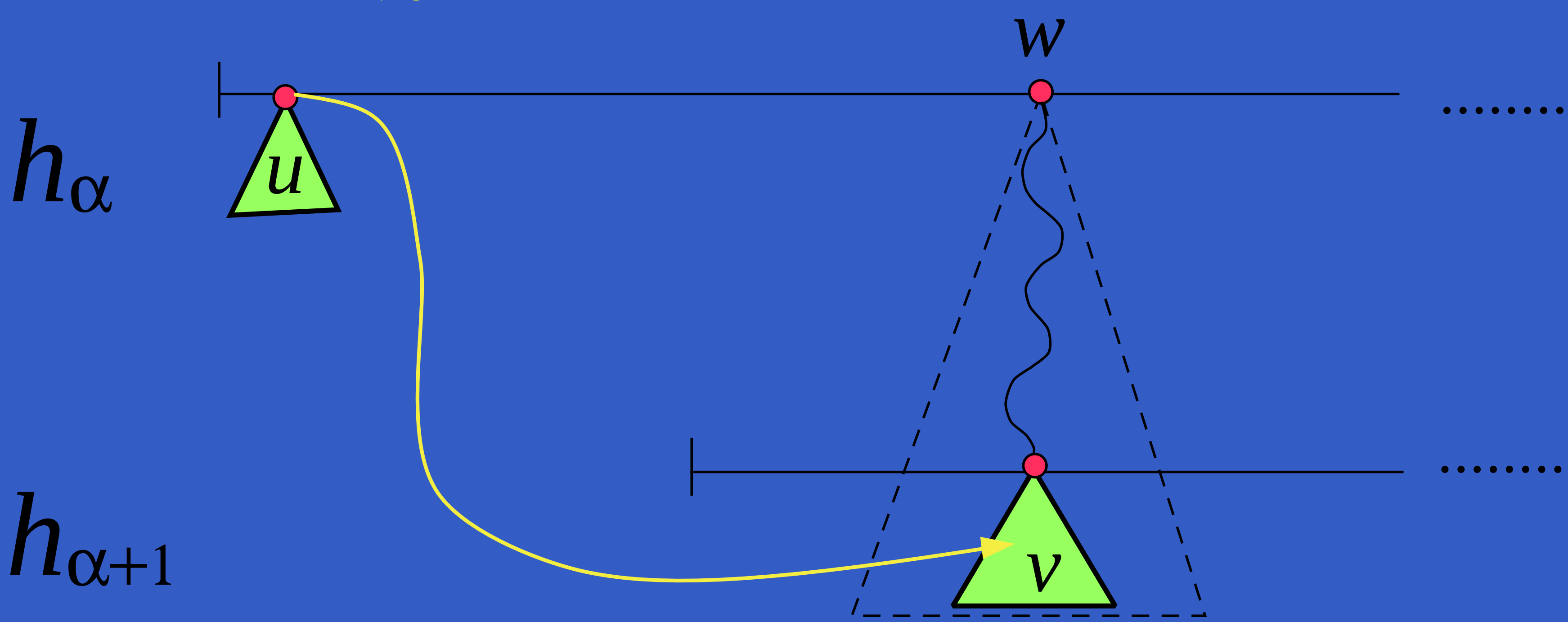
- Reason for root increase in rows: So that $h_{\alpha+1} := h_\alpha{}^\frown h_{\alpha+1}$ is **bad**.

$$u \hookrightarrow v \wedge u^\bullet \precsim w^\bullet \Rightarrow u \hookrightarrow w$$

Gap Embedding: The Construction

- Reason for root increase in rows: So that $h_{\alpha+1} := h_\alpha \hat{}\, h_{\alpha+1}$ is **bad**.

$$u \hookrightarrow v \wedge u^\bullet \precsim w^\bullet \Rightarrow u \hookrightarrow w$$



Gap Embedding: The Construction

- Reason for root increase in columns: So that $h_{\alpha+1} := h_\alpha {}^\frown h_{\alpha+1}$ is **increasing**.

Gap Embedding: The Construction

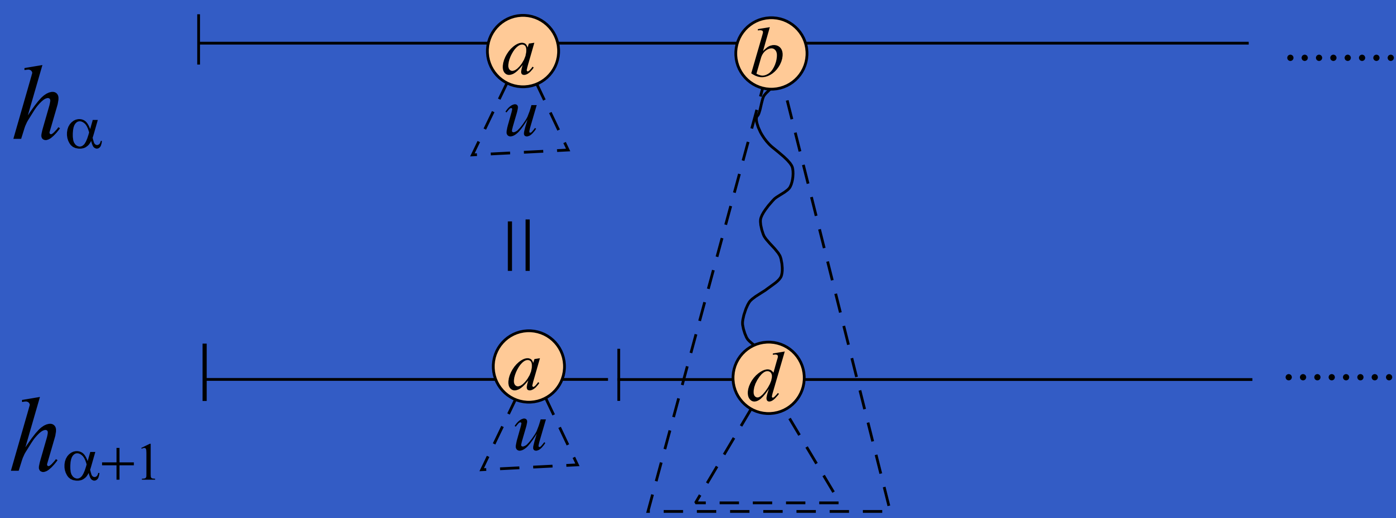
- Reason for root increase in columns: So that $h_{\alpha+1} := h_\alpha \hat{} h_{\alpha+1}$ is **increasing**.

$$a \precsim b \precsim d \Rightarrow a \precsim d$$

Gap Embedding: The Construction

- Reason for root increase in columns: So that $h_{\alpha+1} := h_\alpha \hat{} h_{\alpha+1}$ is **increasing**.

$$a \precsim b \precsim d \Rightarrow a \precsim d$$



Ordinal Labels

- For **Column increase**: Taking only **root increasing** bad sequences suffices in this case.

Ordinal Labels

- For **Column increase**: Taking only **root increasing** bad sequences suffices in this case.
- Each induction step take the lexicographic minimal sequence w.r.t. roots from $Increasing(Subtrees(h_\alpha))$

Ordinal Labels

- For **Column increase**: Taking only **root increasing** bad sequences suffices in this case.
- Each induction step take the lexicographic minimal sequence w.r.t. roots from $Increasing(Subtrees(h_\alpha))$
- This ensures column root increase: Otherwise, had to choose $h_{\alpha+1}(n)$ earlier ($n := min\ dom(h_{\alpha+1})$).

WQO Labels

- For column root increase need the condition that nodes are comparable to their ancestors!

WQO Labels

- For column root increase need the condition that nodes are comparable to their ancestors!
- The previous construction not suffices.

WQO Labels

- For column root increase need the condition that nodes are comparable to their ancestors!
- The previous construction insufficient.
- **Intuition**: There are many lexicographic minimal sequences to choose from at each step $\Rightarrow$ Either doesn't contradict Lex minimality, hence not column increasing, or appending doesn't yields a bad sequence.

WQO Labels

- Solution: Look at all the bad sequences each step, not just root increasing ones!

WQO Labels

- Solution: Look at all the bad sequences each step, not just root increasing ones!
- Then take the lexicographic minimal w.r.t. roots.

WQO Labels

- Solution: Look at all the bad sequences each step, not just root increasing ones!
- Then take the lexicographic minimal w.r.t. roots.
- And only then, take a root increasing subsequence from the chosen one!

WQO Labels

- This approach also **simplifies** the proof: Column   root increase achieved since otherwise contradiction with the **preceding** step alone, and not some early stage.

The Limit Step

- For λ a limit ordinal:

$$h_\lambda(i) := \lim_{\alpha \to \lambda} h_\alpha(i)$$

The Limit Step

- For λ a limit ordinal:

$$h_\lambda(i) := \lim_{\alpha \to \lambda} h_\alpha(i)$$

-

$$h_\lambda := Subtrees(h_\lambda)$$

The Limit Step

- For λ a limit ordinal:

$$h_\lambda(i) := \lim_{\alpha \to \lambda} h_\alpha(i)$$

- $$h_\lambda := Subtrees(h_\lambda)$$

- $$h_\lambda := Increasing\ Lex(h_\lambda)$$

The Limit Step

- For λ a limit ordinal:

$$h_\lambda(i) := \lim_{\alpha \to \lambda} h_\alpha(i)$$

- $$h_\lambda := Subtrees(h_\lambda)$$

- $$h_\lambda := Increasing\ Lex(h_\lambda)$$

It works! (believe me)

THE END