Rewrite Systems

2. Termination

This Week

הקורס כולל: שיקול משוואתי, משפט בירקהוף, מערכות שכתוב וצורות נורמליות, תכונות צ'רץ'-רוסר ולמת-ניומן, זוגות קריטיים ואנכיות, שכתוב מקבילי ונירמול חיצוני, אי-כריעות של תכונות שכתוב, דימויי סדר טובים, למת-היגמן ומשפט העצים של קרוסקל, יחסי סדר מפשטים, יחסי סדר לשקיות, לעצים ולהוכחות, תהליך ההשלמה של קנות ובנדיקס והרחבותיו, הוכחות באינדוקציה, החוג הבוליאני והסקה מסדר ראשון.

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Readings

- Terese: Ch. 6; §§ A1, A2, A5
- Baader&Nipkow: Ch. 5
- Survey: §5
- Plaisted: §4

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Well-Founded Ordering

- Partial ordering
  - Irreflexive
  - Transitive
  - Asymmetric
- Well-founded
  - No infinite decreasing chains

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Termination Proofs

1. Choose well-founded set.
2. Choose cutpoints.
3. Establish invariants.
4. Choose mapping.
5. Show decrease.

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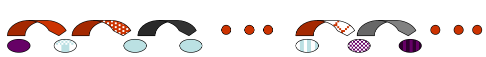
Computation



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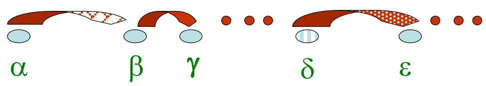
Valuation



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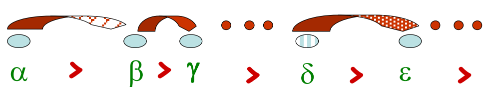
Reduction



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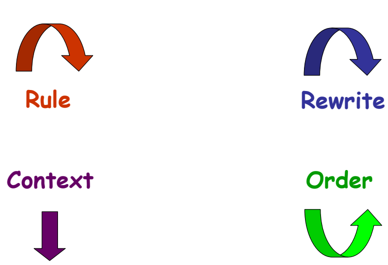
Legend



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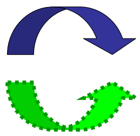
Obvious Method



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Easy!

$$
\begin{aligned}
1 \cdot x &\to x & x \cdot 1 &\to x \\
x^- \cdot x &\to 1 & x \cdot x^- &\to 1 \\
1^- &\to 1 & (x^-)^- &\to x \\
x^- \cdot (x \cdot y) &\to y & x \cdot (x^- \cdot y) &\to y
\end{aligned}
$$

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Loops

$$
\begin{aligned}
x \backslash x &\to e & x \cdot (x \backslash y) &\to y \\
x / x &\to e & (y / x) \cdot x &\to y \\
e \cdot x &\to x & x \backslash (x \cdot y) &\to y \\
x \cdot e &\to x & (y \cdot x) \backslash x &\to y \\
e \backslash x &\to x & x / (y \backslash x) &\to y \\
x / e &\to x & (x / y) \backslash x &\to y \, .
\end{aligned}
$$

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Beware!

$$(x^-) \cdot y \to y \cdot y,$$

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Counting

$$f(f(x)) \quad \to \quad g(f(x))$$

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$$\begin{aligned} w(r(x)) &\to r(w(x)) \\ b(r(x)) &\to r(b(x)) \\ b(w(x)) &\to w(b(x))\,. \end{aligned}$$

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Combining Well-Founded Orderings

Lexicographic Orderings

- The lexicographic combination of well-founded orderings is well-founded.

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Quasi-orderings

- Quasi-ordering
  - Reflexive
  - Transitive

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Lexicographic Ordering

$$\frac{}{\langle\,\rangle \approx_{\mathrm{lex}} \langle\,\rangle} \qquad \frac{\langle s_1,\ldots,s_m\rangle \succsim_{\mathrm{lex}} \langle t_1,\ldots,t_n\rangle}{\langle s_1,\ldots,s_m,s_{m+1}\rangle \succ_{\mathrm{lex}} \langle t_1,\ldots,t_n\rangle}$$

$$\frac{\langle s_1,\ldots,s_n\rangle \approx_{\mathrm{lex}} \langle t_1,\ldots,t_n\rangle \qquad s_{n+1} \approx_{n+1} t_{n+1}}{\langle s_1,\ldots,s_{n+1}\rangle \approx_{\mathrm{lex}} \langle t_1,\ldots,t_{n+1}\rangle}$$

$$\frac{\langle s_1,\ldots,s_m\rangle \approx_{\mathrm{lex}} \langle t_1,\ldots,t_m\rangle \qquad s_{m+1} \succ_{m+1} t_{m+1}}{\langle s_1,\ldots,s_{m+1}\rangle \succ_{\mathrm{lex}} \langle t_1,\ldots,t_{m+1},\ldots,t_n\rangle}$$

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$$\begin{array}{lcl} f(f(x)) & \rightarrow & g(f(x)) \\ g(g(x)) & \rightarrow & f(x) \end{array}$$

$$\langle \text{size}, \text{number of } f\text{s}\rangle.$$

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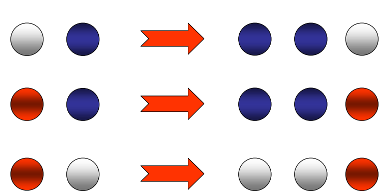
Growth Problem



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$$\begin{aligned} f(g(x)) &\to g(g(f(x))) \\ f(g(x)) &\to g(g(g(x)))\ , \end{aligned}$$

$$\langle \text{number of } f\text{s}, \text{height of rightmost } f, \ldots, \text{height of leftmost } f \rangle .$$

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$$\begin{aligned} D\mathrm{t} &\to 1 \\ D(\text{constant}) &\to 0 \\ D(x+y) &\to Dx + Dy \\ D(x \times y) &\to (y \times Dx) + (x \times Dy) \\ D(x-y) &\to Dx - Dy \\ D(-x) &\to -Dx \\ D(x/y) &\to (Dx/y) - (x \times Dy/y^2) \\ D(\ln x) &\to Dx/x \\ D(x^y) &\to (y \times x^{y-1} \times Dx) + (x^y \times (\ln x) \times Dy) \end{aligned}$$

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$$\begin{aligned} [\![x+y]\!] &= [\![x]\!] + [\![y]\!] & [\![x \times y]\!] &= [\![x]\!] + [\![y]\!] \\ [\![x-y]\!] &= [\![x]\!] + [\![y]\!] & [\![x/y]\!] &= [\![x]\!] + [\![y]\!] \\ [\![x^y]\!] &= [\![x]\!] + [\![y]\!] & [\![Dx]\!] &= [\![x]\!]^2 \\ [\![-x]\!] &= [\![x]\!] + 1 & [\![\ln x]\!] &= [\![x]\!] + 1 \\ [\![\text{constant}]\!] &= 2 & [\![\mathrm{t}]\!] &= 2\ . \end{aligned}$$

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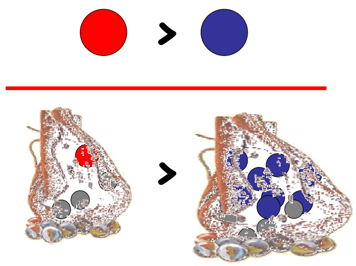
Multiset Ordering



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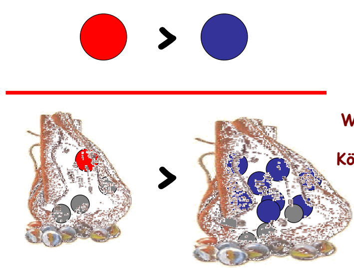
Multiset Ordering



Well-founded
by
König's Lemma

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Konig's Lemma

A tree is infinite
(has infinitely many nodes)

iff

it has a node of infinite degree
or
it has an infinite path

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Multiset Ordering

$$\frac{}{[\,] \approx_{\text{bag}} [\,]} \qquad \frac{[s_1,\ldots,s_n] \approx_{\text{bag}} [t_1,\ldots,t_n] \quad s \approx t}{[s_1,\ldots,s_n,s] \approx_{\text{bag}} [t_1,\ldots,t_n,t]}$$

$$\frac{[s_1,\ldots,s_m] \succsim_{\text{bag}} [t_1,\ldots,t_n] \quad s \succ u_1,\ldots,u_k}{[s_1,\ldots,s_m,s] \succ_{\text{bag}} [t_1,\ldots,t_n,u_1,\ldots,u_k]}$$

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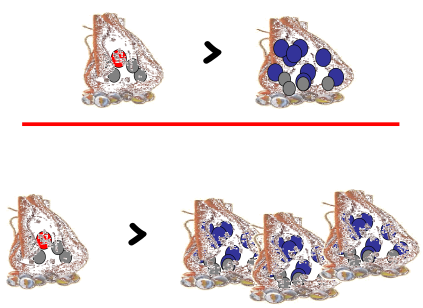
Nested Ordering



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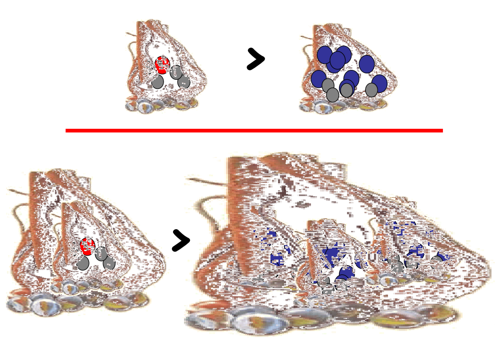
Nested Ordering



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Term Structure

- Orderings
  - Monotonic
  - Path orderings

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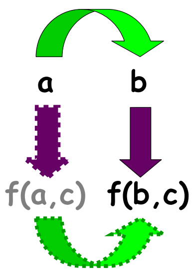
Monotonicity



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Reduction Ordering

- well-founded
- monotonic

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A rewrite system terminates

iff

its rules decrease
for some reduction ordering

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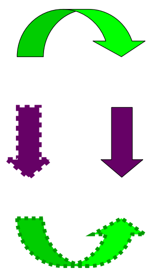
Monotonicity



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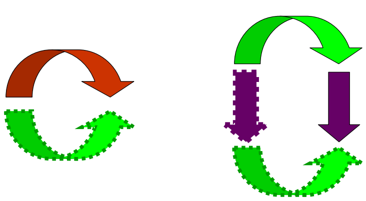
Standard Method



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$$(x \cdot y) \cdot z \quad \rightarrow \quad x \cdot (y \cdot z)\ ;$$

$$[\![x \cdot y]\!] = 2[\![x]\!] + [\![y]\!] \qquad [\![\text{constant}]\!] = 2$$

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$$\begin{aligned}
x \times (y + z) &\rightarrow (x \times y) + (x \times z) \\
(y + z) \times x &\rightarrow (x \times y) + (x \times z) \\
(x \times y) \times z &\rightarrow x \times (y \times z) \\
(x + y) + z &\rightarrow x + (y + z)\ ,
\end{aligned}$$

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n use $\langle [\![\text{term}]\!], [\![\text{term}]\!]' \rangle$, where

$$\begin{aligned}
[\![x \times y]\!] &= [\![x]\!][\![y]\!] & [\![x \times y]\!]' &= 2[\![x]\!]' + [\![y]\!]' \\
[\![x + y]\!] &= [\![x]\!] + [\![y]\!] + 1 & [\![x + y]\!]' &= 2[\![x]\!]' + [\![y]\!]' \\
[\![\text{constant}]\!] &= 2 & [\![\text{constant}]\!]' &= 2\ .
\end{aligned}$$

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$$f(f(x)) \rightarrow f(g(f(x)))$$

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Monotonic Interpretation

$$f(f(x)) \rightarrow f(g(f(x)))$$

[*f*](a,n) = (1,n+a)
[*g*](a,n) = (0,n)
[const] = (0,0)

(x,y) > (x',y') iff x=x',y>y'

[ff](0,n) = (1,n+1) > (1,n) = [fgf](0,n)
[ff](1,n) = (1,n+2) > (1,n+1) = [fgf](1,n)

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Alternate Interpretation

$$f(f(x)) \rightarrow f(g(f(x)))$$

[*f*](m,n) = (n+m,m)
[*g*](m,n) = (n,m)
[const] = (1,1)

[ff](m,n) = (n+2m,n+m) >
(n+2m,m) = [fgf](m,n)

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$$
\begin{aligned}
Dt &\rightarrow 1\\
D(\text{constant}) &\rightarrow 0\\
D(x+y) &\rightarrow Dx+Dy\\
D(x\times y) &\rightarrow (y\times Dx)+(x\times Dy)\\
D(x-y) &\rightarrow Dx-Dy\ .
\end{aligned}
$$

$$
\begin{aligned}
[\![x+y]\!] &= [\![x]\!]+[\![y]\!] & [\![Dx]\!] &= 3^{[\![x]\!]}\\
[\![x-y]\!] &= [\![x]\!]+[\![y]\!] & [\![\mathbf{t}]\!] &= 3\\
[\![x\times y]\!] &= [\![x]\!]+[\![y]\!] & [\![\text{constant}]\!] &= 3
\end{aligned}
$$

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$$
\begin{aligned}
\neg\neg x &\rightarrow x\\
\neg(x\vee y) &\rightarrow (\neg x)\wedge(\neg y)\\
\neg(x\wedge y) &\rightarrow (\neg x)\vee(\neg y)\\
x\wedge(y\vee z) &\rightarrow (x\wedge y)\vee(x\wedge z)\\
(y\vee z)\wedge x &\rightarrow (x\wedge y)\vee(x\wedge z)
\end{aligned}
$$

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$$
\begin{aligned}
[\![x\vee y]\!] &= [\![x]\!]+[\![y]\!]+1 & [\![\neg x]\!] &= 2^{[\![x]\!]}\\
[\![x\wedge y]\!] &= [\![x]\!][\![y]\!] & [\![\text{constant}]\!] &= 2\ .
\end{aligned}
$$

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Ordinals

$0, 1, 2, \ldots,$
$\omega, \omega+1, \omega+2, \ldots,$
$2\omega, 2\omega+1, \ldots, 3\omega, \ldots,$
$\omega^2, \ldots, \omega^2+2\omega+3, \ldots, \omega^3, \ldots,$
$\omega^\omega, \ldots, \omega^{\omega^\omega}, \ldots,$
$\varepsilon_0, \varepsilon_0+1, \ldots, 2\varepsilon_0+\omega^\omega+2\omega+3, \ldots,$
$\varepsilon_1, \ldots, \varepsilon_{\varepsilon_0}, \ldots,$
$\ldots$

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$$
\begin{aligned}
D\mathrm{t} &\rightarrow 1 \\
D(\mathrm{constant}) &\rightarrow 0 \\
D(x+y) &\rightarrow Dx + Dy \\
D(x \times y) &\rightarrow (y \times Dx) + (x \times Dy) \\
D(x-y) &\rightarrow Dx - Dy\,.
\end{aligned}
$$

$$
\begin{aligned}
[\![Dx]\!] &= \omega^{[\![x]\!]} & [\![x+y]\!] &= [\![x]\!] + [\![y]\!] \\
[\![\mathrm{t}]\!] &= 1 & [\![x-y]\!] &= [\![x]\!] + [\![y]\!] \\
[\![\mathrm{constant}]\!] &= 1 & [\![x \times y]\!] &= [\![x]\!] + [\![y]\!]
\end{aligned}
$$

Hydra Rewriting

$$
\begin{aligned}
h_n(x:y) &\rightarrow h_{n+1}(g_n(x:y)) \\
g_n(x:y) &\rightarrow g_n(x):y \\
g_n(x:y) &\rightarrow y \\
g_{n+1}((x:y):z) &\rightarrow y : g_n((x:y):z)
\end{aligned}
$$

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Hercules Defeats Hydra

- Cannot be proved in Peano Arithmetic [Paris & Kirby]
- Requires induction up to ε_0
- Natural numbers do not suffice
- Sophisticated variants require more powerful systems [Friedman]

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Careful Now...

- Consider pairs of pairs
- (1,0) > (0,(1,0)) > (0,(0,(1,0))) > ...
- Need bound

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