Rewrite Systems

4. Dependencies

Monotonic Interpretation

$$f(f(x)) \longrightarrow f(g(f(x)))$$

[f](a,n) = (1,n+a)
[g](a,n) = (0,n)
[const] = (0,0)

(x,y) > (x',y') iff x=x',y>y'

[ff](0,n) = (1,n+1) > (1,n) = [fgf](0,n)
[ff](1,n) = (1,n+2) > (1,n+1) = [fgf](1,n)

2

Alternate Interpretation

$$f(f(x)) \longrightarrow f(g(f(x)))$$

[f](m,n) = (n+m,m)
[g](m,n) = (n,m)
[const] = (1,1)

[ff](m,n) = (n+2m,n+m) >
(n+2m,m) = [fgf](m,n)

3

$S := V$
$R := \emptyset$
do forever
$\quad x :\in S$
$\quad R := R \cup \{x\}$
$\quad S := S \setminus \{x\}$
$\quad W := \{s \in S \mid c(x,s) = \text{white}\}$

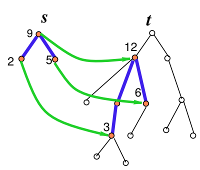
$$S := \begin{cases} W & \textbf{if } |W| = \infty \\ S \setminus W & \textbf{otherwise} \end{cases}$$

$W := \{x \in R \mid \forall y \in R.\ y \neq x \rightarrow c(x,y) = \text{white}\}$

$$\textbf{return} \begin{cases} W & \textbf{if } |W| = \infty \\ R \setminus W & \textbf{otherwise} \end{cases}$$

4

Tree Embedding

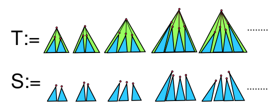


5

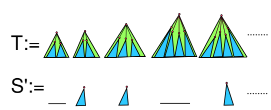
Kruskal's Tree Theorem

- Every infinite sequence of trees (over a finite alphabet) includes an embedding.

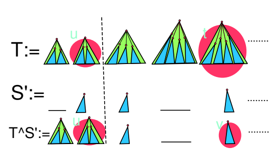
6



7



8



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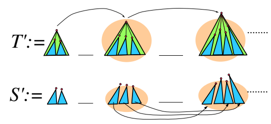

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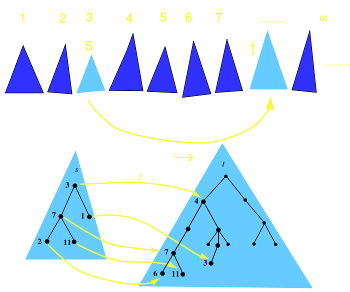
Labels



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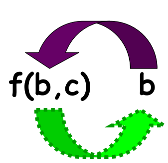
Subterm Property



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Simplification Ordering

- (Weakly) Monotonic
- (Weakly) Subterm
- ...

- They are well-quasi-orders

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Total Ordering

- If monotonic and total
- Then Subterm
- And Well-founded

14

Polynomials over the Reals

- Reals
- Times, plus
- Equalities, inequalities
- Connectives, quantifiers

- Decidable first-order theory!

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Total Simplification Ordering

- Not always possible

$f(a) \rightarrow f(b)$
$g(a) \rightarrow g(b)$

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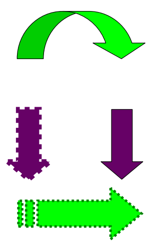
Weak Monotonicity



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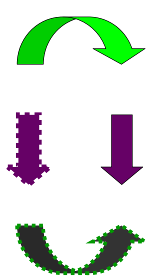
Harmony



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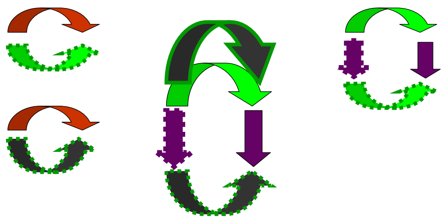
Harmony Method



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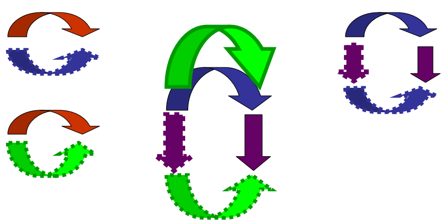
Special Case



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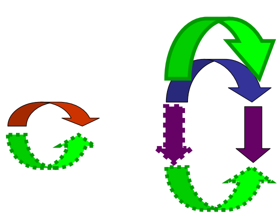
Kamin & Levy Method



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$$s \succ t \text{ and } s \to t \quad \Rightarrow \quad g(\ldots,s,\ldots) \succeq g(\ldots,t,\ldots)$$

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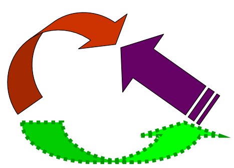
Right Property



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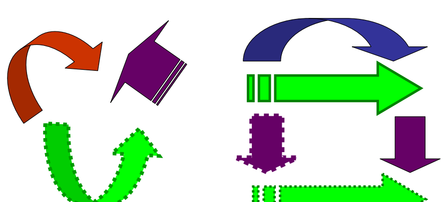
Right Method



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Mortals & Immortals

- Immortal terms can be rewritten forever
- Mortals cannot be

25

Mortals & Immortals

- Immortal terms can be rewritten forever
- Mortals cannot be

1. Mortals remain mortal
2. Mortals beget mortals
3. Constricted immortals remain immortal

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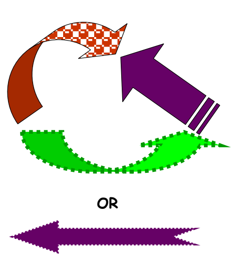
Depend Property



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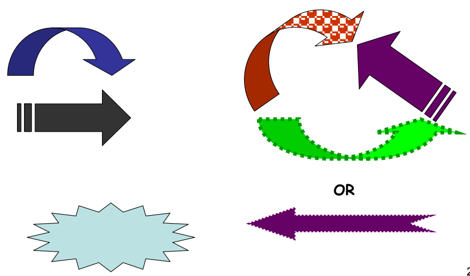
Intermediate Method



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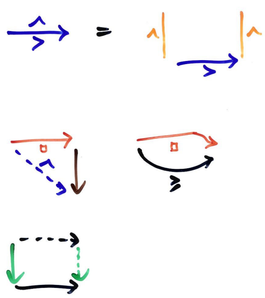
Dependency Pairs



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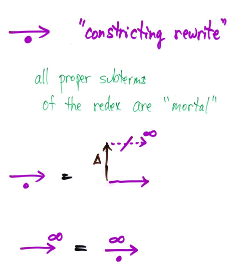
Constricting Derivation



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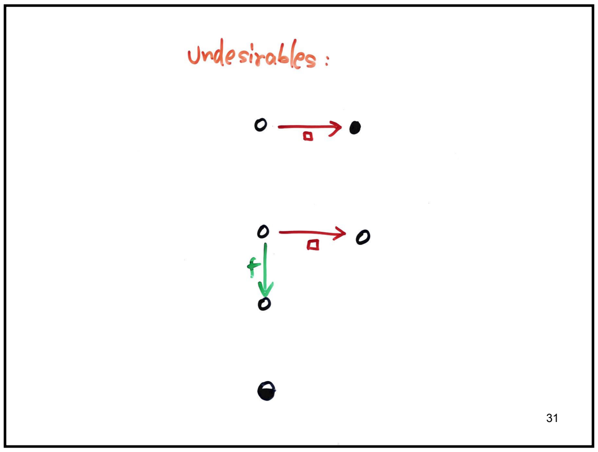
Undesirables :
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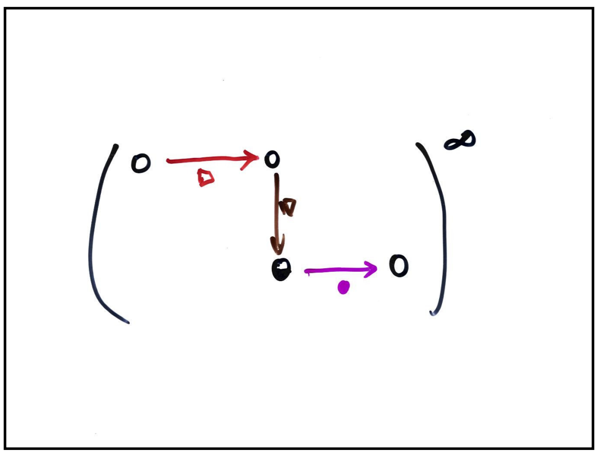

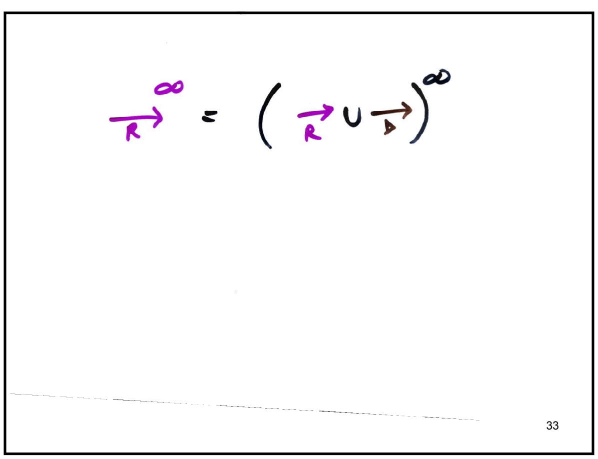
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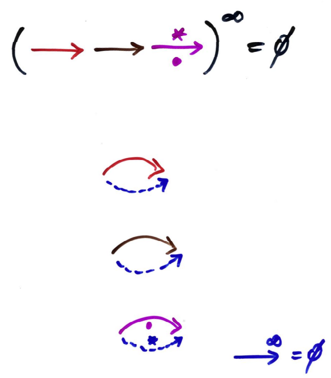

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$$
\begin{aligned}
x \times (y + z) &\rightarrow (x \times y) + (x \times z)\\
(y + z) \times x &\rightarrow (x \times y) + (x \times z)\\
(x \times y) \times z &\rightarrow x \times (y \times z)\\
(x + y) + z &\rightarrow x + (y + z)\,.
\end{aligned}
$$

$\langle$[term], top, left argument, right argument$\rangle$. A natural interpretation, with precedence $\times > +$ does the trick. □

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Semantic Path Ordering

$$\frac{s_i \succsim_{\text{mpo}} t}{f(s_1,\ldots,s_m) \succ_{\text{mpo}} t} \qquad \frac{f \succ g \quad f(s_1,\ldots,s_m) \succ_{\text{mpo}} t_1,\ldots,t_n}{f(s_1,\ldots,s_m) \succ_{\text{mpo}} g(t_1,\ldots,t_n)}$$

$$\frac{f \approx g \quad [s_1,\ldots,s_m] \approx_{\text{bag}} [t_1,\ldots,t_n]}{f(s_1,\ldots,s_m) \approx_{\text{mpo}} g(t_1,\ldots,t_n)} \qquad \frac{f \approx g \quad [s_1,\ldots,s_m] \succ_{\text{bag}} [t_1,\ldots,t_n]}{f(s_1,\ldots,s_m) \succ_{\text{mpo}} g(t_1,\ldots,t_n)}$$

36

$$
\begin{aligned}
1 \cdot x &\rightarrow x & x \cdot 1 &\rightarrow x \\
x^- \cdot x &\rightarrow 1 & x \cdot x^- &\rightarrow 1 \\
1^- &\rightarrow 1 & x^{--} &\rightarrow x \\
y^- \cdot (y \cdot z) &\rightarrow z & y \cdot (y^- \cdot z) &\rightarrow z \\
(x \cdot y) \cdot z &\rightarrow x \cdot (y \cdot z) & (x \cdot y)^- &\rightarrow y^- \cdot x^-
\end{aligned}
$$

$\langle \text{weight}, \text{top}, \text{first argument}, \ldots, \text{last argument} \rangle.$

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Factorial!

$$
\begin{aligned}
p(s(x)) &\rightarrow x \\
\mathit{fact}(0) &\rightarrow s(0) \\
\mathit{fact}(s(x)) &\rightarrow s(x) \times \mathit{fact}(p(s(x))) \\
0 \times y &\rightarrow 0 \\
s(x) \times y &\rightarrow (x \times y) + y \\
x + 0 &\rightarrow x \\
x + s(y) &\rightarrow s(x + y) \ ,
\end{aligned}
$$

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$\langle \mathbf{top}, [\![\mathbf{term}]\!] \rangle$

$$
\begin{aligned}
[\![\mathit{fact}(x)]\!] &= [\![x]\!]! & [\![x \times y]\!] &= [\![x]\!][\![y]\!] \\
[\![s(x)]\!] &= [\![x]\!] + 1 & [\![x + y]\!] &= [\![x]\!] + [\![y]\!] \\
[\![p(x)]\!] &= [\![x]\!] - 1 & [\![0]\!] &= 0 \ .
\end{aligned}
$$

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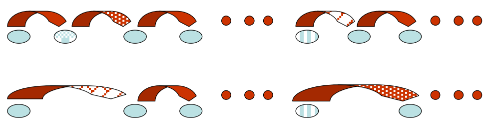
Composition



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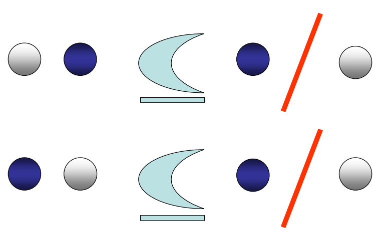
Closure



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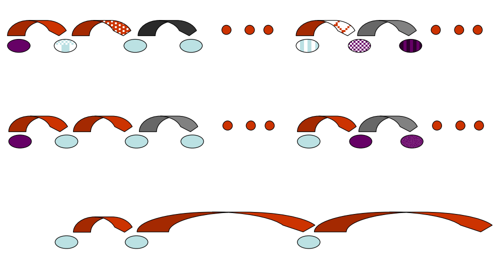
Extraction



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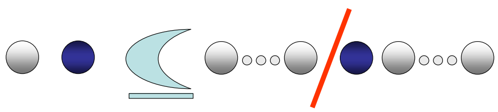
Alternative (1)



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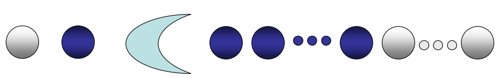
Alternative (2)



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Abstraction

- Invariant (Place)
  - abstract location and values
  - finitely many
- Variant (Time)
  - abstract values
  - well-founded set

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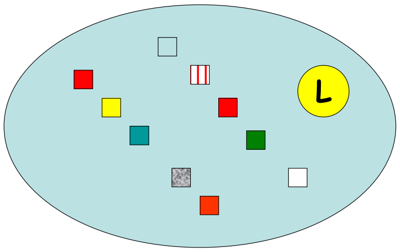
State



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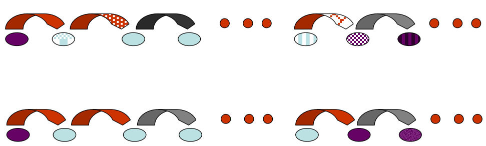
Abstraction



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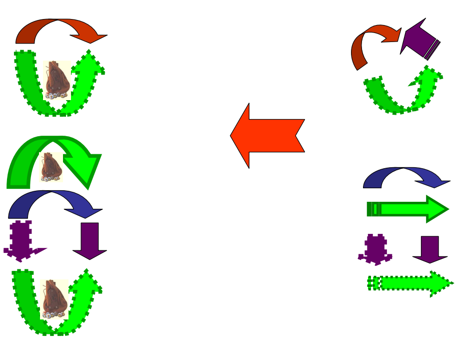
K&L ⇒ Right



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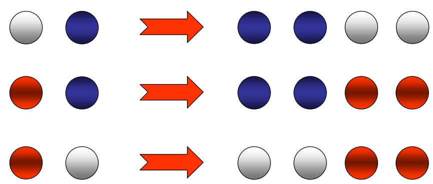
Duplication Problem



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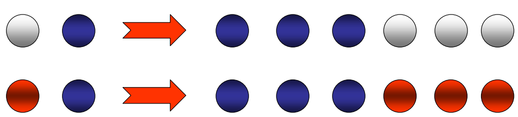
Toyama's Problem



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Tools

- *TermiLog* [Sagiv, Lindenstrauss, & Serebrenik]
- *TerminWeb* [Codish & al.]
- *cTI* [Mesnard & Neumerkel]
- *Hasta la Vista* [Serebrenik & De Schreye]
- *TALP* [Ohlebusch, Claves & Marché]
- *AProVE* [Giesl & al.]

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