Well-founded Unions Verified

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Abstract

We describe how we jointly answered and machine-checked the following fundamental mathematical question: given two or more well-founded binary relations $R_1, R_2, \ldots, R_n$, when is their union $R_1 \cup R_2 \cup \cdots \cup R_n$ also well-founded? We first present two sets of incomparable conditions and the pen-and-paper proofs for $n = 3$ suggested by Dershowitz, extending the condition of Doornbos and von Karger for $n = 2$. We then describe conditions and machine-checked proofs for $n = 4$ suggested by Dawson and Goré. Finally, we give general conditions and machine-checked proofs for arbitrary n suggested by Dawson and Goré. We describe how all proofs were machine-checked using Isabelle/HOL 2005 and our reasons for using such an old version of Isabelle.

Keywords and phrases well-founded relations, termination of term rewriting, union of relations

Digital Object Identifier 10.4230/LIPIcs...

1 Introduction

A binary relation R over some set is well-founded if there is no infinite *ascending* chain $x_0 \ R \ x_1 \ R \ \cdots \ R \ x_{n-1} \ R \ x_n \ R \ \cdots$. Thus, by *well-founded*, we mean the absence of infinite forward-pointing paths. Lescanne [11] documents the early use of Ramsey's Theorem for this purpose. Blass and Gurevich [2] cover early history. Given $n \geq 3$ well-founded binary relations $R_1, R_2, \cdots, R_n$ over some set, when is their union $\bigcup_{i=1}^{n} R_i$ also well-founded?

We first tackle just three relations, A, B, and C, and refer to the relations as "colours". We extend our results to $n = 4$ and finally describe the general pattern for arbitrary n.

To ensure correctness, we wanted to machine-check all proofs using an interactive theorem prover. The natural way was to build upon the substantial work of the first author, which covers almost 20 years of collaboration with the third author. Unfortunately, continuously bringing large slabs of previous code up-to-date with the latest incarnation of Isabelle is cumbersome, and involves a substantial amount of work, which only needs to be repeated at the next incarnation. We therefore deliberately stuck with Isabelle/HOL 2005, which presumably is still a reliable interactive theorem prover.

The proofs are in `http://users.cecs.anu.edu.au/~jeremy/isabelle/2005/gen/`. The proofs described in this paper are contained in `tripartite.{thy,ML}`. Other results on the unions of well-founded relations are in `WfUn.{thy,ML}`, and more general results on well-foundedness are in `Wfss.{thy,ML}`. The formalisation task (which reused parts of previous proofs of similar results, such as those in [8]) took approximately two weeks of full-time work for Dawson, who has over 25 years of experience in formal proof.

The paper describes a genuine collaboration over email between two distinct groups. We therefore deliberately present the results in a manner which allows the reader to ascertain which contributions are due to each group, and avoid the temptation to "dress-up" the results into one uniform notation (which would be intelligible to only one community).

The main difference between the term-rewriting community and the Isabelle/HOL community is that the notation for "relational composition" and "infinite chains" are opposites.

Leibniz International Proceedings in Informatics
Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

Fortunately, when combined, these two reversals cancel each other. We therefore begin with definitions from the former to make the paper accessible to this community. We deliberately swap the notation to that of Isabelle/HOL 2005 to faithfully record our methodology and avoid transcription errors. Finally, we return to the original notation at the end of the paper.

2 Well-Founded Tripartite Unions

Assuming that A, B and C are all well-founded binary relations over some set of elements, let us use the following shorthand notations:

AB	stands for	$\{(x,z) \mid \exists y. (x,y) \in A \,\&\, (y,z) \in B\}$
$AB \cup CD$	stands for	$(AB) \cup (CD)$
C^+	stands for	the transitive closure of C
C^*	stands for	the reflexive-transitive closure of C.

▶ **Theorem 1** ([9]). *The union $A \cup B \cup C$ of well-founded relations A, B, and C is well-founded if*

$$(A \cup B \cup C)(A \cup B \cup C) \subseteq A \cup B \cup C \tag{1}$$

Proof. The infinite version of Ramsey's Theorem applies when the union is transitive, so that every two (distinct) nodes within an infinite chain in the union of the colours has a coloured (directed) edge. Then, there must lie an infinite monochrome subchain within any infinite chain, contradicting the well-foundedness of each colour alone. ◀

▶ **Theorem 2** ([6]). *The union $A \cup B \cup C$ of well-founded relations A, B, and C is well-founded if*

$$BA \cup CA \cup CB \quad \subseteq \quad A \cup B \cup C\,. \tag{2}$$

Proof. When the union is not well-founded, there is an infinite forward-pointing path $X = \{x_i\}_i$ with each edge from x_i to x_{i+1} one of A, B, or C. Extract a maximal subsequence $Y = \{x_{i_j}\}_j$ of X such that $x_{i_j}\ A\ x_{i_{j+1}}$ for each j. If it's finite, then repeat at the first opportunity in the tail, and add the intervening steps to Y. If any is infinite, we have our contradiction. If they're all finite, then consider the first occurrence of $x\ (B \cup C)\ y\ A\ z$ in Y. Since we could not take an A-step from x or else we would have, the conditions tell us that $x\ (B \cup C)\ z$. Swallowing up all such (non-initial) A-steps in this way, we are left with an infinite chain in $B \cup C$, for which we also know that no A-steps are possible anywhere. Now extract maximal B-chains in the same fashion and then erase them, replacing $x\ C\ y\ B\ z$ with $x\ C\ z$ (A- and B-steps having been precluded), leaving an infinite chain coloured C. ◀

▶ **Corollary 3** ([6]). *If A, B, and C are transitive and $BA \cup CA \cup CB \subseteq A \cup B \cup C$, then, there is an infinite monochromatic clique whenever there is an infinite path in $A \cup B \cup C$.*

Proof. By Theorem 2, (at least) one of A, B, C is not well-founded. By transitivity, the elements of any infinite chain in that non-well-founded colour form an infinite clique. ◀

Let's call the existence of an infinite ascending chain in the union $A \cup B \cup C$-*immortality* and elements of such a chain, *immortal*. We can do considerably better than Theorem 2:

▶ **Theorem 4** (Tripartite [6]). *The union $A \cup B \cup C$ of well-founded relations A, B, and C is well-founded if both*

$$\begin{aligned} (B \cup C)A &\subseteq A(A \cup B \cup C)^* \cup B \cup C \\ CB &\subseteq A(A \cup B \cup C)^* \cup B^+ \cup C\,. \end{aligned} \tag{3}$$

Proof. We first construct an infinite chain $X = \{x_i\}_i$, in which an A-step is always preferred over B or C, as long as immortality is maintained. To do this, we start with an immortal element x_0 in the underlying set. At each stage in the construction, if the chain so far ends in x_i, we look to see if there is any y such that $x_i\ A\ y$ and from which proceeds some infinite chain in the union, in which case y is chosen to be x_{i+1}. Otherwise, x_{i+1} is any immortal element z such that $x_i\ B\ z$ or $x_i\ C\ z$.

If there are infinitely many B's and/or C's in X, use them – by means of the first condition – to remove all subsequent A-steps, leaving only B- and C-steps going out of points from which A leads of necessity to mortality. From what remains, if there is any C-step at a point where one could take one or more B-steps to anyplace later in the chain, take the latter route instead. What remains now are C-steps at points where B^+ detours are also precluded. If there are infinitely many such C-steps, then applying the condition for CB will result in a pure C-chain, because neither $A(A \cup B \cup C)^*$ nor B^+ are options. ◀

Dropping C from the conditions of the previous theorem, one gets the criterion $BA \subseteq A(A \cup B)^* \cup B$ of Doornbos and von Karger [8] for well-foundedness of the union of two well-founded relations A and B, called "jumping" [5]. Sans the B possibility on the right, this is equivalent to the *quasi-commutation* of Bachmair and Dershowitz a condition that comes into play in many rewriting situations [10, 4, 1]. Applying the jumping criterion twice, one gets somewhat different (incomparable) conditions for well-foundedness.

▶ **Theorem 5** (Jumping I [6]). *The union $A \cup B \cup C$ of well-founded relations A, B, and C is well-founded if both:*

$$\begin{aligned} BA &\subseteq A(A \cup B)^* \cup B \\ C(A \cup B) &\subseteq (A \cup B)(A \cup B \cup C)^* \cup C\,. \end{aligned}$$

Proof. The first inequality is the jumping criterion. The second is the same with C for B and $(A \cup B)$ in place of A. ◀

For two relations, jumping provides a substantially weaker criterion for well-foundedness than does the appeal to Ramsey. But for three, whereas jumping allows more than one step for BA (in essence, AA^*B^*), it doesn't allow for C, which Ramsey does.

Switching rôles, starting with jumping for $(B \cup C)$ before combining with A:

▶ **Theorem 6** (Jumping II [6]). *The union $A \cup B \cup C$ of well-founded relations A, B, and C is well-founded if both:*

$$\begin{aligned} CB &\subseteq B(B \cup C)^* \cup C \\ (B \cup C)A &\subseteq A(A \cup B \cup C)^* \cup B \cup C\,. \end{aligned}$$

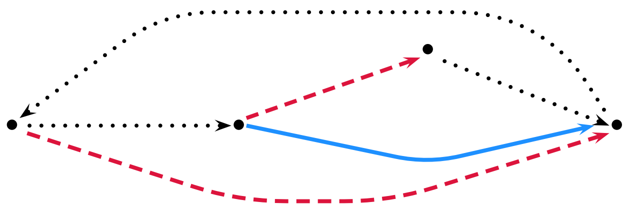
Both this version of jumping and our tripartite condition allow

$$\begin{aligned} (B \cup C)A &\subseteq A(A \cup B \cup C)^* \cup B \cup C \\ CB &\subseteq B^+ \cup C\,. \end{aligned}$$

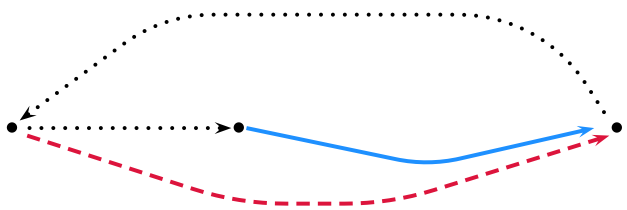
They differ in that jumping also allows the condition shown below on the left whereas tripartite has the one shown on the right instead:

jumping allows	tripartite allows
$CB \subseteq B(B \cup C)^*$	$CB \subseteq A(A \cup B \cup C)^*$

Sadly, we cannot have the best of both worlds. Let's color edges A, B, and C with (solid) azure, (dotted) black, and (dashed) crimson ink, respectively. The graph below only has multicolored loops despite satisfying both $(B \cup C)A \subseteq C$ and $CB \subseteq A \cup B(B \cup C)^*$:

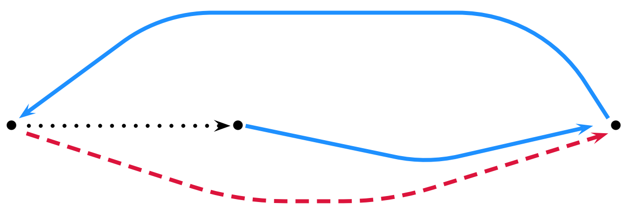


Even $(B \cup C)A \subseteq C$ and $CB \subseteq B(A \cup B)^*$ doesn't work. To wit, the double loop in the graph below harbours no monochrome subchain:



By the same token, the putative hypothesis below left is countered by the graph below right:

$$\begin{array}{rcl} BA \cup CB & \subseteq & C \\ CA & \subseteq & BA^* \end{array}$$



3 Verifying these Theorems in Isabelle/HOL 2005

In Section 2, we started with Theorems 1 and 2, and proved Theorem 4, which is a considerable improvement over Theorems 1 and 2. Now, we start with Theorem 4, which we call `wf_tri`. But we first "push" the Isabelle/HOL notation onto the stack.

▶ **Definition 7.** In Isabelle/HOL 2005, the well-foundedness and composition of relations are defined as follows, where $x <_R y$ means $(x, y) \in R$:

1. A relation R is *well-founded* if there is no infinite descending chain

$$\cdots <_R x_n <_R x_{n-1} <_R \cdots <_R x_1 <_R x_0$$

The Isabelle/HOL 2005 definition below states the positive form, which is that a relation R is well-founded iff the principle of well-founded induction holds for R:

```
wf ?R == ALL P.
           (ALL x. (ALL y. (y, x) : ?R --> P y) --> P x)
         --> (ALL x. P x)
```

Next, we give its equivalent, which says that a relation R is well-founded if every non-empty set Q has an R-minimal member:

```
wf ?R = (ALL Q x. x : Q --> (EX z:Q. ALL y. (y, z) : ?R --> y ~: Q))
```

Then the expression which states precisely that `wf` R iff there are no infinite descending chains is as follows, where `Suc` signifies successor in the naturals:

```
wf ?R = (~ (EX f. ALL i. (f (Suc i), f i) : ?R))
```

2. The *composition* of relations R and S is defined by

$$R \circ S = \{(x, z) \mid \exists y. (x, y) \in S \;\&\; (y, z) \in R\}$$

```
   ?R O ?S == {(x, z). EX y. (x, y) : ?S & (y, z) : ?R}
```

In Definition 7, the question mark symbol `?` indicates implicit universal quantification and so `?R` and `?S` are free variables (parameters) that are instantiated. The notation `~` encodes classical negation and `:` encodes $\in$, so `~:` encodes $\notin$.

▶ Remark. Our notation RS from Section 2 and the Isabelle/HOL 2005 notation $R \circ S$ for "relational composition" use their operands in opposite order, obeying $RS = (R^{-1} \circ S^{-1})^{-1}$:

$$\begin{aligned} RS = S \circ R \quad &= \quad \{(a, c) \mid \exists b. (a, b) \in R \;\&\; (b, c) \in S\} \\ \text{that is,} \quad &\qquad a \xrightarrow{R} b \xrightarrow{S} c \\ (RS)^{-1} = R^{-1} \circ S^{-1} \quad &= \quad \{(c, a) \mid \exists b. (c, b) \in S^{-1} \;\&\; (b, a) \in R^{-1}\} \end{aligned}$$

Following Doornbos and von Karger [8], we define a binary condition *dvk_cond* D A as the Isabelle/HOL 2005 analogue of jumping.

▶ **Definition 8** (`dvk_cond`).

$$D \circ A \subseteq (A \circ (A \cup D)^*) \cup D$$

```
dvk_cond ?D ?A == ?D O ?A <= (?A O (?A Un ?D)^*) Un ?D
```

In Definition 8, `O` is the relational composition operator in Isabelle/HOL 2005 from Definition 7; `Un` is Isabelle's set-union operator; and `<=` is $\subseteq$.

▶ Remark. Since the definition of composition $\circ$ and the definition `wf` of well-founded used in Isabelle are *both* mirror images of those used in Sections 1 and 2, the theorems in the sequel capture the same notions. Were only one different, we would have to reverse the order of relation composition to make the two notions coincide.

We separate Dershowitz's proof of the Tripartite Theorem 4 from Section 2 into a number of lemmas. Generally, statements in the proof of the form "there exists an infinite chain" are translated into a (contrapositive) lemma with hypotheses and conclusion that certain relations are well-founded, or that something is in the "well-founded part" of a relation.

▶ **Definition 9** (`wfp R`). Given a binary relation R, the *well-founded part* $\texttt{wfp}(R)$ is the set whose elements are not the head of any infinite descending R-chain.

```
(?z : wfp ?R) = (~ (EX f. f 0 = ?z & (ALL i. (f (Suc i), f i) : ?R)))
```

The set `wfp R` is defined using Isabelle's inductively defined sets. Thus, the members of the set `wfp R` are precisely those elements which can be shown to be in `wfp R` by repeated applications of the rule below:

```
    ALL x. (x, ?y) : ?R --> x : wfp ?R ==> ?y : wfp ?R
```

Provably equivalent is the other characterisation of `wfp` R above in terms of an infinite descending chain. So a relation R over an underlying set V is well-founded if and only if every member of the underlying set V is in the well-founded part of R. The set-complement operation is written as `-` in Isabelle; thus `- wfp R` is the set $\overline{wfp}(R)$ of those elements which are the head of an infinite descending R-chain. We write $\overline{wfp}(R)$ rather than $\overline{wfp(R)}$ to avoid clutter in the technical equations that follow.

▶ **Definition 10** (Immortal [5]). An R-*immortal* element is the head of an infinite descending R-chain and "*immortal*" means $(A \cup R)$-immortal, or $(A \cup B \cup C)$-immortal, as relevant.

```
immortal_def:    "immortal ?R ?y == (?y : (- wfp ?R))"
```

▶ Remark. The Isabelle "immortal" is the mirror image of that used in Section 1 and 2.

▶ **Definition 11.** The (forward) *image* $R(s)$ (``R `` s`` in the code) of a set s under a relation R consists of those z such that some $y \in s$ has $(y, z) \in R$.

```
Image_def: "?R `` ?s == {z. EX y:?s. (y, z) : ?R}"
```

In Definition 11, the expression `y:?s` encodes $y \in s$ and `(y, z) : ?R` encodes $(y, z) \in R$; thus in the next Definition 12, `~:` is $\notin$.

▶ **Definition 12** (Constriction R^- [5, 3]).

$$\begin{aligned} R^- &= R \setminus \{(w, z) \mid \exists y. (y, z) \in A \ \&\ y \text{ is } (A \cup R)\text{-immortal } \} \\ &= R \setminus \{(w, z) \mid \exists y. (y, z) \in A \ \&\ y \in \overline{wfp}(A \cup R)\} \\ &= R \cap \{(w, z) \mid z \notin A(\overline{wfp}(A \cup R))\} \end{aligned}$$

```
 rminus ?R ?A == (?R Int {(w, z). z ~: ?A `` (- wfp (?A Un ?R))})
```

The idea of constriction was inspired by Plaisted [12]. Thus R^- excludes from R all steps of the form (w, z) where there exists some y such that $(y, z) \in A$ and y is $(A \cup R)$-immortal.

▶ **Lemma 13** (`wfp_tri_lem_ch` [6, 3]). *If $A \cup R^-$ is well-founded then $A \cup R$ is well-founded.*

```
wfp_tri_lem_ch:
   "?x : wfp (?A Un (rminus ?R ?A)) ==> ?x : wfp (?A Un ?R)"
```

Proof. This is restated as Corollary 14: we prove it in those terms by constructing an infinite descending $(A \cup R)$-chain using A where possible and using R only where necessary. ◀

▶ **Corollary 14** ([3]). *If there is an infinite descending $(A \cup R)$-chain then there is an infinite descending $(A \cup R)$-chain where each step $(w, z) \in R$ in that chain has the property that there is no $(y, z) \in A$ such that y is the head of an infinite descending $(A \cup R)$-chain.*

▶ **Lemma 15** (wfp_tri_lem_dvk [3]). *If R and A satisfy the Doornbos and von Karger [8] condition* dvk_cond *(Definition 8 with R for D), and A and R^- are well-founded, then $A \cup R$ is well-founded.*

```
wfp_tri_lem_dvk: "[| dvk_cond ?R ?A;
      wf ?A; wf (rminus ?R ?A) |] ==> ?x : wfp (?A Un ?R)"
```

We give an intuitive proof, then a proof which closely approximates the Isabelle proof.

Intuitive Proof. Consider an infinite descending $(A \cup R)$-chain

$$\cdots \quad <_{A\cup R} \quad x_2 \quad <_{A\cup R} \quad x_1 \quad <_{A\cup R} \quad x_0$$

As A is well-founded, from somewhere after any point in this chain there must be an alternative R-step. Wherever possible replace an R-step $z >_R v$ followed by an A-step $v >_A u$ by an A-step (possibly followed by a number of A- or R-steps) in the given chain. That is, use the fact that $R \circ A \subseteq (A \circ (A \cup R)^*) \cup \cdots$ of the dvk_cond.

Alternatively, if possible, replace the remainder of the chain from the head z of an R-step $z >_R v$ by any other infinite descending chain $z >_A y >_{A\cup R} \cdots$. Again, this will not always be possible since A is well-founded.

So, where this is not possible, we can replace an R-step followed by an A-step by a single R-step, which is possible using the part $R \circ A \subseteq \ldots \cup R$ of dvk_cond.

Doing this repeatedly could absorb any number of A-steps, but, again, as A is well-founded there cannot be infinitely many such A-steps, so eventually we reach another R-step. Repeating this gives the required infinite R-chain, with the given property.

The next proof more closely approximates the Isabelle proof.

Proof. Assume that A is well-founded. Define R^- to be R, excluding steps of the form (w, z) where there exists some y such that $(y, z) \in A$ and y is immortal. Assume that R^- is well-founded.

Suppose that there is an infinite descending $(A \cup R)$-chain. Choose a head z of such a chain, choosing z to be A-minimal, and also to be R^--minimal among A-minimal immortal elements. As z is A-minimal, the first step of an infinite chain must be $(y, z) \in R$ (say), and also in fact $(y, z) \in R^-$. As A is well-founded, let y be A-minimal among possible choices for y. Then, by the R^--minimality of z, although y is immortal, it is not A-minimal among immortal members. So we have $(x, y) \in A$, where x is immortal. Since $R \circ A \subseteq (A \circ (A \cup R)^*) \cup R$, we could replace $z >_R y >_A x$ in the infinite chain by

$\boldsymbol{(x, z) \in A \circ (A \cup R)^*}$: say $(x, y') \in (A \cup R)^*$ and $(y', z) \in A$ where x, y' and z are immortal. But this would contradict the A-minimality of z and our consequent inference that $(y, z) \notin A$, or

$\boldsymbol{(x, z) \in R}$: which would contradict our choice of y which was to be A-minimal, since x could have been chosen instead of y.

Thus, either way we get a contradiction, so $A \cup R$ is well-founded. ◀

▶ **Corollary 16** ([3]). *If R and A satisfy the Doornbos and von Karger [8] condition* dvk_cond*, and A is well-founded, and there is an infinite descending $(A \cup R)$-chain, then there is an infinite descending R-chain whose members are not of the form z where $(y, z) \in A$ and y is immortal, that is, is the head of an infinite descending $(A \cup R)$-chain.*

▶ **Definition 17.** Given some fixed relation X, define C^- to be C, excluding steps of the form (w,z) where there exists some y such that $(y,z) \in A$ and y is $(X \cup B \cup C)$-immortal.

$$\begin{aligned} C^- &= C \setminus \{(w,z) \mid \exists y.\, (y,z) \in A \;\&\; y \text{ is } (X \cup B \cup C)\text{-immortal}\} \\ &= C \cap \{(w,z) \mid z \notin A(\overline{wfp}(X \cup B \cup C))\} \end{aligned}$$

```
cminus ?C ?B ?A ?X ==
     (?C Int {(w, z). z ~: ?A `` (- wfp (?X Un ?B Un ?C))})
```

For the purposes of Theorem 19, X will be A, but we will use the lemma with more general X when there are four or more relations.

▶ **Lemma 18** (`wf_tri_lem_BCm` [3]). *If B and C^- are well-founded, and condition (c) of Theorem 19 is satisfied, then $B \cup C^-$ is well-founded.*

```
wf_tri_lem_BCm:
 "[| wf ?B;
    wf (cminus ?C ?B ?A ?X);
    ?C O ?B <= (?A O (?X Un ?B Un ?C)^*) Un (?B O ?B^*) Un ?C |]
  ==> wf (?B Un (cminus ?C ?A ?B ?X))"
```

Proof. Assume that B and C^- are well-founded.

For a contradiction, suppose that $B \cup C^-$ is not well-founded. So there exists an $(B \cup C^-)$-immortal z. Choose z to be B-minimal such, and also to be C^--minimal among possible B-minimal choices.

As z is B-minimal, the first step of an infinite $B \cup C^-$-chain must be $(y,z) \in C^-$ (say). As B is well-founded, let y be B-minimal among possible choices for y. Then, by the C^--minimality of z, although y is $(B \cup C^-)$-immortal, it is not B-minimal among $(B \cup C^-)$-immortal members. So we have $(x,y) \in B$, where x is $(B \cup C^-)$-immortal. Since $C \circ B \subseteq (A \circ (X \cup B \cup C)^*) \cup (B \circ B^*) \cup C$, we could replace $z >_C y >_B x$ in the infinite chain by

$(x,z) \in A \circ (X \cup B \cup C)^*$: say $(x,y') \in (X \cup B \cup C)^*$ and $(y',z) \in A$ – but x is $(B \cup C^-)$-immortal, and so is the head of an infinite descending $X \cup B \cup C$-chain, contradicting that $(y,z) \in C^-$; or

$(x,z) \in B \circ B^*$: which would contradict our choice of z to be B-minimal; or

$(x,z) \in C$: and so $(x,z) \in C^-$ which would contradict our choice of y to be B-minimal, since x could have been chosen instead of y.

Thus, in each case we get a contradiction, so $B \cup C^-$ is well-founded. ◀

▶ **Theorem 19** (`wf_tri`, Tripartite [6, Theorem 3]). *Given three binary relations A, B and C, their tripartite union $A \cup B \cup C$ is well-founded if all of the following hold:*

(a) each of A, B, C is well-founded;
(b) dvk_cond $(B \cup C)$ A;
(c) $C \circ B \subseteq (A \circ (A \cup B \cup C)^*) \cup (B \circ B^*) \cup C$.

```
wf_tri:  "[|
   wf ?A ; wf ?B ; wf ?C ;
   dvk_cond (?B Un ?C) ?A ;
   ?C O ?B <= (?A O (?A Un ?B Un ?C)^*) Un (?B O ?B^*) Un ?C
   |] ==> wf (?A Un ?B Un ?C)"
```

Proof. By Corollary 16 it is enough to prove that $(B \cup C)^-$ is well-founded. By Lemma 18 we have that $B \cup C^-$ is well-founded, and clearly $(B \cup C)^- \subseteq B \cup C^-$. ◀

4 Quadripartite Well-Foundedness in Isabelle/HOL 2005

Now we show how to extend this result to four relations, still using the Isabelle notions of "well-founded" and "composition". While this is a special case of the subsequent result for an arbitary number of relations, the specific case of four relations displays clearly the pattern of the general result and its proof. In particular, it was the way that the general result was found. It's always good to learn to walk before learning to run.

▶ **Theorem 20** (wf_four [3]). *$Uall = A \cup B \cup C \cup D$ is well-founded if*

(a) *each of A, B, C and D is well-founded, and*
(b) $(B \cup C \cup D) \circ A \subseteq (A \circ (Uall)^*) \cup (B \cup C \cup D)$
(c) $D \circ C \subseteq (A \circ (Uall)^*) \cup (C \circ C^*) \cup D$
(d) $(C \cup D) \circ B \subseteq (A \circ (Uall)^*) \cup (B \circ B^*) \cup C \cup D$

```
val wf_four =
 "[| ?uall = ?A Un ?B Un ?C Un ?D;
     wf ?A; wf ?B; wf ?C; wf ?D ;
     dvk_cond (?B Un ?C Un ?D) ?A;
     ?D O ?C <= (?A O ?uall^*) Un (?C O ?C^*) Un ?D;
     (?C Un ?D) O ?B <= (?A O ?uall^*) Un (?B O ?B^*) Un (?C Un ?D)
  |] ==> wf ?uall"
```

Proof. We adapt the previous definition of R^- for a relation R: that is, R^- is R, excluding steps of the form (w, z) where there exists some y such that $(y, z) \in A$ and y is $(A \cup B \cup C \cup D)$-immortal:

$$\begin{aligned} R^- &= R \setminus \{(w,z) \in R \mid \exists y.\, (y,z) \in A \text{ and } y \text{ is } (A \cup B \cup C \cup D)\text{-immortal}\}. \\ &= R \cap \{(w,z) \mid z \notin A(\overline{wfp}(A \cup B \cup C \cup D))\} \end{aligned}$$

As D is well-founded, *a fortiori*, D^- is well-founded. Setting A, X, B, C to $A, A \cup B, C, D$, respectively, in Lemma 18 and using condition (c), we obtain that $C \cup D^-$ is well-founded, and so *a fortiori* $(C \cup D)^-$ is well-founded. Setting A, X, B, C to $A, A, B, C \cup D$, respectively in Lemma 18 and using condition (d) we obtain that $B \cup (C \cup D)^-$ is well-founded and so *a fortiori* $(B \cup C \cup D)^-$ is well-founded. Finally, by Corollary 16 $A \cup B \cup C \cup D$ is well-founded. ◀

5 The General Result in Isabelle/HOL 2005

We now extend Theorem 20 to an arbitrary number of relations. The proof is a relatively straightforward generalization of our previous ones, but we need some new definitions.

Foremost is the condition we call `tri_cond` in the formal proofs, which is the condition satisfied by the list $[B, C, D]$ in the case of four relations. It is defined recursively below and yields (for example) conditions (c) and (d) of Theorem 20. The point is that we require the union of $B, C, D, \ldots$ to be well-founded so as to apply `dvk_cond` in conjunction with A, but were we to simply use the same method to establish this, we would not be allowed to introduce any A-steps in the inclusions for compositions of pairs from $B, C, D, \ldots$.

▶ **Definition 21.** For an arbitrary binary (infix) operator f, define the right-fold operator for list $[B, C, D]$ with initial value Z to be

$$\text{foldr } f \ [B, C, D] \ Z = B \ f \ (C \ f \ (D \ f \ Z))$$

The code below assumes prefix notation and generalises to arbitrary lists.

```
foldr.simps : [
   "foldr ?f [] ?z = ?z",
   "foldr ?f (?x # ?xs) ?z = ?f ?x (foldr ?f ?xs ?z)"]
```

Here `#` is the (cons) operator which adds an element x to the list $xs = [x_1, x_2, ..., x_n]$ of elements to give the list $[x, x_1, x_2, ..., x_n]$.

▶ **Definition 22** ([3])**.** For all `Auall`, the condition *`tri_cond Auall L`* is true if
Base Case: L is the empty list
Inductive Case: L=B.Bs and

- uBs $= \bigcup_{B_i \in \texttt{Bs}} B_i$ and
- uBs $\circ$ B $\subseteq$ Auall $\cup$ (B $\circ$ B*) $\cup$ uBs and
- *`tri_cond Auall Bs`* holds.

```
tri_cond_Nil  : "tri_cond ?Auall [] = True"
tri_cond_Cons : "tri_cond ?Auall (?B # ?Bs) =
   ((let uBs = foldr (op Un) ?Bs {}
      in uBs O ?B <= ?Auall Un (?B O ?B^*) Un uBs)
   & tri_cond ?Auall ?Bs)"
```

We then need a separate lemma which is proved by structural induction on the list, say $[B, C, D]$, of relations. For the general case we work with a list $B_1, B_2, \ldots, B_n$.

▶ **Definition 23** ([3])**.** Define Ubs^- to be Ubs, excluding steps of the form (w, z) where there exists some y such that $(y, z) \in A$ and y is $(X \cup Ubs)$-immortal for some given relation X:

$$\begin{aligned} Ubs^- &= Ubs \setminus \{(w, z) \in Ubs \mid \exists y. (y, z) \in A \text{ and } y \text{ is } (X \cup Ubs)\text{-immortal } \} \\ &= Ubs \cap \{(w, z) \in Ubs \mid z \notin \overline{wfp}(X \cup Ubs)\} \end{aligned}$$

▶ **Lemma 24** ([3])**.** *If $Bs = [B_1, B_2, \ldots, B_n]$, and we are given some fixed relation X, then Ubs^- is well-founded if*

(a) *$B_1, B_2, \ldots, B_n$ are all well-founded; and*
(b) *$Ubs = (B_1 \cup B_2 \cup \cdots \cup B_n)$; and*
(c) `tri_cond` $(A \circ (X \cup Ubs)^*)$ $[B_1, B_2, \ldots, B_n]$ *holds*

```
tri_n_lem :
 "[| ALL B:(set ?Bs) . wf B ;
     ?uBs = foldr (op Un) ?Bs {};
     ?Auall = ?A O (?X Un ?uBs)^*;
     tri_cond ?Auall ?Bs |] ==> wf (uminus ?uBs ?A ?X)"
```

Proof. The proof is by structural induction on the list $Bs = [B_1, B_2, \ldots, B_n]$. That is, the base case is when Bs is the empty list $[\ ]$, and the inductive step is when $Bs = [B_i, B_{i+1}, \cdots, B_n]$ and X is an arbitrary X_i and then the inductive hypothesis used is that the lemma holds for $Bs = [B_{i+1}, \cdots, B_n]$ and $X = X_i \cup B_i$. The inductive step is proved using Lemma 18.

The value used for X varies throughout, so if its final value is X_f, then the lemma is proved successively for

$[\,]$	$X = X_f \cup B_1 \cup B_2 \cup \cdots \cup B_n$
$[B_n]$	$X = X_f \cup B_1 \cup B_2 \cup \cdots \cup B_{n-1}$
$\vdots$	$\vdots$
$[B_2, \ldots, B_n]$	$X = X_f \cup B_1$
$[B_1, B_2, \ldots, B_n]$	$X = X_f$

Note that the value of $X \cup \mathit{Ubs}$ (as used in the definition of Ubs^- above) therefore remains unchanged, and that at each step we also use, in addition to Lemma 18, the fact that if $B \cup V^-$ is well-founded, then $(B \cup V)^-$ is well-founded. ◀

We have reached our main result.

▶ **Theorem 25** ([3]). *A union $A \cup B_1 \cup B_2 \cup \cdots \cup B_n$ $(n \geq 1)$ is well-founded if*
(a) A and $B_1, B_2, \ldots, B_n$ are each well-founded;
(b) `tri_cond` $(A \circ (A \cup \mathit{Ubs})^*)$ $[B_1, B_2, \ldots, B_n]$ holds, where $\mathit{Ubs} = B_1 \cup B_2 \cup \cdots \cup B_n$; and
(c) $\mathit{Ubs} \circ A \subseteq (A \circ (A \cup \mathit{Ubs})^*) \cup \mathit{Ubs}$.

```
tri_n "[|
  wf ?A;  ALL B : (set ?Bs). wf B
  ?uBs = (foldr (op Un) ?Bs {});
  ?Auall = ?A O (?A Un ?uBs)^*; tri_cond ?Auall ?Bs;
  dvk_cond ?uBs ?A |] ==> wf (?A Un ?uBs)"
```

Proof. By Lemma 24, we get that Ubs^- is well-founded, where we use $X = A$ in the definition of Ubs^-. Then Corollary 16 gives the result. ◀

6 Preferential Commutation versus Jumping

We now "pop" the notation stack and return to our original definitions of "well-founded" and "composition" from Section 1 and Section 2.

Consider Theorem 25, letting $A = B_0$ and $B = B_0 \cup \cdots \cup B_n$. The conditions (b) and (c) of Theorem 25 give our new combined condition for B to be well-founded. Re-expressing this combination of conditions in our original notation from Sections 1 and 2 and using subscripts for partial unions, we obtain our new notion of *Preferential Commutation*:

$$B_{i+1\,..\,n} B_i \subseteq B_0 B^*_{0\,..\,n} \cup B^+_i \cup B_{i+1\,..\,n} \text{ for all } i = 0\,..\,n-1.$$

Condition `tri_cond` generalises the conjunction of conditions (c) and (d), but not (b), of Theorem 20; Preferential Commutation, on the other hand, generalises the conjunction of conditions (b), (c) and (d) of Theorem 20.

Compare Preferential Commutation with repeated application of Jumping II (Theorem 6), which yields the equivalent of

$$B_{i+1\,..\,n} B_i \subseteq B_i B^*_{i+1\,..\,n} \cup B^+_i \cup B_{i+1\,..\,n} \text{ for all } i = 0\,..\,n-1$$

The beauty of Preferential Commutation lies in that it allows initial "preferred" B_0-steps and even more.

7 Finite Rearrangement

As noted above (and implicit in Theorems 5 and 6), the “jumping” condition is sufficient for the union of two well-founded relations to be well-founded.

From [5, Theorem 54] we get that this condition is also sufficient for any finite chain $(x, z) \in A \cup B$ to be rearranged as $(x, z) \in A^* \circ B^*$, *provided that* A and B are well-founded. We get similar results corresponding to Theorems 20 and 25.

▶ **Theorem 26** (`wf_rearr_four`). *If*

(a) each of A, B, C and D is well-founded, and
(b) $(B \cup C \cup D) \circ A \subseteq (A \circ (A \cup B \cup C \cup D)^*) \cup (B \cup C \cup D)$
(c) $D \circ C \subseteq (A \circ (A \cup B \cup C \cup D)^*) \cup (C \circ C^*) \cup D$
(d) $(C \cup D) \circ B \subseteq (A \circ (A \cup B \cup C \cup D)^*) \cup (B \circ B^*) \cup C \cup D$

and $(w, z) \in (A \cup B \cup C \cup D)^*$ then $(w, z) \in A^* \circ B^* \circ C^* \circ D^*$

```
val wf_rearr_four =
   "[| ?uall = ?A Un ?B Un ?C Un ?D; dvk_cond (?B Un ?C Un ?D) ?A;
      ?D O ?C <= (?A O ?uall^*) Un (?C O ?C^*) Un ?D;
     ?C Un ?D O ?B <= (?A O ?uall^*) Un (?B O ?B^*) Un (?C Un ?D);
       wf ?A; wf ?B; wf ?C; wf ?D; (?w, ?z) : ?uall^* |] ==>
     (?w, ?z) : ?A^* O ?B^* O ?C^* O ?D^*"
```

Proof. From Theorem 20 we get that $A \cup B \cup C \cup D$ is well-founded.

Therefore we can take z to be an $(A \cup B \cup C \cup D)$-minimal counterexample.

If $w = z$ of course the result is trivial, so let the chain be $w, \ldots, y, z$, and (having fixed x and z), chose y be $(A \cup B \cup C \cup D)$-minimal: that is, y is minimal subject to the conditions that $(y, z) \in A \cup B \cup C \cup D$ and $(w, y) \in (A \cup B \cup C \cup D)^*$.

(A): Now, by the minimality of z, y has the property that we can rearrange steps between w and y, so that $(w, y) \in A^* \circ B^* \circ C^* \circ D^*$. So if $(y, z) \in A$, then $(w, z) \in A^* \circ B^* \circ C^* \circ D^*$, as required.

If $w = y$ then again the result is trivial, so let the chain $(w, y) \in A^* \circ B^* \circ C^* \circ D^*$ be $w, \ldots, x, y$, where $(x, y) \in A \cup B \cup C \cup D$. This gives cases

- $(w, x) \in A^* \circ B^* \circ C^* \circ D^*$, $(x, y) \in A$
- $(w, x) \in B^* \circ C^* \circ D^*$, $(x, y) \in B$ (and $(x, y) \notin A$)
- $(w, x) \in C^* \circ D^*$, $(x, y) \in C$ (and $(x, y) \notin A \cup B$)
- $(w, x) \in D^*$, $(x, y) \in D$ (and $(x, y) \notin A \cup B \cup C$)

So let $(y, z) \in B \cup C \cup D$. If $(x, y) \in A$ then we have $(x, z) \in (B \cup C \cup D) \circ A \subseteq (A \circ (A \cup B \cup C \cup D)^*) \cup (B \cup C \cup D)$ by condition (b), which gives us two cases.

(B): If $(x, z) \in A \circ (A \cup B \cup C \cup D)^*$ then we have y' such that $(y', z) \in A$ and $(x, y') \in (A \cup B \cup C \cup D)^*$, hence $(w, y') \in (A \cup B \cup C \cup D)^*$, and the argument at (A) above applies.

The other case is $(x, z) \in B \cup C \cup D$, which contradicts the minimality of y.

So we now have $(x, y) \in B \cup C \cup D$.

Now, with (x, y) and (y, z) in $B \cup C \cup D$, certain cases are easy. For example, if $(x, y) \in C$ and $(w, x) \in C^* \circ D^*$, then $(w, y) \in C^* \circ D^*$, so if $(y, z) \in B \cup C$, this gives $(w, z) \in B^* \circ C^* \circ D^* \subseteq A^* \circ B^* \circ C^* \circ D^*$, as required.

The remaining cases are

- $(y, z) \in D, (x, y) \in C$, $(w, x) \in C^* \circ D^*$ and
- $(y, z) \in C \cup D, (x, y) \in B$, $(w, x) \in B^* \circ C^* \circ D^*$.

So in these cases, conditions (c) or (d), respectively, apply. For example, for the first case condition (c) gives one of:

- $(x, z) \in A \circ (A \cup B \cup C \cup D)^*$, and the argument at (B) above applies;
- $(x, z) \in C \circ C^*$: since $(w, x) \in C^* \circ D^*$, we have $(w, z) \in C^* \circ D^* \subseteq A^* \circ B^* \circ C^* \circ D^*$, as required.
- $(x, z) \in D$: this contradicts the minimality of y.

The second case is similar. ◀

Now we generalise the result to any number of relations.

▶ **Theorem 27** (`wf_rearr_n`). *A union $A \cup B_1 \cup B_2 \cup \dots \cup B_n$ $(n \geq 1)$ is well-founded if*
(a) A and $B_1, B_2, \dots, B_n$ are each well-founded;
(b) `tri_cond` $(A \circ (A \cup Ubs)^*)$ $[B_1, B_2, \dots, B_n]$ holds, where $Ubs = B_1 \cup B_2 \cup \dots \cup B_n$; and
(c) $Ubs \circ A \subseteq (A \circ (A \cup Ubs)^*) \cup Ubs$,
and $(w, z) \in (A \cup Ubs)^*$, then, where $Obs = B_1^* \circ B_2^* \circ \dots \circ B_n^*$, $(w, z) \in Obs$.

```
val wf_rearr_n =
  "[| ?uBs = foldr op Un ?Bs {}; ?Auall = ?A O (?A Un ?uBs)^*;
     dvk_cond ?uBs ?A; tri_cond ?Auall ?Bs; wf ?A; Ball (set ?Bs) wf;
    (?w, ?z) : (?A Un ?uBs)^* |] ==>
    (?w, ?z) : ?A^* O foldr op O (map rtrancl ?Bs) Id"
```

Proof. From Theorem 25 we get that $A \cup Ubs$ is well-founded.

Therefore we can take z to be an $(A \cup Ubs)$-minimal counter-example.

If $w = z$ of course the result is trivial, so let the chain be $w, \dots, y, z$. and (having fixed x and z), chose y be $(A \cup Ubs)$-minimal: that is, y is minimal subject to the conditions that $(y, z) \in A \cup Ubs$ and $(w, y) \in (A \cup Ubs)^*$.

(A): Now, by the minimality of z, y has the property that $(w, y) \in A^* \circ Obs$. So if $(y, z) \in A$, then $(w, z) \in A^* \circ Obs$, as required.

If $w = y$ then again the result is trivial, so let the chain $(w, y) \in A^* \circ Obs$ be $w, \dots, x, y$, where $(x, y) \in A \cup Ubs$.

So let $(y, z) \in Ubs$. If $(x, y) \in A$ then we have $(x, z) \in (Ubs) \circ A \subseteq (A \circ (A \cup Ubs)^*) \cup Ubs$ which gives us two cases.

(B): If $(x, z) \in A \circ (A \cup Ubs)^*$ then we have y' such that $(y', z) \in A$ and $(x, y') \in (A \cup Ubs)^*$, hence $(w, y') \in (A \cup Ubs)^*$, and the argument at (A) above applies.

The other case is $(x, z) \in Ubs$, which contradicts the assumption of the minimality of y.

So we now have $(x, y) \in Ubs$, and $(w, x) \in Obs$.

Now, with (y, z) in Ubs, certain cases are easy: let $(x, y) \in B_i$ and $(w, y) \in B_i^* \circ B_{i+1}^* \circ \dots B_n^*$, then if $(y, z) \in B_j$ for $j \leq i$, this gives $(w, z) \in B_j^* \circ B_{j+1}^* \circ \dots B_n^* \subseteq A^* \circ Obs$, as required.

Write $B_{>i}$ for $B_{i+1} \cup \dots \cup B_n$. The remaining cases are $(y, z) \in B_j$, $(x, y) \in B_i$ and $(w, y) \in B_i^* \circ B_{i+1}^* \circ \dots B_n^*$ where $i < j$. So in these cases, the condition (b), that ***tri_cond*** $(A \circ (A \cup Ubs)^*)$ $[B_1, B_2, \dots, B_n]$ holds, and in particular its "case" $B_{>i} \circ B_i \subseteq (A \circ (A \cup Ubs)^*) \cup (B_i \circ B_i^*) \cup B_{>i}$

Since $(x, z) \in B_{>i} \circ B_i$, this gives one of the following cases.

- $(x, z) \in A \circ (A \cup Ubs)^*$, and the argument at (B) above applies
- $(x, z) \in B_i \circ B_i^*$: since $(w, x) \in B_i^* \circ B_{i+1}^* \circ \dots B_n^*$ we have $(w, z) \in B_i^* \circ B_{i+1}^* \circ \dots B_n^* \subseteq A^* \circ Obs$, as required.
- $(x, z) \in B_k$ for some $k > i$: this contradicts the minimality of y.

◀

8 Devil's Advocate Questions

The first author started using Isabelle in 1996 and actively brought his proof scripts up to date with every new incarnation of Isabelle. By 2005, his proof-scripts had become so voluminous that it became impossible to keep bringing them up to date. Moreover, we have build upon our previous work, as stated earlier. We therefore decided to keep working in Isabelle/HOL 2005. We now discuss the objections that will no doubt arise:

Isabelle/HOL 2005 is too old: Why? All of the work done in 2005 uses this version. Does every new incarnation of Isabelle invalidate all of the old formalisations?

One cannot check the development presented: Isabelle/HOL 2005 has checked it!

The pen and paper proofs don't correspond to the Isabelle proofs: We deliberately presented them that way because they are different!

Applications are missing: So we should not publish this fundamental result until we have applications?

How does this relate with other formalization works in Isabelle/HOL or other provers:? Who cares? We have presented and machine-checked a fundamental new result which significantly goes beyond the current known results on well-founded unions.

9 Conclusion and Further Work

Whereas previous work has provided sufficient conditions for the union of two well-founded orderings to be well-founded, we have found a corresponding result for the union of three well-founded orderings. We have discussed how our sufficient conditions differ from those which result merely from the repeated application of the result for two orderings.

Our methodology was to repackage the proof of this result for three orderings and extend it to four orderings to look for a pattern. We then extended it to the union of any number of well-founded orderings – defining a condition called Preferential Commutation.

We have presented proofs of these results, which have been verified using the Isabelle/HOL 2005 theorem prover. Of course an attempt at a formal proof is most valuable when it fails, pinpointing a flaw in the less formal proof. Where that is not the case, as here, the process of proving the results in Isabelle/HOL 2005 is valuable because it forces one to clearly set out the steps of reasoning and the assumptions upon which each step depends. As here, it also sometimes allows us to give different proofs, using "positive" notions (such as `wfp`) rather than "negative" notions (such as "no infinite chains"). Compare, for example the two proofs of Lemma 15. Further, as always, formalising a proof confirms that no details have been overlooked or other errors made.

The choice of prover depends on a number of (sometimes conflicting) criteria:

- ease of understanding the mathematical syntax used
- ease of understanding the steps in a proof
- ease of constructing a proof
- speed of performing the proof on normal hardware
- likelihood that the proof script will be useful in 10, 20 or 50 years time.

As noted above, this work uses Isabelle 2005 for good reasons. Interactive theorem proving often requires a major re-proving effort when minor changes to a previous theorem are required. Surprisingly, the Isabelle proofs performed for this paper were not unduly difficult since it was easy to adapt our previous proofs of results (such as [8]).

Further matters to be explored are:

- Can we "extract" any code (semi-)automatically? For example, if we express results in the contrapositive, then given an infinite descending chain in one relation, to obtain an infinite descending chain in another, as in the written proof of Lemma 13.
- Can we obtain a better understanding of the `tri_cond` condition which might allow the results reported here to be extended?
- What effect would transitivity of the individual relations have on the conditions for well-foundedness? It is known to allow weakening of the Jumping criterion [5].
- What conditions guarantee that, if there is a chain in the union of well-founded relations from s to t, then there is one that takes steps from the relations one after the other?
- One of the motivations for this work is the search for novel termination orderings, particularly for term rewriting. The tripartite condition has a direct application to a path ordering based on Takeuti's ordinal diagrams [7].
- Isabelle/HOL is based on classical higher-order logic with the axiom of choice. Is it possible to get rid of some of these axioms to make the proof more constructive? Technically, the results on infinite sequences should be easily and naturally encoded into modern (versions of) Isabelle or Coq, using co-induction. We leave this as further work.

Acknowledgements. We thank the reviewers of a previous (unsuccessful) submission for their Devilish Advocacy.

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