Rewrite Systems

13. Semantic Unification

Tentative Course Outline

Sources

- Terese, Sect. 7.7
- Jouannaud, Sect. 6.2-3
- Baader & Snyder Handout

A Simple Calendar: 12 months of 30 days

months: $0, 1, \ldots, 11$
days: $0, 1, \ldots, 29$
years: $\ldots, -2, -1, 0, 1, 2, \ldots$

$$\textbf{fixed-from-simple}\left(\boxed{month \mid day \mid year}\right) \stackrel{\text{def}}{=}$$

$$\textbf{simple-epoch} - 1$$
$$+360 \cdot year$$
$$+30 \cdot month$$
$$+day$$

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A Simple Calendar: 12 months of 30 days

$$\textbf{simple-from-fixed}\,(date) \stackrel{\text{def}}{=} \boxed{month \mid day \mid year}$$

where

$$n = date - \textbf{simple-epoch}$$
$$year = \lfloor n/360 \rfloor$$
$$month = \lfloor (n - 360 \cdot year)/30 \rfloor$$
$$day = n \bmod 30$$

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Goal Solving

- Given N,
- find d,m,y such that

1. fixed-from-simple([d,m,y]) =? N
2. $0 \leq d < 30$ =? T
3. $0 \leq m < 12$ =? T

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Syntactic Unification

- We ask whether two terms s,t can be made identical:

$$\exists\sigma.\ Eq \models s\sigma = t\sigma$$

Syntactic Equality

$$\frac{s1 = t1, \ldots, sn = tn}{f(s1,\ldots,sn) = f(t1,\ldots,tn)}$$

Trivial Goal

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s =? s

Decompose Goal

s1 =? t1 ,..., sn =? tn

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f(s1,...,sn) =? f(t1,...,tn)

Clash

F

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f(s1,...,sn) =? g(t1,...,tn)

Occurs Check

F

========== x within t

x =? t[x]

Eliminate

$$\frac{x = t,\ S\{x \mapsto t\}}{x =?\ t,\ S}\ x \text{ not in } t$$

Semantic Unification

- We ask whether two terms s,t can be made equal:

$$\exists\sigma.\ E \models s\sigma = t\sigma$$

British Museum Method

- Try every substitution σ
- Check if $E \models s\sigma = t\sigma$

Mutate Goal

$$\frac{s1 =? u1, \ldots, sn =? un,\ r =? t}{f(s1,\ldots,sn) =? t}\ f(u1,\ldots,un)=r \in E$$

Expand Goal

$$\frac{x1 =? u1, \ldots, xn =? un,\ r =? t,\ x = f(x1,\ldots,xn)}{x =? t}\ f(u1,\ldots,un)=r \in E$$

Completeness Proof Sketch

- Suppose s =? t has a solution σ
- Consider proof

  $s\sigma = \ldots = v = w = \ldots = t\sigma$

  where each step uses an axiom in E
- Show that applying rule to s =? t yields smaller solvable subgoals
  - Measure by number of non-eliminated variables in goal and number of symbols in proof

Case Analysis

1. No top application of axiom
   - Decompose and consider proofs of $s_i\sigma = t_i\sigma$
2. v = w is first top application
   - Mutate and consider $s_i\sigma = u_i\sigma$, $r\tau = t\sigma$
3. x =? t (no x in t)
   - Eliminate x and $x\sigma$ is $t\sigma$
4. x =? t
   - Expand and consider $x_i\sigma = t_i\sigma$, $r\tau = t\sigma$

Narrowing

- Refine the set of solutions to a problem
- If every term has a unique nf, then we need only look for normal-form substitutions

Narrowing

- Apply a minimal substitution that allows for a rewrite
- u[s] =? v narrows to $u\sigma[r\sigma]$ =? $v\sigma$ if
  1. s is not a variable
  2. $l \rightarrow r$ is a rule
  3. $\sigma = mgu(s,l)$

$s1 =? u1, ..., sn =? un, r =? t$

---------------------- $f(u1,...,un) \to r \in E$

$f(s1,...,sn) =? t$

Basic Strategy

$s1 =? u1, ..., sn =? un, r =? t$

---------------------- $f(u1,...,un) \to r \in E$

$f(s1,...,sn) =? t$

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Decidable Cases

- Given a terminating, confluent system for E
- Unification is still undecidable. Why?

Decidable Cases

- Every non-ground right side is a variable [Hullot, 1980].
- Every non-ground right side is a constructor term [Dershowitz *et al.*, 1992].
- Every non-ground right side is a proper subterm of its left side [Narendran, Pfenning and Statman, 1997].
- Every non-ground right side is either a constructor term or a proper subterm of its left side [Mitra, 1994].
- Every right side is composed of constructors and proper subterms of its left side [Mitra, 1994].

Cont'd

- All variables are shallow on the left side [Christian, 1992].
- The system is linear and every variable that appears on both sides is shallow on both sides (convergence is unnecessary) [Nieuwenhuis, 1998].
- The system is linear and the right side of every f-rule is either a constructor term or a proper subterm of the left side, except for at most one right side that may be a value context with a single subterm $g(\ldots, r_i, \ldots)$, where every r_i is either a variable or a value [Dershowitz and Mitra, 1992].
- The system is linear and the right side of every f-rule is a constructor term, except for at most one right side that may be a constructor context with a single subterm $g(\ldots, r_i, \ldots)$, where every r_i is either a variable or a value [Mitra, 1994].

Function Inversion

- One side of goal equation has no variables
- Assume system is left-linear (or else same as unification)
- Assume confluent (for ground terms)

Basics

$$x + 0 \to x \qquad s(x) < s(y) \to x < y$$
$$x + s(y) \to s(x + y) \qquad 0 < s(y) \to T$$
$$x \times 0 \to 0 \qquad 0 < 0 \to F$$
$$x \times s(y) \to (x \times y) + x \qquad s(x) < 0 \to F$$

$$s(x) =^? s(y) \to x =^? y$$
$$s(x) =^? 0 \to F$$
$$0 =^? s(y) \to F$$

Method

Decompose: $\dfrac{f(s_1, \ldots, s_n) \to^? f(t_1, \ldots, t_n)}{s_1 \to^? t_1, \ldots, s_n \to^? t_n}$

Mutate: $\dfrac{f(s_1, \ldots, s_n) \to^? t}{s_1 \to^? l_1\rho, \ldots, s_n \to^? l_n\rho}$ ρ is a solution to $r \to^? t$; $f(l_1, \ldots, l_n) \to r$ is a (renamed) rule in R

Eliminate: $\dfrac{x \to^? t}{x \equiv^? t}$ x is a variable

Ignore: $\dfrac{s \to^? x}{}$ x is a variable

Winning Strategy

Repeat
- **Try**
  - *Ignore*
  - *Eliminate*
  - **Combine**
    - *Decompose*
    - *Mutate*

Higher-Order Unification

- We ask whether two lambda terms s,t can be made equal:

$$\exists\sigma.\ \lambda \models s\sigma = t\sigma$$