Maximally Paraconsistent Logics

What Do We Mean By “Logic”?

1. A formal language $\mathcal{L}$, in which $\mathcal{L}$-formulas are constructed. *We assume that $\mathcal{L}$ includes a unary connective $\neg$.*
   We denote by $F_{\mathcal{L}}$ the set of well-formed formulas of $\mathcal{L}$.
2. A consequence relation $\vdash$ for $\mathcal{L}$.

A consequence relation (cr) for $\mathcal{L}$ is a binary relation $\vdash: 2^{F_{\mathcal{L}}} \times F_{\mathcal{L}}$, having the following properties:

strong reflexivity: if $\psi \in \Gamma$ then $\Gamma \vdash \psi$.
monotonicity: if $\Gamma \vdash \psi$ and $\Gamma \subseteq \Gamma'$, then $\Gamma' \vdash \psi$.
transitivity (cut): if $\Gamma \vdash \psi$ and $\Gamma, \psi \vdash \varphi$ then $\Gamma \vdash \varphi$.

Properties of Consequence Relations

- A cr $\vdash$ for $\mathcal{L}$ is structural if for every uniform $\mathcal{L}$-substitution σ and every Γ and ψ: if $\Gamma \vdash \psi$ then $\sigma[\Gamma] \vdash \sigma[\psi]$.
  Example: $p \wedge q \vdash p$ implies $\psi \wedge \varphi \vdash \psi$ for every $\varphi, \psi \in F_{\mathcal{L}}$.

- A cr $\vdash$ for $\mathcal{L}$ is consistent if there exist a non-empty Γ and ψ, such that $\Gamma \not\vdash \psi$.

- A cr $\vdash$ for $\mathcal{L}$ is finitary if whenever $\Gamma \vdash \psi$, there exists some finite $\Gamma' \subseteq \Gamma$, such that $\Gamma' \vdash \psi$.

- A propositional logic is a pair $\langle \mathcal{L}, \vdash \rangle$, where $\vdash$ is a structural, consistent and finitary cr for $\mathcal{L}$.

Paraconsistent Logics

- In classical logic (and most other logics), the explosive non-contradiction principle $\varphi, \neg\varphi \vdash \psi$ allows us to derive any formula out of a contradiction. This makes any inconsistent theory trivial, and so no sensible reasoning can take place in the presence of contradictions.

- Paraconsistent logics do allow non-trivial inconsistent theories.

- A logic $\langle \mathcal{L}, \vdash \rangle$ is called $\neg$-*paraconsistent* if there are formulas ψ, ϕ in $F_{\mathcal{L}}$, such that $\psi, \neg\psi \nvdash \phi$.
  As $\vdash$ is structural, it is enough to require that there are atoms p, q *such that* $p, \neg p \nvdash q$.

But What Is Negation?

- *Paraconsistency is characterized by a 'negation connective'. But there is no general agreement about the properties that such a connective should satisfy.*
- We make some very minimal assumptions about the interpretation of negation.
- We say that $\neg$ is a *pre-negation* for $\mathbf{L} = \langle \mathcal{L}, \vdash \rangle$, if there is some atom p in $\mathcal{L}$ such that $p \nvdash \neg p$.

Defining Paraconsistent Logics: Many-valued Matrices

A *many-valued matrix* for a language $\mathcal{L}$ is a triple $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where

- $\mathcal{V}$ is a non-empty set of truth values,
- $\mathcal{D}$ is a non-empty proper subset of $\mathcal{V}$, called the *designated* elements of $\mathcal{V}$, and
- $\mathcal{O}$ includes an n-ary function $\widetilde{\diamond}_{\mathcal{M}} : \mathcal{V}^n \to \mathcal{V}$ for every n-ary connective $\diamond$ of $\mathcal{L}$.

We denote $\overline{\mathcal{D}} = \mathcal{V} \setminus \mathcal{D}$.

Logics Induced by Matrices

- A valuation v in a matrix $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ is any function from $\mathcal{L}$-formulas to $\mathcal{V}$, which satisfies:

$$v(\diamond(\psi_1, ..., \psi_n)) = \tilde{\diamond}(v(\psi_1), ..., v(\psi_n))$$

- v is a model of an $\mathcal{L}$-formula ψ in $\mathcal{M}$, denoted by $v \models_{\mathcal{M}} \psi$, if $v(\psi) \in \mathcal{D}$. The set of models of ψ is denoted by $mod_{\mathcal{M}}(\psi)$. v is a model of theory Γ in $\mathcal{M}$, denoted by $v \models_{\mathcal{M}} \Gamma$, if v is a model of every $\psi \in \Gamma$.
- $\Gamma \vdash_{\mathcal{M}} \psi$ if for every valuation v in $\mathcal{M}$: $v \models_{\mathcal{M}} \Gamma$ implies $v \models_{\mathcal{M}} \psi$.
- For any (finite) $\mathcal{M}$ for $\mathcal{L}$, $\langle \mathcal{L}, \vdash_{\mathcal{M}} \rangle$ is a propositional logic.
- *We say that $\mathcal{M}$ is paraconsistent if so is the logic it induces.*

Matrices and Negation

Reminder: $\neg$ *is a* pre-negation *for* $\mathbf{L} = \langle \mathcal{L}, \vdash \rangle$, *if there is some atom* p *in* $\mathcal{L}$ *such that* $p \not\vdash \neg p$.

Proposition 1: Let $\mathbf{L}_{\mathcal{M}} = \langle \mathcal{L}, \vdash_{\mathcal{M}} \rangle$ be a logic induced by a matrix $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ for $\mathcal{L}$ with pre-negation. $\neg$ is a pre-negation for $\mathbf{L}_{\mathcal{M}}$ iff there is an element $x \in \mathcal{D}$ such that $\tilde{\neg} x \in \overline{\mathcal{D}}$.

3-valued Paraconsistent Matrices

Proposition 2: A 3-valued matrix $\mathcal{M}$ with a pre-negation is paraconsistent iff it is isomorphic to a matrix $\langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ in which $\mathcal{V} = \{t, \top, f\}$, $\mathcal{D} = \{t, \top\}$, $\tilde{\neg} t = f$, and $\tilde{\neg} \top \neq f$.

Proof: Suppose that $\mathbf{L}_{\mathcal{M}}$ is $\neg$-paraconsistent. Since $\neg$ is a pre-negation for $\mathbf{L}_{\mathcal{M}}$, there is an element in $\mathcal{D}$, denote it t, such that $\tilde{\neg} t \notin \mathcal{D}$. So let $f \in \overline{\mathcal{D}}$ such that $\tilde{\neg} t = f$. Also, since $\mathbf{L}_{\mathcal{M}}$ is $\neg$-paraconsistent, we have that $p, \neg p \nvdash_{\mathcal{M}} q$ for some $p, q \in \mathcal{A}_{\mathcal{L}}$, and so $mod_{\mathcal{M}}(\{p, \neg p\}) \neq \emptyset$. In this case t cannot be the only designated element. Let $\top$ be another one. It follows that $\mathcal{V} = \{t, \top, f\}$, where $\top \in \mathcal{D}$, and f is the only non-designated element. Also for $\nu \in mod_{\mathcal{M}}(\{p, \neg p\})$ necessarily $\nu(p) = \top$. This implies that $\nu(\neg p) = \tilde{\neg} \top \in \mathcal{D}$, and so $\tilde{\neg} \top \neq f$.

The proof of the converse is easy.

The Maximality Problem

- N. da Costa formulated the *maximality problem*: finding paraconsistent logics which are maximal with respect to classical logic.

- A logic $\mathbf{L} = \langle \mathcal{L}, \vdash \rangle$ is *weakly maximally paraconsistent* if every logic $\langle \mathcal{L}, \Vdash \rangle$ that extends $\mathbf{L}$ (*i.e., a logic in the same language of* $\mathbf{L}$ *such that* $\vdash \subseteq \Vdash$), and whose set of theorems *properly includes* that of $\mathbf{L}$, is not paraconsistent.

- A logic $\mathbf{L}$ is *strongly maximally paraconsistent* if every logic $\langle \mathcal{L}, \Vdash \rangle$ that *properly extends* $\mathbf{L}$ is not paraconsistent.

- *Strong maximal paraconsistency implies weak maximal paraconsistency. Does the converse hold?*

Weak Max. Par. $\not\Rightarrow$ Strong Max. Par.

Consider Sobociński's three-valued matrix $\mathcal{S} = \langle \{t, f, \top\}, \{t, \top\}, \{\tilde{\rightarrow}, \tilde{\neg}\} \rangle$, where the $\neg$ is an involutive negation (i.e., $\tilde{\neg} t = f$, $\tilde{\neg} f = t$, and $\tilde{\neg} \top = \top$), and the implication is interpreted as follows:

$$a \tilde{\rightarrow} b \; = \begin{cases} \top & \text{if } a = b = \top, \\ f & \text{if } a >_{\mathsf{t}} b \;\; (\text{where } t >_{\mathsf{t}} \top >_{\mathsf{t}} f), \\ t & \text{otherwise.} \end{cases}$$

Weak Max. Par. $\not\Rightarrow$ Strong Max. Par.

- Sobociński's has axiomatized the set of valid sentences of $\mathcal{S}$ by an Hilbert-type system $\mathsf{H}_{\mathcal{S}}$ such that for every $\mathcal{T}, \varphi, \psi$ in the language of $\{\neg, \rightarrow\}$:
  - *ψ is provable in $\mathsf{H}_{\mathcal{S}}$ iff ψ is valid in $\mathcal{S}$*
  - *$\mathcal{T}, \varphi \vdash_{\mathsf{H}_{\mathcal{S}}} \psi$ iff either $\mathcal{T} \vdash_{\mathsf{H}_{\mathcal{S}}} \psi$ or $\mathcal{T} \vdash_{\mathsf{H}_{\mathcal{S}}} \varphi \rightarrow \psi$*
- $\langle \mathcal{L}, \vdash_{\mathsf{H}_{\mathcal{S}}} \rangle$ is weakly maximally paraconsistent: any extension of the set of theorems of $\mathsf{H}_{\mathcal{S}}$ by a non-provable axiom yields either classical logic or a trivial logic.
- $\vdash_{\mathcal{S}}$ is a proper paraconsistent extension of $\vdash_{\mathsf{H}_{\mathcal{S}}}$, since $\neg(p \rightarrow q) \vdash_{\mathcal{S}} p$, while $\neg(p \rightarrow q) \nvdash_{\mathsf{H}_{\mathcal{S}}} p$.
- It follows that Sobociński's logic $\langle \mathcal{L}, \vdash_{\mathsf{H}_{\mathcal{S}}} \rangle$ is maximally paraconsistent in the weak sense but *not* in the strong sense!

Natural 3-valued Logics Are Maximally Paraconsistent

Theorem 1: Let $\mathcal{M}$ be a three-valued paraconsistent matrix for $\mathcal{L}$ with a pre-negation $\neg$. Suppose that there is a formula $\Psi(p, q)$ in $\mathcal{L}$ such that for for every $\mathcal{M}$-valuation ν, $\nu(\Psi) = t$ in case $\nu(p) \neq \top$ or $\nu(q) \neq \top$. Then $\langle \mathcal{L}, \vdash_{\mathcal{M}} \rangle$ is maximally paraconsistent.

Two Important Particular Cases when $\neg f = t$:

1. A three-valued paraconsistent matrix with a binary connective $+$ such that for every $x \in \mathcal{V}$, $x + t = t + x = t$:

$$\Psi(p, q) = (p + \neg p) + (q + \neg q)$$

2. A three-valued paraconsistent matrix with a propositional constant f (for which $\nu(\mathsf{f}) = \mathsf{f}$ for every $\nu \in \Lambda_{\mathcal{M}}$):

$$\Psi(p, q) = \neg \mathsf{f}$$

Proof of Theorem 1

By Proposition 2, $\mathcal{M}$ has designated elements t and $\top$, $\tilde{\neg} t = f$, and $\tilde{\neg}\top \in \mathcal{D}$.

Let $\langle \mathcal{L}, \vdash \rangle$ be a (finitary) propositional logic that is strictly stronger than $\langle \mathcal{L}, \vdash_{\mathcal{M}} \rangle$. Then there is a finite theory Γ and a formula ψ such that $\Gamma \vdash \psi$ but $\Gamma \not\vdash_{\mathcal{M}} \psi$. In particular, there is a valuation ν such that $\nu \models \Gamma$, and $\nu(\psi) = f$.

Consider the substitution θ, defined for $p \in \mathsf{Atoms}(\Gamma \cup \{\psi\})$ as follows:

$$\theta(p) = \begin{cases} q_0 & \text{if } \nu(p) = t, \\ \neg q_0 & \text{if } \nu(p) = f, \\ p_0 & \text{if } \nu(p) = \top, \end{cases}$$

Note that $\theta(\Gamma)$ and $\theta(\psi)$ contain (at most) the variables p_0, q_0, and that for every valuation μ, if $\mu(p_0) = \top$ and $\mu(q_0) = t$ then $\mu(\theta(\phi)) = \nu(\phi)$ for every formula ϕ such that $\mathsf{Atoms}(\{\phi\}) \subseteq \mathsf{Atoms}(\Gamma \cup \{\psi\})$. Thus,

$(\star)$ *any* valuation μ such that $\mu(p_0) = \top$, $\mu(q_0) = t$
is an $\mathcal{M}$-model of $\theta(\Gamma)$ that does not $\mathcal{M}$-satisfy $\theta(\psi)$.

Case I. There is a formula $\phi(p,q)$ such that for every μ, $\mu(\phi) \neq \top$ if $\mu(p) = \mu(q) = \top$.

In this case, let $\mathtt{tt} = \Psi(\mathsf{q}_0, \phi(\mathsf{p}_0, \mathsf{q}_0))$. Note that $\mu(\mathtt{tt}) = \mathtt{t}$ for every μ such that $\mu(p_0) = \top$. Now, as $\vdash$ is structural, $\Gamma \vdash \psi$ implies that

$$\theta(\Gamma)\ [\mathtt{tt}/\mathsf{q}_0] \vdash \theta(\psi)\ [\mathtt{tt}/\mathsf{q}_0] \tag{1}$$

Also, by the property of $\mathtt{tt}$ and by $(\star)$, any $\mu \in \Lambda_{\mathcal{M}}$ for which $\mu(p_0) = \top$ is a model of $\theta(\Gamma)\ [\mathtt{tt}/\mathsf{q}_0]$ but does not $\mathcal{M}$-satisfy $\theta(\psi)\ [\mathtt{tt}/\mathsf{q}_0]$. Thus,

- $p_0, \neg p_0 \vdash_{\mathcal{M}} \theta(\gamma)\ [\mathtt{tt}/\mathsf{q}_0]$ for every $\gamma \in \Gamma$.
  As $\langle \mathcal{L}, \vdash \rangle$ is stronger than $\langle \mathcal{L}, \vdash_{\mathcal{M}} \rangle$, this implies that

$$p_0, \neg p_0 \vdash \theta(\gamma)\ [\mathtt{tt}/\mathsf{q}_0] \text{ for every } \gamma \in \mathsf{\Gamma}. \tag{2}$$

- The set $\{p_0,\ \neg p_0,\ \theta(\psi)[\mathtt{tt}/\mathsf{q}_0]\}$ is not $\mathcal{M}$-satisfiable, thus

$$p_0, \neg p_0, \theta(\psi)\ [\mathtt{tt}/\mathsf{q}_0] \vdash_{\mathcal{M}} \mathsf{q}_0$$

  Again, as $\langle \mathcal{L}, \vdash \rangle$ is stronger than $\langle \mathcal{L}, \vdash_{\mathcal{M}} \rangle$, we have that

$$p_0,\ \neg p_0,\ \theta(\psi)\ [\mathtt{tt}/\mathsf{q}_0] \vdash \mathsf{q}_0. \tag{3}$$

By (1)–(3) and the transitivity property, $p_0, \neg p_0 \vdash q_0$, thus $\langle \mathcal{L}, \vdash \rangle$ is not $\neg$-paraconsistent.

Case II. For every formula ϕ in p, q and for every μ, if $\mu(p) = \mu(q) = \top$ then $\mu(\phi) = \top$.

As $\vdash$ is structural, the assumption that $\Gamma \vdash \psi$ implies that

$$\theta(\Gamma)\ [\Psi(q_0, q_0)/q_0] \vdash \theta(\psi)\ [\Psi(q_0, q_0)/q_0] \tag{4}$$

In addition, $(\star)$ above entails that any valuation μ such that $\mu(p_0) = \top$ and $\mu(q_0) \in \{t, f\}$ is a model of $\theta(\Gamma)\ [\Psi(q_0, q_0)/q_0]$ which is not a model of $\theta(\psi)\ [\Psi(q_0, q_0)/q_0]$. Thus, the only $\mathcal{M}$-model of $\{p_0, \neg p_0, \theta(\psi)\ [\Psi(q_0, q_0)/q_0]\}$ is the one in which both of p_0 and q_0 are assigned the value $\top$. It follows that $p_0, \neg p_0, \theta(\psi)\ [\Psi(q_0, q_0)/q_0] \vdash_{\mathcal{M}} q_0$. Hence:

$$p_0, \neg p_0, \theta(\psi)\ [\Psi(q_0, q_0)/q_0] \vdash q_0. \tag{5}$$

By using $(\star)$ again (for $\mu(q_0) \in \{t, f\}$) and the condition of case II (for $\mu(q_0) = \top$), we have:

$$p_0, \neg p_0 \vdash \theta(\gamma)\ [\Psi(q_0, q_0)/q_0] \text{ for every } \gamma \in \Gamma. \tag{6}$$

Again, (4)–(6) and the transitivity property of $\vdash$ entail that $p_0, \neg p_0 \vdash q_0$, and so $\langle \mathcal{L}, \vdash \rangle$ is not $\neg$-paraconsistent in this case either. $\square$

Sette's Logic P1 is Strongly Maximally Paraconsistent

$\mathsf{P1} = \langle \{\mathsf{t}, \mathsf{f}, \top\}, \{\mathsf{t}, \top\}, \mathcal{O}\rangle$ for the language of $\{\neg, \vee, \wedge, \rightarrow\}$ where:

$\tilde{\vee}$	t	f	$\top$
t	t	t	t
f	t	f	t
$\top$	t	t	t

$\tilde{\wedge}$	t	f	$\top$
t	t	f	t
f	f	f	f
$\top$	t	f	t

$\tilde{\rightarrow}$	t	f	$\top$
t	t	f	t
f	t	t	t
$\top$	t	f	t

$\tilde{\neg}$	
t	f
f	t
$\top$	t

Priest's Logic LP is Strongly Maximally Paraconsistent

LP $= \langle \{\mathsf{t}, \mathsf{f}, \top\}, \{\mathsf{t}, \top\}, \mathcal{O}\rangle$ for the language of $\{\neg, \vee, \wedge\}$ with the following standard Kleene interpretations of its connectives:

$\tilde{\vee}$	t	f	$\top$
t	t	t	t
f	t	f	$\top$
$\top$	t	$\top$	$\top$

$\tilde{\wedge}$	t	f	$\top$
t	t	f	$\top$
f	f	f	f
$\top$	$\top$	f	$\top$

$\tilde{\neg}$	
t	f
f	t
$\top$	$\top$

Sobociński's Logic is Strongly Maximally Paraconsistent

$\mathcal{S} = \langle \{t, f, \top\}, \{t, \top\}, \{\tilde{\rightarrow}, \tilde{\neg}\} \rangle$, for the language of $\{\neg, \rightarrow\}$ where $\tilde{\neg} t = f$, $\tilde{\neg} f = t$, and $\tilde{\neg} \top = \top$, and the implication is interpreted as follows:

$$a \tilde{\rightarrow} b \;=\; \begin{cases} \top & \text{if } a = b = \top, \\ f & \text{if } a >_{\mathsf{t}} b \;\; (\text{where } t >_{\mathsf{t}} \top >_{\mathsf{t}} f), \\ t & \text{otherwise.} \end{cases}$$

And Many More...

Every extension of P1, LP, or Sobociński's logic is also a strongly maximally paraconsistent logic. This includes:

1. PAC, extending LP by an implication connective $\supset$, defined by $x \supset y = y$ if $x \in \{t, \top\},$ otherwise $x \supset y = t$.

2. J_3, obtained from PAC by adding the constant f.

3. The logic of the *maximally monotonic* language that consists of the connectives of LP and the constants f and T.

4. The logic of the *functionally complete* language $\mathcal{L}_3^\star$, consisting of the connectives of PAC and the constants f and T.

Non-deterministic Matrices

A non-deterministic matrix (Nmatrix) for $\mathcal{L}$ is a tuple $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$:

- $\mathcal{V}$ - the set of truth-values,
- $\mathcal{D}$ - the set of designated truth-values,
- $\mathcal{O}$ - contains an interpretation function $\tilde{\diamond} : \mathcal{V}^n \to P^+(\mathcal{V})$ for every n-ary connective $\diamond$ of $\mathcal{L}$.

Ordinary matrices correspond to the case when each $\widetilde{\diamond}$ *is a function taking singleton values only* (*then it can be treated as a function* $\widetilde{\diamond} : \mathcal{V}^n \to \mathcal{V}$).

Logics Induced by Nmatrices

- A valuation v in an Nmatrix $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ is a function from $\mathcal{L}$-formulas to $\mathcal{V}$, satisfying the following condition:

$$v(\diamond(\psi_1, ..., \psi_n)) \in \tilde{\diamond}(v(\psi_1), ..., v(\psi_n)).$$

- A satisfaction relation **($\models_{\mathcal{M}}$)** and the consequence relation induced by $\mathcal{M}$ **($\vdash_{\mathcal{M}}$)** are defined as in the deterministic case.

- For any finite Nmatrix $\mathcal{M}$ for propositional language $\mathcal{L}$, $\mathbf{L}_{\mathcal{M}} = \langle \mathcal{L}, \vdash_{\mathcal{M}} \rangle$ is a propositional logic.

Nmatrices and Negation

Reminder: $\neg$ is a pre-negation for $\mathbf{L} = \langle \mathcal{L}, \vdash \rangle$, if there is some atom p in $\mathcal{L}$ such that $p \not\vdash \neg p$.

Proposition. Let $\mathbf{L}_{\mathcal{M}} = \langle \mathcal{L}, \vdash_{\mathcal{M}} \rangle$ be a logic induced by an Nmatrix $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ for a language $\mathcal{L}$ with a pre-negation $\neg$. Then $\neg$ is a pre-negation for $\mathbf{L}_{\mathcal{M}}$ iff there is an element $x \in \mathcal{D}$ s.t. $\tilde{\neg} x \cap \overline{\mathcal{D}} \neq \emptyset$.

Paraconsistent Nmatrices

An Nmartix is $\neg$-paraconsistent if so is $\mathbf{L}_{\mathcal{M}} = \langle \mathcal{L}, \vdash_{\mathcal{M}} \rangle$.

Proposition. Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be an Nmatrix for a language $\mathcal{L}$ with a pre-negation $\neg$. Then $\mathcal{M}$ is $\neg$-paraconsistent iff there is some $x \in \mathcal{D}$ such that $\tilde{\neg} x \cap \mathcal{D} \neq \emptyset$.

Can Paraconsistent Nmatrices Be Maximal?

Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be a paraconsistent Nmatrix for $\mathcal{L}$ with pre-negation. If one of the following conditions holds, then $\mathcal{M}$ is not maximally paraconsistent:

1. $\mathcal{D}$ is a singleton (and so no two-valued Nmatrix is paraconsistent!)

2. There is some $x \in \mathcal{D}$ such that $x \in \tilde{\neg} x$ and $\tilde{\neg} x \cap \overline{\mathcal{D}} \neq \emptyset$.

3. $\mathcal{M}$ is 3-valued Nmatrix which is not isomorphic to an Nmatrix $\langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ in which $\mathcal{V} = \{t, \top, f\}$, $\mathcal{D} = \{t, \top\}$, $\tilde{\neg} t = \{f\}, \tilde{\neg} \top = \{t, f\}$ and $\tilde{\neg} f = \{f\}$ or $\tilde{\neg} f = \{t\}$.

Guidelines for a Proof – Simple Refinements

An Nmatrix $\mathcal{M}_1 = \langle \mathcal{V}, \mathcal{D}, \mathcal{O}_1 \rangle$ for $\mathcal{L}$ is a simple $\diamond$-refinement of an Nmatrix $\mathcal{M}_2 = \langle \mathcal{V}, \mathcal{D}, \mathcal{O}_2 \rangle$ for $\mathcal{L}$, if $\tilde{\diamond}_{\mathcal{M}_1}(\overline{x}) \subseteq \tilde{\diamond}_{\mathcal{M}_2}(\overline{x})$ for all $\overline{x} \in \mathcal{V}^n$.

$\mathcal{M}_1$ is a simple refinement of $\mathcal{M}_2$, if it is a simple $\diamond$-refinement of $\mathcal{M}_2$ for every $\diamond$ in $\mathcal{L}$.

Proposition. If $\mathcal{M}_1$ is a simple refinement of $\mathcal{M}_2$ then $\vdash_{\mathcal{M}_2} \subseteq \vdash_{\mathcal{M}_1}$.

Corollary. A paraconsistent Nmatrix $\mathcal{M}$ is *non-maximal* if it is refined by a paraconsistent Nmatrix $\mathcal{M}^*$ (in which $\neg$ is still a pre-negation) and $\vdash_{\mathcal{M}^*} \setminus \vdash_{\mathcal{M}} \neq \emptyset$.

Guidelines for a Proof – An Example

Consider an Nmatrix $\mathcal{M} = \langle \{t, \top, f\}, \{t, \top\}, \mathcal{O} \rangle$, where $\tilde{\neg} t = \{\top, f\}$ and $\tilde{\neg} \top = \{t, f\}$.

Proposition. $\mathcal{M}$ is not maximally paraconsistent.

Proof. Consider a simple refinement $\mathcal{M}^* = \langle \{t, \top, f\}, \{t, \top\}, \mathcal{O} \rangle$ of $\mathcal{M}$, in which $\tilde{\neg} t = \{f\}$ and $\tilde{\neg} \top = \{t\}$. Then

- $\neg$ is still a pre-negation in $\mathcal{M}^*$,
- $\mathcal{M}^*$ is still paraconsistent,
- $p, \neg p, \neg\neg p \vdash_{\mathcal{M}^*} q$ while $p, \neg p, \neg\neg p \nvdash_{\mathcal{M}} q$ (thus $\vdash_{\mathcal{M}} \subset \vdash_{\mathcal{M}^*}$).

Can Paraconsistent Nmatrices Be Maximal? (Cont'd)

It remains to consider two cases for the interpretation of $\neg$:

a) $\tilde{\neg}t = \{f\}$, $\tilde{\neg}\top = \{t, f\}$, and $\tilde{\neg}f = \{t\}$

b) $\tilde{\neg}t = \{f\}$, $\tilde{\neg}\top = \{t, f\}$, and $\tilde{\neg}f = \{f\}$.

- If $\neg$ is the only connective in $\mathcal{L}$, then in both cases the corresponding Nmatrix is maximally paraconsistent.
- If there is another connective with a proper non-deterministic interpretation, maximal paraconsistency cannot be achieved.
- If apart of $\neg$ all the other connectives have deterministic interpretations, there is no unique answer:
  - If all complex formulas can get only values in $\{t, f\}$, the logic induced by $\mathcal{M}$ may be maximal.
  - If there is a $\top$-free connective, $\mathcal{M}$ is not maximal.

A Final Note

For characterizing three-valued maximally paraconsistent logics it is enough to consider only deterministic matrices.

Theorem. Let $\mathcal{M}$ be an n-valued maximally paraconsistent Nmatrix. Then there is a deterministic matrix $\mathcal{M}^*$ which induces the same logic.

Open Questions

1. Are *all* the 3-valued paraconsistent logics induced by deterministic matrices maximal?

2. Is every maximally paraconsistent n-valued Nmatrix reducible to a 3-valued matrix?