The Equivalence of Contests

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Some basic terms

- 2-player contest
  - i, j are the players
  - x_i , x_j are their efforts for winning

- Contest designer (e.g. government, company,…)

- Lottery contest success function:

$$P_i(x_i, x_j) = \begin{cases} x_i/(x_i + x_j) & \text{if } x_i + x_j \neq 0 \\ 1/2 & \text{if } x_i = x_j = 0 \end{cases}$$

Some basic terms (cont'd)

- Payoff function:

$$\pi_i(x_i, x_j) = \begin{cases} W + \alpha_1 x_i + \beta_1 x_j & \textit{with prob.} \quad P_i(x_i, x_j) \\ L + \alpha_2 x_i + \beta_2 x_j & \textit{with prob.} \quad 1 - P_i(x_i, x_j) \end{cases}$$

$\{W, L, \alpha_1, \alpha_2, \beta_1, \beta_2\}$ is the parameter space

- $W > L \geq 0$ are the winning/losing prizes
- α_1, α_2 are the cost parameters

  Reflect the effect of own effort x_i on payoff π_i $\begin{cases} \xrightarrow{\alpha_1} \text{winning} \\ \xrightarrow{\alpha_2} \text{losing} \end{cases}$

- β_1, β_2 are the spillover parameters

  Reflect the effect of rival's effort x_j on payoff π_i $\begin{cases} \xrightarrow{\beta_1} \text{winning} \\ \xrightarrow{\beta_2} \text{losing} \end{cases}$

Example – Tullock-type contest

- $\{W, L, \alpha_1, \alpha_2, \beta_1, \beta_2\}$ =
  $\{W, 0, -1, -1, 0, 0\}$

- Payoff function

$$\pi_i(x_i, x_j) = \begin{cases} W - x_i & \textit{with prob.} \quad P_i(x_i, x_j) \\ -x_i & \textit{with prob.} \quad 1 - P_i(x_i, x_j) \end{cases}$$

Some basic terms (cont'd)

- <u>Expected Payoff</u>:

$$E(\pi_i(x_i, x_j)) = \frac{x_i}{x_i + x_j}(W + \alpha_1 x_i + \beta_1 x_j) + \frac{x_j}{x_i + x_j}(L + \alpha_2 x_i + \beta_2 x_j)$$

maximize with respect to x_i :

- <u>Best Response Function</u>:

$$x_i^{BRF} = -x_j + \sqrt{\frac{\{(\alpha_1 - \alpha_2) - (\beta_1 - \beta_2)\} x_j^{\,2} - (W - L) x_j}{\alpha_1}}$$

enforce symmetric equilibrium ($x_i^* = x_j^*$) + some restrictions on parameters:

- <u>Equilibrium efforts</u>:

$$x_i^* = x_j^* = \frac{W - L}{-(3\alpha_1 + \alpha_2) - (\beta_1 - \beta_2)}$$

Main theme of the paper

- Different contests may be equivalent in some aspects:
  - <u>Strategically equivalent</u>
    same family of BRF x_i^{BRF}
  - <u>Revenue equivalent</u>
    same revenue (for the contest designer) $\sum x_i^*$
  - <u>Payoff equivalent</u>
    same expected payoffs (for players) $E(\pi_i(x_i, x_j))$

Relation between equivalences

<u>Payoff</u> $\xleftrightarrow{\text{definition}}$ same expected payoffs

X *

<u>Strategically</u> $\xleftrightarrow{\text{definition}}$ same family of BRF

Same BRF
$\Downarrow$
Same x_i^* ‘s
$\Downarrow$
Same $\sum x_i^*$

<u>Revenue</u> $\xleftrightarrow{\text{definition}}$ same revenue ($\sum x_i^*$)

* If 2 contests are <u>strategically</u> eq. but not <u>payoff</u> eq., contest designer may prefer one contest over the other. This is known as <u>Pareto improvement</u>:

A change in the allocation of a resource to a set of individuals that is an improvement for at least one and no worse for any other

Algorithm for Finding equivalences

Given a contest, to find <u>strategically</u> ($\Rightarrow$ revenue) equivalent contests:

<u>Step1</u> Express BRF for the contest $x_i^{BRF} = \ldots$

<u>Step2</u> Derive restrictions on parameters, which generate the same BRF

$$x_i^{BRF} = -x_j + \sqrt{\frac{\{(\alpha_1 - \alpha_2) - (\beta_1 - \beta_2)\}x_j^{\,2} - (W - L)x_j}{\alpha_1}}$$

Tullock-type contest revisited

- $\{W, L, \alpha_1, \alpha_2, \beta_1, \beta_2\}$ =
  $\{W, 0, -1\ , -1, 0,\ \ 0\ \}$

- Payoff function

$$\pi_i(x_i, x_j) = \begin{cases} W - x_i & \textit{with prob.} \quad P_i(x_i, x_j) \\ -x_i & \textit{with prob.} \quad 1 - P_i(x_i, x_j) \end{cases}$$

- BRF:

$$x_i^{BRF} = -x_j + \sqrt{W x_j}$$

- When is a contest <u>strategically</u> equivalent to $\{W, 0, -1, -1, 0, 0\}$?

Equivalence to Tullock-type contest

When is a contest strategically equivalent to $\{W, 0, -1, -1, 0, 0\}$?

$$\{W, L, \alpha_1, \alpha_2, \beta_1, \beta_2\} \overset{?}{\equiv} \{W, 0, -1, -1, 0, 0\}$$

$$x_i^{BRF} = x_i^{BRF}$$

$$x_i^{BRF} = -x_j + \sqrt{\frac{\{\underbrace{(\alpha_1 - \alpha_2) - (\beta_1 - \beta_2)}_{0}\} x_j^{\,2} - (\underbrace{W - L}_{W}) x_j}{\underbrace{\alpha_1}_{-1}}} = -x_j + \sqrt{W x_j}$$

$W - L = W$
$\alpha_1 = -1$
$\beta_2 - \beta_1 - \alpha_2 = 1$

restrictions on parameters, which generate the same BRF

Example 1

Some interesting cases of Tullock $\{W,0,-1,-1,0,0\}$ equivalence:

- $\{W+\Delta, \Delta, -1, -1, 0, 0\}$

$$\pi_i(x_i,x_j) = \begin{cases} W+\Delta - x_i & \textit{with prob.} \quad P_i(x_i,x_j) \\ \Delta - x_i & \textit{with prob.} \quad 1-P_i(x_i,x_j) \end{cases}$$

Conclusion: the same Δ in the winning and losing prize does not affect equilibrium efforts x_i^*.

Use: the contest designer may wish to change the prizes values while ensuring no change in behavior of players.

Example 2

Some interesting cases of Tullock $\{W,0,-1,-1,0,0\}$ equivalence:

- $\{W, 0\ , -1, 0, -1, 0\}$ that is winner reimburses loser:

$$\pi_i(x_i, x_j) = \begin{cases} W - x_i - x_j & \textit{with prob.} \quad P_i(x_i, x_j) \\ 0 & \textit{with prob.} \quad 1 - P_i(x_i, x_j) \end{cases}$$

(Also payoff equivalent)

Conclusion: if winner reimburses loser, this affects neither equilibrium efforts , nor payoffs.

Use: a contest where only the winner expends resources may be simpler to implement.

Example 3

Some interesting cases of Tullock $\{W,0,-1,-1,0,0\}$ equivalence:

- "Input spillover": $\{W, 0, -1, -1, \beta, \beta\}$ $\quad \beta \in (-1,1)$

$$\pi_i(x_i, x_j) = \begin{cases} W - x_i + \beta x_j & \textit{with prob.} \quad P_i(x_i, x_j) \\ -x_i + \beta x_j & \textit{with prob.} \quad 1 - P_i(x_i, x_j) \end{cases}$$

- Not payoff equivalent: $\beta > 0$ → payoff > Tullock

  $\beta < 0$ → payoff < Tullock

Conclusion: symmetric spillovers do not affect equilibrium efforts

but do affect the payoffs of players

Use: the contest designer may wish to enhance positive spillovers.

Contests with complementarities

E_i is the total amount of resources of player i
x_i is the total amount of effort of player i
Φ is the fraction of resources destroyed during the conflict

War

$\{\phi(E_i\text{-}x_i), 0, \text{-}\phi, \text{-}\phi, 0, 0\}$

$$\pi_i = \begin{cases} \Phi(E_i - x_i) + \Phi(E_j - x_j) & winer \\ 0 & loser \end{cases}$$

$$x_i^{BRF} = -x_j + \sqrt{(E_i + E_j)x_j}$$

Skaperdas (1992), Garfinkel and Skaperdas (2000)

$\equiv$

Harmful residual

$\{\phi E_i, \text{-}\phi E_j\, 0, \text{-}\phi, 0, 0, \phi\}$

$$\pi_i = \begin{cases} \Phi(E_i - x_i) & winer \\ -\Phi(E_j - x_j) & loser \end{cases}$$

$\Leftarrow the \quad same$

Also payoff

$\equiv$

Conditions for equivalence

The contests shown assume these conditions:

1. Two-player contest

2. Lottery success function: $P(x_i, x_j) = \dfrac{x_i}{x_i + x_j}$

3. Linear cost and spillover $\ldots + \alpha x_i + \beta x_j$

4. Risk neutrality

When relaxing these conditions

(e.g. risk aversion, $P(x_i, x_j) = \dfrac{x_i^r}{x_i^r + x_j^r}, \quad r \neq 1$, $\ldots + \alpha x_i^2 + \beta \sqrt{x_j}$),

equivalences shown may not hold.

Conclusions

- Intuitively and structurally different contests can be strategically and revenue equivalent

- Contest designer may replace contest parameter space, keeping the same revenue, while:
    - reducing operational cost.
    - Improving payoffs for contestants
    - Reducing risk for contestants
    - Avoiding regulatory restrictions

- The type of contests shown in the paper cover the majority of contests in the literature.

Thank you !