Improved Bounds for Incidences between Points and Circles*

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ABSTRACT

We establish an improved upper bound for the number of incidences between m points and n arbitrary circles in three dimensions. The previous best known bound, originally established for the planar case and later extended to any dimension ≥ 2, is $O^*\left(m^{2/3}n^{2/3} + m^{6/11}n^{9/11} + m + n\right)$ (where the $O^*(\cdot)$ notation hides sub-polynomial factors). Since all the points and circles may lie on a common plane or sphere, it is impossible to improve the bound in $\mathbb{R}^3$ without first improving it in the plane.

Nevertheless, we show that if the set of circles is required to be "truly three-dimensional" in the sense that no sphere or plane contains more than q of the circles, for some $q \ll n$, then the bound can be improved to

$$O^*\left(m^{3/7}n^{6/7} + m^{2/3}n^{1/2}q^{1/6} + m^{6/11}n^{15/22}q^{3/22} + m + n\right).$$

For various ranges of parameters (e.g., when $m = \Theta(n)$ and $q = o(n^{7/9})$), this bound is smaller than the best known two-dimensional worst-case lower bound $\Omega^*(m^{2/3}n^{2/3} + m + n)$.

We present several extensions and applications of the new bound: (i) For the special case where all the circles have the same radius, we obtain the improved bound $O^*\left(m^{5/11}n^{9/11} + m^{2/3}n^{1/2}q^{1/6} + m + n\right)$. (ii) We present an improved analysis that removes the subpolynomial factors from the bound when $m = O(n^{3/2-\varepsilon})$ for any fixed $\varepsilon > 0$. (iii) We use our results to obtain the improved bound $O(m^{15/7})$ for the number of mutually similar triangles determined by any set of m points in $\mathbb{R}^3$.

Our result is obtained by applying the polynomial partitioning technique of Guth and Katz using a constant-degree partitioning polynomial (as was also recently used by Solymosi and Tao). We also rely on various additional tools from analytic, algebraic, and combinatorial geometry.

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1. INTRODUCTION

Recently, Guth and Katz [20] presented the *polynomial partitioning technique* as a major technical tool in their solution of the famous planar distinct distances problem of Erdős [16]. This problem can be reduced to an incidence problem involving points and lines in $\mathbb{R}^3$ (following the reduction that was proposed in [15]), which can be solved by applying the aforementioned polynomial partitioning technique. The Guth-Katz result prompted various other incidence-related studies that rely on polynomial partitioning (e.g., see [24, 25, 38, 43]). One consequence of these studies is that they have led to further developments and enhancements of the technique itself (as seen for example in the use of induction in [38], and the use of two partitioning polynomials in [24, 43]). Also, the technique was recently applied to some problems that are not incidence related: it was used to provide an alternate proof of the existence of spanning trees with small crossing number in any dimension [25], and to obtain improved algorithms for range searching with semialgebraic sets [2]. Thus, it seems fair to say that applications and enhancements of the polynomial partitioning technique form an active contemporary area of research in combinatorial and computational geometry.

In this paper we study incidences between points and circles in three dimensions. Let $\mathcal{P}$ be a set of m points and $\mathcal{C}$ a set of n circles in $\mathbb{R}^3$. We denote the number of point-circle incidences in $\mathcal{P} \times \mathcal{C}$ as $I(\mathcal{P}, \mathcal{C})$. When the circles have arbitrary radii, the current best bound for any dimension $d \geq 2$ (originally established for the planar case in [3, 8, 28], and later extended to higher dimensions by Aronov et al. [7]) is

$$I(\mathcal{P},\mathcal{C}) = O^*\left(m^{2/3}n^{2/3} + m^{6/11}n^{9/11} + m + n\right). \quad (1)$$

Here in fact the $O^*(\cdot)$ notation only hides polylogarithmic factors; the precise best known upper bound is $O(m^{2/3}n^{2/3} + m^{6/11}n^{9/11}\log^{2/11}(m^3/n) + m + n)$ [28].

Since the three-dimensional case also allows $\mathcal{P}$ and $\mathcal{C}$ to lie on a single common plane or sphere[1], the point-circle incidence bound in $\mathbb{R}^3$ cannot be improved without first improving the planar bound (1) (which is an open problem for about 10 years). Nevertheless, as we show in this paper, an improved bound can be obtained if the configuration of points and circles is "truly three-dimensional" in the sense that no sphere or plane contains too many circles from $\mathcal{C}$. (Guth and Katz [20] use a similar assumption on the maximum number of lines that can lie in a common plane or regulus.) Our main result is given in the following theorem.

THEOREM 1.1. *Let $\mathcal{P}$ be a set of m points and let $\mathcal{C}$ be a set of n circles in $\mathbb{R}^3$, let ε be an arbitrarily small positive constant, and let $q < n$ be an integer. If no sphere or plane contains more than q circles of $\mathcal{C}$, then*

$$I(\mathcal{P},\mathcal{C}) = O\left(m^{3/7+\varepsilon}n^{6/7} + m^{2/3+\varepsilon}n^{1/2}q^{1/6} + m^{6/11+\varepsilon}n^{15/22}q^{3/22} + m + n\right),$$

where the constant of proportionality depends on ε.

Remarks. (1) In the planar case, the best known lower bound for the number of incidences between points and circles is $\Omega^*(m^{2/3}n^{2/3} + m + n)$ (e.g., see [33])[2]. Theorem 1.1 implies that for certain ranges of m, n, and q, a smaller upper bound holds in $\mathbb{R}^3$. This is the case, for example, when $m = \Theta(n)$ and $q = o(n^{7/9})$.
(2) When $m > n^{3/2}$, we have $m^{3/7}n^{6/7} < m$ and also $m^{6/11}n^{15/22}q^{3/22} < m^{2/3}n^{1/2}q^{1/6}$. Hence, we have $I(\mathcal{P},\mathcal{C}) = O(m^{2/3+\varepsilon}n^{1/2}q^{1/6} + m^{1+\varepsilon})$. In fact, as the analysis in this paper will show, the first term in this bound only arises from bounding incidences on certain potentially "heavy" planes or spheres. For $q = O(m^2/n^3)$ we have $I(\mathcal{P},\mathcal{C}) = O(m^{1+\varepsilon})$.
(3) When $m \le n^{3/2}$, any of the terms except for m can dominate the bound. However, if in addition $q = O\left(\left(\frac{n^3}{m^2}\right)^{3/7}\right)$ then the bound becomes $I(\mathcal{P},\mathcal{C}) = O(m^{3/7+\varepsilon}n^{6/7} + n)$. Note also that the interesting range of parameters is $m = \Omega^*(n^{1/3})$ and $m = O^*(n^2)$; in the complementary ranges both the old and new bounds become (almost) linear in $m + n$. In the interesting range, the new bound is asymptotically smaller than the planar bound given in (1) for q sufficiently small (e.g., when $q = O\left(\left(\frac{n^3}{m^2}\right)^{3/7}\right)$ as above), and as noted, it is also smaller than the best known worst-case lower bound in the planar case for certain ranges of m and n.
(4) Interestingly, the "threshold" value $m = \Theta(n^{3/2})$ where a quantitative change in the bound takes place (as noted in Remarks (2) and (3) above) also arises in the study of incidences between points and lines in $\mathbb{R}^3$ [14, 19, 20]. See Section 5 for a discussion of this threshold phenomenon.

[1]There is no real difference between the cases of coplanarity and cosphericality of the points and circles, since the latter case can be reduced to the former (and vice versa) by means of the stereographic projection.

[2]It is in fact larger than the explicit expression by a fractional logarithmic factor.

The proof of Theorem 1.1 is based on the polynomial partitioning technique of Guth and Katz [20], where we use a constant-degree partitioning polynomial in a manner similar to that used by Solymosi and Tao [38]. (The use of constant-degree polynomials and the induction arguments it leads to are essentially the only similarities with the technique of [38], which does not apply to circles in any dimension since it cannot handle situations where arbitrarily many curves can pass between any specific pair of points.) The application of this technique to incidences involving circles leads to new problems involving the handling of points that are incident to many circles that are entirely contained in the zero set of the partitioning polynomial. To handle this situation we turn these circles into lines using an inversion transformation. We then analyze the geometric and algebraic structure of the transformed zero set using a variety of tools such as *flecnode polynomials* (as used in [20]), additional classical 19$^{\text{th}}$-century results in analytic geometry from [35] (mostly related to *ruled surfaces*), a very recent technique for analyzing surfaces that are "ruled" by lines and circles [32], and some traditional tools from combinatorial geometry.

Removing the epsilons. One disadvantage of the current use of constant-degree partitioning polynomials is that ε's appear in some exponents in the resulting bound, as stated in Theorem 1.1. In Section 3.1 we present a new method, with a more involved analysis, for partially removing these ε's. It yields the following theorem:

THEOREM 1.2. *Let $\mathcal{P}$ be a set of m points and let $\mathcal{C}$ be a set of n circles in $\mathbb{R}^3$, let $q < n$ be an integer, and let $m = O(n^{3/2-\varepsilon})$, for some fixed arbitrarily small $\varepsilon > 0$. If no sphere or plane contains more than q circles of $\mathcal{C}$, then*

$$I(\mathcal{P},\mathcal{C}) \le A_{m,n}\left(m^{3/7}n^{6/7} + m^{2/3}n^{1/2}q^{1/6} + m^{6/11}n^{15/22}q^{3/22}\log^{2/11} m + m + n\right), \quad (2)$$

where $A_{m,n} = A^{\left\lceil \frac{3}{2}\cdot\frac{\log(m/n^{1/3})}{\log(n^{3/2}/m)}\right\rceil}$, for some absolute constant $A > 1$.

Recently, several other cases in which such ε's can be removed were described in [44]. Our method seems to be sufficiently generic, so that variants of it may possibly yield similar improvements of other bounds that were obtained with constant-degree partitioning polynomials, such as the ones in [38].

Unit circles. In the special case where all the circles of $\mathcal{C}$ have the same radius, we derive the following improved bound.

THEOREM 1.3. *Let $\mathcal{P}$ be a set of m points and let $\mathcal{C}$ be a set of n unit circles in $\mathbb{R}^3$, let ε be an arbitrarily small positive constant, and let $q < n$ be an integer. If no plane or sphere contains more than q circles of $\mathcal{C}$, then*

$$I(\mathcal{P},\mathcal{C}) = O\left(m^{5/11+\varepsilon}n^{9/11} + m^{2/3+\varepsilon}n^{1/2}q^{1/6} + m + n\right),$$

where the constant of proportionality depends on ε.

This improvement is obtained through the following steps. (i) We use the planar (or spherical) bound $O(m^{2/3}n^{2/3} +$

$m + n$) for incidences with unit circles (e.g., see [39]). (ii) We show that the number of unit circles incident to at least three points in a given set of m points in $\mathbb{R}^3$ is only $O(m^{5/2})$. (iii) We use this bound as a bootstrapping tool for deriving the bound asserted in the theorem. The details are presented in the full version of this paper [37]. Here too the ε's can be (partially) removed from the bound; we omit the details of this improvement in this abstract.

An application: similar triangles. Given a finite point set $\mathcal{P}$ in $\mathbb{R}^3$ and a triangle Δ, we denote by $F(\mathcal{P}, \Delta)$ the number of triangles that are spanned by points of $\mathcal{P}$ and are similar to Δ. Let $F(m) = \max_{|\mathcal{P}|=m,\Delta} F(\mathcal{P}, \Delta)$. The problem of obtaining good bounds for $F(m)$ is motivated by questions in exact pattern matching, and has been studied in several previous works (see [1, 4, 6, 10]). Theorem 1.2 implies the bound $F(m) = O(m^{15/7})$, which slightly improves upon the previous bound of $O^*(m^{58/27})$ from [6] (in the previous bound, the $O^*()$ notation only hides polylogarithmic factors); see also [1]. The new bound is an almost immediate corollary of Theorem 1.2, while the previous bound requires a more complicated analysis. This application is presented in Section 5.

2. ALGEBRAIC PRELIMINARIES

We briefly review in this section the machinery needed for our analysis, including the polynomial partitioning technique of Guth and Katz and several basic tools from algebraic geometry.

Polynomial partitioning. Consider a set $\mathcal{P}$ of m points in $\mathbb{R}^d$. Given a polynomial $f \in \mathbb{R}[x_1, \ldots, x_d]$, we define the *zero set* of f to be $Z(f) = \{p \in \mathbb{R}^d \mid f(p) = 0\}$. For $1 < r \le m$, we say that $f \in \mathbb{R}[x_1, \ldots, x_d]$ is an *r-partitioning polynomial* for $\mathcal{P}$ if no connected component of $\mathbb{R}^d \setminus Z(f)$ contains more than m/r points of $\mathcal{P}$. Notice that there is no restriction on the number of points of $\mathcal{P}$ that lie in $Z(f)$. The following result is due to Guth and Katz [20]. A detailed proof can also be found in [25].

THEOREM 2.1. **(Polynomial partitioning [20])** *Let $\mathcal{P}$ be a set of m points in $\mathbb{R}^d$. Then for every $1 < r \le m$, there exists an r-partitioning polynomial $f \in \mathbb{R}[x_1, \ldots, x_d]$ of degree $O(r^{1/d})$.*

To use such a partitioning effectively, we also need a bound on the maximum possible number of cells of the partition. Such a bound is provided by the following theorem.

THEOREM 2.2. **(Warren's theorem [41])** *Given a polynomial $f \in \mathbb{R}[x_1, \ldots, x_d]$ of degree k, the number of connected components of $\mathbb{R}^d \setminus Z(f)$ is $O((2k)^d)$.*

Consider an r-partitioning polynomial f for a point-set $\mathcal{P}$, as provided in Theorem 2.1. The number of cells in the partition is equal to the number of connected components of $\mathbb{R}^d \setminus Z(f)$. By Theorem 2.2, this is $O((r^{1/d})^d) = O(r)$ (recall that f is of degree $O(r^{1/d})$ and that d is treated as a fixed constant — 3 in our case). It follows that the bound on the number of points in each cell, namely m/r, is asymptotically best possible.

Since this paper studies incidences in a three-dimensional space, we will only apply the above theorems for $d = 3$.

Bézout's theorem. We also need the following basic property of zero sets of polynomials in the plane (for further discussion see [12, 13]).

THEOREM 2.3. **(Bézout's theorem)** *Let f, g be two polynomials in $\mathbb{R}[x_1, x_2]$ of degrees D_f and D_g, respectively. (i) If $Z(f)$ and $Z(g)$ have a finite number of common points, then this number is at most $D_f D_g$. (ii) If $Z(f)$ and $Z(g)$ have an infinite number of (or just more than $D_f D_g$) common points, then f and g have a common (nontrivial) factor.*

The following is an extension of Bézout's theorem to complex projective spaces of any dimension (e.g., see [18]).

THEOREM 2.4. **(Higher dimensional extension of Bézout's theorem)** *Let Z_1 and Z_2 be pure-dimensional varieties (every irreducible component of a pure-dimensional variety has the same dimension) in the d-dimensional complex projective space, with $\operatorname{codim} Z_1 + \operatorname{codim} Z_1 = d$. Then if $Z_1 \cap Z_2$ is a zero-dimensional set of points, this set is finite.*

The following lemma is a consequence of Theorem 2.3. Its proof is given in [14, Proposition 1] and [19, Corollary 2.5].

LEMMA 2.5. **(Guth and Katz [19])** *Let f and g be two polynomials in $\mathbb{R}[x_1, x_2, x_3]$ of respective degrees D_f and D_g, such that f and g have no common factor. Then there are at most $D_f D_g$ lines on which both f and g vanish identically.*

Flecnode polynomial. A *flecnode* of a surface Z in $\mathbb{R}^3$ is a point $p \in Z$ for which there exists a line that passes through p and agrees with Z at p to order three. That is, if $Z = Z(f)$ (here f is the lowest degree polynomial whose zero-set is Z) and the direction of the line is $v = (v_1, v_2, v_3)$ then

$$f(p) = 0, \quad \nabla_v f(p) = 0, \quad \nabla_v^2 f(p) = 0, \quad \nabla_v^3 f(p) = 0,$$

where $\nabla_v f, \nabla_v^2 f, \nabla_v^3 f$ are, respectively, the first, second, and third-order derivatives of f in the direction v. That is $\nabla_v f = \nabla f \cdot v$, $\nabla_v^2 f = v^T H_f v$, where H_f is the *Hessian matrix* of f, and $\nabla_v^3 f$ is similarly defined, although its explicit expression in terms of the third-order partial derivatives of f is somewhat more involved.

The *flecnode polynomial* of f, denoted FL_f, is the polynomial obtained by eliminating v from the last three equations. Note that the corresponding polynomials of the system are homogeneous in v. We thus have a system of three equations in six variables. Eliminating the variables v_1, v_2, v_3 results in a single polynomial equation in $p = (x_1, x_2, x_3)$, which is the desired flecnode polynomial. By construction, the flecnode polynomial of f vanishes on all the flecnodes of $Z(f)$. The following results, also mentioned in [20, Section 3], are taken from Salmon [35, Chapter XVII, Section III].

LEMMA 2.6. *Let $Z \subset \mathbb{R}^3$ be a surface, with $Z = Z(f)$ for a polynomial $f \in \mathbb{R}[x_1, x_2, x_3]$ of degree $d \ge 3$. Then FL_f has degree at most $11d - 24$.*

An algebraic surface S in a three-dimensional space (we restrict our attention to $\mathbb{R}^3$, $\mathbb{C}^3$, and the complex projective space $\mathbb{CP}^3$) is said to be *ruled* if every point of S is incident to a straight line that is fully contained in S. Equivalently, S is a (two-dimensional) union of lines.[3] We say that an irreducible surface S is *triply ruled* if for every point on S there are (at least) three straight lines contained in S and passing through that point. As is well known (e.g., see [17,

[3]We do not insist on the more restrictive definition used in differential geometry, which requires the ruling lines to form a smooth 1-parameter family; cf. [9, Chapter III].

Lecture 16]), the only triply ruled surfaces are planes. We say that an irreducible surface S is *doubly ruled* if it is not triply ruled and for every point on S there are (at least) two straight lines contained in S and passing through that point. It is well known that the only doubly ruled surfaces are the hyperbolic paraboloid and the hyperboloid of one sheet (again, see [17, Lecture 16]). Finally, we say that an irreducible ruled surface is *singly ruled* if it is neither doubly nor triply ruled.

LEMMA 2.7. *Let $Z \subset \mathbb{R}^3$ be a surface with $Z = Z(f)$ for a polynomial $f \in \mathbb{R}[x_1, x_2, x_3]$ of degree $d \geq 3$. Then every line that is fully contained in Z is also fully contained in $Z(\mathsf{FL}_f)$.*

THEOREM 2.8. **(Cayley-Salmon [35])** *Let $Z \subset \mathbb{R}^3$ be a surface with $Z = Z(f)$ for a polynomial $f \in \mathbb{R}[x_1, x_2, x_3]$ of degree $d \geq 3$. Then Z is ruled if and only if $Z \subseteq Z(\mathsf{FL}_f)$.*

COROLLARY 2.9. *Let $Z \subset \mathbb{R}^3$ be a surface with $Z = Z(f)$ for an irreducible polynomial $f \in \mathbb{R}[x_1, x_2, x_3]$ of degree $d \geq 3$. If Z contains more than $d(11d - 24)$ lines then Z is a ruled surface.*

Proof. Lemma 2.5 and Lemma 2.7 imply that in this case f and FL_f have a common factor. Since f is irreducible, f divides FL_f, and Theorem 2.8 completes the proof. ◻

A modern treatment (and generalization) of the Cayley-Salmon theorem can be found in a more recent work by Landsberg [26].

THEOREM 2.10. **(Landsberg [26])** *Let Z be a surface in either $\mathbb{C}^3$ or $\mathbb{C}\mathbf{P}^3$ (the three-dimensional complex projective space), and let $Z = Z(f)$ for a polynomial f of degree $d \geq 3$. Then Z is ruled if and only if $Z \subseteq Z(\mathsf{FL}_f)$.*

3. PROOF OF THEOREM 1.1

The proof proceeds by induction on $m + n$. Specifically, we prove by induction that, for any fixed $\varepsilon > 0$, there exist constants α_1, α_2 such that

$$I(\mathcal{P}, \mathcal{C}) \leq \alpha_1 \left(m^{3/7+\varepsilon} n^{6/7} + m^{2/3+\varepsilon} n^{1/2} q^{1/6} + m^{6/11+\varepsilon} n^{15/22} q^{3/22} \right) + \alpha_2(m + n).$$

Let n_0 be a constant. The base case where $m + n < n_0$ can be dealt with by choosing α_1 and α_2 sufficiently large.

We start by recalling a well-known simple, albeit weaker bound. The incidence graph $G \subseteq \mathcal{P} \times \mathcal{C}$ whose edges are the incident pairs in $\mathcal{P} \times \mathcal{C}$ cannot contain $K_{3,2}$ as a subgraph, because two circles have at most two intersection points. By the Kővári-Sós-Turán theorem (e.g., see [29, Section 4.5]), $I(\mathcal{P}, \mathcal{C}) = |G| = O\left(n^{2/3} m + n\right)$. This immediately implies the theorem if $m = O\left(n^{1/3}\right)$. Thus we may assume that $n = O\left(m^3\right)$.

We next apply the polynomial partitioning technique. That is, we set r as a sufficiently large constant (whose value depends on ε and will be determined later), and apply the polynomial partitioning theorem (Theorem 2.1) to obtain an r-partitioning polynomial f. According to the theorem, f is of degree $D = O\left(r^{1/3}\right)$ and $Z(f)$ partitions $\mathbb{R}^3$ into maximal connected cells, each containing at most m/r points of $\mathcal{P}$. As already noted, Warren's theorem (Theorem 2.2) implies that the number of cells is $O(r)$.

Let $\mathcal{C}_0$ denote the subset of circles of $\mathcal{C}$ that are fully contained in $Z(f)$, and let $\mathcal{C}' = \mathcal{C} \setminus \mathcal{C}_0$. Similarly, set $\mathcal{P}_0 = \mathcal{P} \cap Z(f)$ and $\mathcal{P}' = \mathcal{P} \setminus \mathcal{P}_0$. Notice that

$$I(\mathcal{P}, \mathcal{C}) = I(\mathcal{P}_0, \mathcal{C}_0) + I(\mathcal{P}_0, \mathcal{C}') + I(\mathcal{P}', \mathcal{C}'). \tag{3}$$

The terms $I(\mathcal{P}_0, \mathcal{C}')$ and $I(\mathcal{P}', \mathcal{C}')$ can be bounded using techniques (detailed below) that are by now fairly standard. On the other hand, bounding $I(\mathcal{P}_0, \mathcal{C}_0)$ is the main technical challenge in this proof. Other works that have applied the polynomial partitioning technique, such as [24, 25, 38, 43, 44], also spend most of their efforts on incidences with curves that are fully contained in the zero set of the partitioning polynomial (where these curves are either original input curves or the intersections of input surfaces with the zero set).

Bounding $I(\mathcal{P}_0, \mathcal{C}')$ and $I(\mathcal{P}', \mathcal{C}')$. For a circle $C \in \mathcal{C}'$, let Π_C be the plane that contains C, and let f_C denote the restriction of f to Π_C. Since C is not contained in $Z(f_C)$, f_C and the irreducible quadratic equation of C within Π_C do not have any common factor. Thus by Bézout's theorem (Theorem 2.3), C and $Z(f_C)$ have at most $2 \cdot \deg(f_C) = O\left(r^{1/3}\right)$ common points. This immediately implies

$$I(\mathcal{P}_0, \mathcal{C}') = O\left(nr^{1/3}\right). \tag{4}$$

Next, let us denote the cells of the partition as $K_1, \ldots, K_s$ (recall that $s = O(r)$ and that the cells are open). For $i = 1, \ldots, s$, put $\mathcal{P}_i = \mathcal{P} \cap K_i$ and let $\mathcal{C}_i$ denote the set of circles in $\mathcal{C}'$ that intersect K_i. Put $m_i = |\mathcal{P}_i|$, $n_i = |\mathcal{C}_i|$, for $i = 1, \ldots, s$. Note that $|\mathcal{P}'| = \sum_{i=1}^{s} m_i$, and recall that $m_i \leq m/r$ for every i. The above bound of $O\left(r^{1/3}\right)$ on the number of intersection points of a circle $C \in \mathcal{C}'$ and $Z(f)$ implies that each circle crosses $O\left(r^{1/3}\right)$ cells (a circle has to intersect $Z(f)$ when moving from one cell to another). This implies $\sum_i n_i = O\left(nr^{1/3}\right)$.

Notice that $I(\mathcal{P}', \mathcal{C}') = \sum_{i=1}^{s} I(\mathcal{P}_i, \mathcal{C}_i)$, so we proceed to bound the number of incidences within a cell K_i. From the induction hypothesis, we get

$$\begin{aligned} I(\mathcal{P}', \mathcal{C}') &\leq \sum_{i=1}^{s} \left(\alpha_1 \left(m_i^{3/7+\varepsilon} n_i^{6/7} + m_i^{2/3+\varepsilon} n_i^{1/2} q^{1/6} + m_i^{6/11+\varepsilon} n_i^{15/22} q^{3/22} \right) + \alpha_2(m_i + n_i) \right) \\ &\leq \sum_{i=1}^{s} \left(\alpha_1 \left(\left(\frac{m}{r}\right)^{3/7+\varepsilon} n_i^{6/7} + \left(\frac{m}{r}\right)^{2/3+\varepsilon} n_i^{1/2} q^{1/6} + \left(\frac{m}{r}\right)^{6/11+\varepsilon} n_i^{15/22} q^{3/22} \right) \right) \\ &\quad + \alpha_2 \left(|\mathcal{P}'| + \sum_{i=1}^{s} n_i \right). \end{aligned} \tag{5}$$

Since $\sum_i n_i = O\left(nr^{1/3}\right)$, Hölder's inequality implies

$$\sum_{i=1}^{s} n_i^{6/7} = O\left(\left(nr^{1/3}\right)^{6/7} \cdot r^{1/7} \right) = O\left(n^{6/7} r^{3/7}\right).$$

Similarly, $\sum_{i=1}^{s} n_i^{1/2} = O\left(n^{1/2} r^{2/3}\right)$ and $\sum_{i=1}^{s} n_i^{15/22} =$

$O\left(n^{15/22}r^{6/11}\right)$. By combining this with (5), we obtain

$$I(\mathcal{P}',\mathcal{C}') \le \alpha_1 \cdot O\left(\frac{m^{3/7+\varepsilon}n^{6/7} + m^{2/3+\varepsilon}n^{1/2}q^{1/6}}{r^\varepsilon} + \frac{m^{6/11+\varepsilon}n^{15/22}q^{3/22}}{r^\varepsilon}\right) + \alpha_2\left(|\mathcal{P}'| + O\left(nr^{1/3}\right)\right).$$

Notice that the bound in (4) is subsumed in this bound, and it is dominated by $O(m^{3/7}n^{6/7})$ since we assumed that $n = O\left(m^3\right)$ and that r is constant. Taking r to be sufficiently large, and α_1 to be sufficiently larger than $\alpha_2 r^{1/3}$, we have

$$I(\mathcal{P}_0 \cup \mathcal{P}',\mathcal{C}') \le \frac{\alpha_1}{3}\left(m^{3/7+\varepsilon}n^{6/7} + m^{2/3+\varepsilon}n^{1/2}q^{1/6} + m^{6/11+\varepsilon}n^{15/22}q^{3/22}\right) + \alpha_2|\mathcal{P}'|. \quad (6)$$

Bounding $I(\mathcal{P}_0, \mathcal{C}_0)$: Handling shared points. We are left with the task of bounding the number of incidences between the set $\mathcal{P}_0$ of points of $\mathcal{P}$ that are contained in $Z(f)$ and the set $\mathcal{C}_0$ of circles of $\mathcal{C}$ that are fully contained in $Z(f)$. We call a point of $\mathcal{P}_0$ *shared* if it is contained in the zero sets of at least two distinct irreducible factors of f, and otherwise we call it *private*. We first consider the case of shared points.

Let $\mathcal{P}_s$ denote the subset of points in $\mathcal{P}_0$ that are shared, and put $m_s = |\mathcal{P}_s|$. Let $f' = \nabla_e f$, where e is a generic choice of a unit vector, and $\nabla_e f$ denotes the directional derivative of f in direction e. Then $\deg(f') < D$. We may assume that f is the lowest-degree polynomial whose zero-set is $Z(f)$, and thus in particular, f is square-free. Therefore, $Z(f')$ contains the singular set of $Z(f)$. By definition, a shared point is necessarily a singular point of f (because, as is easily checked, all first-order partial derivatives of f vanish at a shared point), and thus $\mathcal{P}_s \subset Z(f) \cap Z(f')$. Since f is square-free, $Z(f) \cap Z(f')$ has dimension at most 1. We claim that at most $\frac{1}{2}D^2$ circles can be contained in $Z(f) \cap Z(f')$. Indeed, a generic projection of $Z(f) \cap Z(f')$ onto $\mathbb{R}^2$ yields a (planar) algebraic curve[4] of degree at most D^2, and every circle contained in $Z(f) \cap Z(f')$ is a distinct ellipse contained in the projected curve. A planar algebraic curve of degree at most D^2 can contain at most $\frac{1}{2}D^2$ ellipses, from which the claim follows. Thus there are at most $\frac{1}{2}D^2 m_s$ incidences between points in $\mathcal{P}_s$ and circles contained in $Z(f) \cap Z(f')$.

It remains to bound the number of incidences between points in $\mathcal{P}_s$ and circles of $\mathcal{C}_0$ not contained in $Z(f) \cap Z(f')$ (that is, circles that are contained in $Z(f)$ but not in $Z(f')$). Consider such a circle C and let Π_C be the plane containing C. The intersection $Z(f') \cap \Pi_C$ is therefore a planar algebraic curve of degree at most $D-1$, and by assumption this curve does not contain C. According to Bézout's theorem (Theorem 2.3), C intersects $Z(f') \cap \Pi_C$ at most $2D-2$ times, so C meets $Z(f) \cap Z(f')$ at most $2D-2$ times. This in turn implies that $|C \cap \mathcal{P}_s| < 2D$. Therefore, by taking α_1 and α_2 to be sufficiently large, we have

$$I(\mathcal{P}_s,\mathcal{C}_0) \le \frac{1}{2}D^2 m_s + 2Dn \le \alpha_2(m_s + n/3). \quad (7)$$

Bounding $I(\mathcal{P}_0, \mathcal{C}_0)$: Handling private points. Let $\mathcal{P}_p = \mathcal{P}_0 \setminus \mathcal{P}_s$ denote the set of private points in $\mathcal{P}_0$. Recall that each private point is contained in the zero set of a single irreducible factor of f. Let $f_1, f_2, \dots, f_t$ be the factors of f whose zero sets are planes or spheres. For $i = 1,\dots,t$, set $\mathcal{P}^{(1)}_{p,i} = \mathcal{P}_p \cap Z(f_i)$ and $m_{p,i} = |\mathcal{P}^{(1)}_{p,i}|$. Put $\mathcal{P}^{(1)}_p = \bigcup_{i=1}^t \mathcal{P}^{(1)}_{p,i}$ and $m^{(1)}_p = |\mathcal{P}^{(1)}_p| = \sum_{i=1}^t m_{p,i}$. Let $n_{p,i}$ denote the number of circles of $\mathcal{C}_0$ that are fully contained in $Z(f_i)$. Notice that (i) $t \le D = O\left(r^{1/3}\right)$, (ii) $n_{p,i} \le q$ for every i, and (iii) $\sum_i n_{p,i} \le n$ (we may ignore circles that are fully contained in more than one component, since these will not have incidences with private points). Applying (1) and using the fact that there are no hidden polylogarithmic terms in the linear part of (1), we obtain[5]

$$\begin{aligned}
I(\mathcal{P}^{(1)}_p,\mathcal{C}_0) &= \sum_{i=1}^t \left(O^*\left(m_{p,i}^{2/3}n_{p,i}^{2/3} + m_{p,i}^{6/11}n_{p,i}^{9/11}\right) + O(m_{p,i}+n_{p,i})\right)\\
&= \sum_{i=1}^t \left(O^*\left(m_{p,i}^{2/3}n_{p,i}^{1/3}q^{1/3} + m_{p,i}^{6/11}n_{p,i}^{5/11}q^{4/11}\right) + O(m_{p,i}+n_{p,i})\right)\\
&= O^*\left(m^{2/3}n^{1/3}q^{1/3} + m^{6/11}n^{5/11}q^{4/11}\right) + O\left(m^{(1)}_p + n\right),
\end{aligned}$$

where the last step uses Hölder's inequality; it bounds (twice) $\sum_i m_{p,i} = m^{(1)}_p$ by m. Since $q \le n$, it follows that when n_0 (and thus n), α_1, and α_2 are sufficiently large, we have

$$I(\mathcal{P}^{(1)}_p,\mathcal{C}_0) \le \frac{\alpha_1}{3}\left(m^{2/3+\varepsilon}n^{1/2}q^{1/6} + m^{6/11+\varepsilon}n^{15/22}q^{3/22}\right) + \alpha_2(m^{(1)}_p + n/3). \quad (8)$$

Let $\mathcal{P}^{(2)}_p = \mathcal{P}_p \setminus \mathcal{P}^{(1)}_p$ be the set of private points that lie on the zero sets of factors of f which are neither planes nor spheres, and put $m^{(2)}_p = \left|\mathcal{P}^{(2)}_p\right|$. To handle incidences with these points we require the following lemma, which constitutes a major component of our analysis and which is proved in Section 4 (somewhat similar results can be found in [23, 27]). Let g be an irreducible polynomial in $\mathbb{R}[x_1,x_2,x_3]$ such that $Z(g)$ is a surface. We say that a point $p \in Z(g)$ is *popular* if it is incident to at least $44(\deg g)^2$ circles that are fully contained in $Z(g)$.

LEMMA 3.1. *An irreducible algebraic surface that is neither a plane nor a sphere cannot contain more than two popular points.*

The lemma implies that the number of incidences between popular points of $\mathcal{P}^{(2)}_p$ (within their respective irreducible components of $Z(f)$) and circles of $\mathcal{C}_0$ is at most $2(D/2)n =$

[4]Technically, we need to argue that over $\mathbb{C}$, $Z(f)$ and $Z(f')$ have one–dimensional intersection. However, this follows from the the fact that f is the lowest-degree polynomial whose zero-set is $Z(f)$, and that over $\mathbb{R}$, $Z(f) \cap Z(f')$ has dimension 1. See [42] for further details.
The fact that the degree of the projected curve is at most $D(D-1) < D^2$ is a consequence of the proof of Bézout's theorem (Theorem 2.3), which makes use of the *resultant* of f and f'; e.g., see [13, Section 8.7].

[5]Notice that the dependency of this bound in n and q is better than the one in the bound of the theorem. We compromise on the worse bound so that the partitioning would work.

$Dn \le \alpha_2 n/3$ (the latter inequality holds if α_2 is chosen sufficiently large with respect to $D = O(r^{1/3})$). The number of incidences between non-popular points of $\mathcal{P}_p^{(2)}$ and circles of $\mathcal{C}_0$ is at most $m_p^{(2)} \cdot 44D^2 \le \alpha_2 m_p^{(2)}$ (again for a sufficiently large value of α_2). Combining this with (3), (6), (7), and (8), we get

$$I(\mathcal{P}, \mathcal{C}) \le \alpha_1 \Big(m^{3/7+\varepsilon} n^{6/7} + m^{2/3+\varepsilon} n^{1/2} q^{1/6} + m^{6/11+\varepsilon} n^{15/22} q^{3/22} \Big) + \alpha_2 (m+n).$$

This establishes the induction step, and thus completes the proof of the theorem. □

Remark. It is not immediately clear from the induction step why $m^{3/7+\varepsilon} n^{6/7}$ is the best choice for the leading term. Let us denote the leading term as $m^{a+\varepsilon} n^b$ and observe the following restrictions on a and b: (i) For r to cancel itself in the analysis of incidences within the cells of the partition (up to a power of ε), we require $a \ge 1 - 2b/3$. (ii) For $n = O(m^3)$ to imply $n = O(m^a n^b)$, we must have $a + 3b \ge 3$. Combining both constraints, with equalities, results in $a = 3/7$ and $b = 6/7$.

3.1 Removing the epsilons

In this section we will show that, for any $\varepsilon > 0$, when $m = O(n^{3/2-\varepsilon})$, the epsilons from the bound of Theorem 1.1 can be removed. This is what Theorem 1.2 asserts; we repeat its statement for the convenience of the reader.

Theorem 1.2. *Let $\mathcal{P}$ be a set of m points and let $\mathcal{C}$ be a set of n circles in $\mathbb{R}^3$, let $q < n$ be an integer, and let $m = O(n^{3/2-\varepsilon})$, for some fixed arbitrarily small $\varepsilon > 0$. If no sphere or plane contains more than q circles of $\mathcal{C}$, then*

$$I(\mathcal{P}, \mathcal{C}) \le A_{m,n} \Big(m^{3/7} n^{6/7} + m^{2/3} n^{1/2} q^{1/6} + m^{6/11} n^{15/22} q^{3/22} \log^{2/11} m + m + n \Big),$$

where $A_{m,n} = A^{\left\lceil \frac{3}{2} \cdot \frac{\log(m/n^{1/3})}{\log(n^{3/2}/m)} \right\rceil}$, for some absolute constant $A > 1$.

Proof. We define $\mathcal{P}$, $\mathcal{P}_0$, $\mathcal{C}$, $\mathcal{C}'$, etc., as in the proof of Theorem 1.1. The proof is similar to the one of Theorem 1.1, except that it works in stages, so that in each stage we enlarge the range of m where the (improved) bound applies. At each stage we construct a partitioning polynomial, as before but of a non-constant degree, use the bound obtained in the previous stage for the incidence count within the cells of the polynomial partitioning, and then use a separate argument (essentially the one given in the second part of of the proof of Theorem 1.1) to bound the number of incidences with the points that lie on the zero set of the polynomial. Each stage increases the constant of proportionality in the bound by a constant factor, which is why the "constant" $A_{m,n}$ increases as the m-range approaches $n^{3/2}$. The j-th stage, for $j = 1, 2, \ldots$, asserts the bound specified in the theorem when $m \le n^{\alpha_j}$, for some sequence of exponents $\alpha_j < 3/2$ that increase from stage to stage, and approach $3/2$. Each stage has its own constant of proportionality $A^{(j)}$. The specific values of the exponents α_j (and the constants of proportionality) will be set later. For the 0-th, vacuous stage we use $\alpha_0 = 1/3$, and the bound $O(n)$ that was noted above for $m \le n^{\alpha_0}$, with an implied initial constant of proportionality $A^{(0)}$.

In handling the j-th stage, we assume that $n^{\alpha_{j-1}} < m \le n^{\alpha_j}$; if $m \le n^{\alpha_{j-1}}$ there is nothing to do as we can use the bound from the previous stage. As in the proof of Theorem 1.1, we consider an r-partitioning polynomial f and put $\alpha = \alpha_{j-1}$. To apply the bound from the previous stage uniformly within each cell, we want to have a uniform bound on the number of circles crossing a cell. The average number of such crossings per cell is proportional to $n/r^{2/3}$ (assuming that the number of cells is $\Theta(r)$, an assumption made only for the sake of intuition). A cell crossed by $tn/r^{2/3}$ circles, for $t > 1$, induces $\lceil t \rceil$ subproblems, each involving all the points in the cell and up to $n/r^{2/3}$ crossing circles. It is easily checked that the number of subproblems remains $O(r)$, with a slightly larger constant of proportionality, and each subproblem now involves at most m/r points and at most $n/r^{2/3}$ circles. Moreover, in cells that have strictly fewer than $n/r^{2/3}$ circles, we will assume that there are exactly $n/r^{2/3}$ circles, e.g., by adding dummy circles. This can only increase the number of incidences.

We assume that the number of cells is at most br, for some absolute constant b. To apply the bound from the previous stage, we need to choose r that will guarantee that

$$\frac{m}{r} \le \left(\frac{n}{r^{2/3}}\right)^{\alpha}, \quad \text{or} \quad r^{1-2\alpha/3} \ge \frac{m}{n^{\alpha}}, \quad \text{or} \quad r \ge \frac{m^{3/(3-2\alpha)}}{n^{3\alpha/(3-2\alpha)}}.$$

We choose r to be equal to the last expression. We note that (i) $r \ge 1$, because m is assumed to be greater than n^{α} and $\alpha < 3/2$, and (ii) $r \le m$, because $m \le n^{3/2}$. Because of the somewhat weak bound that we will derive below on the number of incidences with points that lie on $Z(f)$, this choice of r will work only when m is not too large. The resulting constraint on m, of the form $m \le n^{\alpha_j}$, will define the new range in which the bound derived in the present stage applies. The number of incidences within the partition cells is thus

$$\begin{aligned} I(\mathcal{P}', \mathcal{C}') \le A^{(j-1)} \sum_{i=1}^{br} \Bigg(& \left(\frac{m}{r}\right)^{3/7} \left(\frac{n}{r^{2/3}}\right)^{6/7} \\ & + \left(\frac{m}{r}\right)^{2/3} \left(\frac{n}{r^{2/3}}\right)^{1/2} q^{1/6} \\ & + \left(\frac{m}{r}\right)^{6/11} \left(\frac{n}{r^{2/3}}\right)^{15/22} q^{3/22} \log^{2/11}\left(\frac{m}{r}\right) \\ & + \frac{m}{r} + \frac{n}{r^{2/3}} \Bigg) \\ \le bA^{(j-1)} \Big(& m^{3/7} n^{6/7} + m^{2/3} n^{1/2} q^{1/6} \\ & + m^{6/11} n^{15/22} q^{3/22} \log^{2/11} m + m + nr^{1/3} \Big). \end{aligned}$$

We claim that the above choice of r ensures that $nr^{1/3} \le m^{3/7} n^{6/7}$. That is,

$$r^{1/3} = \frac{m^{1/(3-2\alpha)}}{n^{\alpha/(3-2\alpha)}} \le \frac{m^{3/7}}{n^{1/7}}.$$

Indeed, this is easily seen to hold because $1/3 \le \alpha < 3/2$ and $m \le n^{3/2}$. Recall that we also have $I(\mathcal{P}_0, \mathcal{C}') \le A' n r^{1/3}$ for some constant A' (see (4)). By choosing $A^{(0)} > A'$ (so that $A^{(j-1)} > A'$ for every j), we have

$$\begin{aligned} I(\mathcal{P},\mathcal{C}') &= I(\mathcal{P}_0,\mathcal{C}') + I(\mathcal{P}',\mathcal{C}') \\ &\le bA^{(j-1)}\Big(3m^{3/7}n^{6/7} + m^{2/3}n^{1/2}q^{1/6} \\ &\qquad + m^{6/11}n^{15/22}q^{3/22}\log^{2/11} m + m\Big). \end{aligned} \tag{9}$$

As proved in Theorem 1.1,

$$I(\mathcal{P}_s,\mathcal{C}_0) + I(\mathcal{P}_p^{(2)},\mathcal{C}_0) \le 45mr^{2/3} + 3nr^{1/3} \tag{10}$$

(this follows by substituting $D = O(r^{1/3})$ in the bounds in the proof of Theorem 1.1, which are $I(\mathcal{P}_s,\mathcal{C}_0) \le mD^2/2 + 2nD$ and $I(\mathcal{P}_p^{(2)},\mathcal{C}_0) \le 44mD^2 + nD$).

It remains to bound $I(\mathcal{P}_p^{(1)},\mathcal{C}_0)$. For this, we again use an analysis similar to the one in Theorem 1.1. Let $f_1, f_2, \ldots, f_t$ be the factors of f whose zero sets are planes or spheres. For $i = 1,\ldots,t$, set $\mathcal{P}_{p,i}^{(1)} = \mathcal{P}_p \cap Z(f_i)$ and $m_{p,i} = |\mathcal{P}_{p,i}^{(1)}|$. Let $n_{p,i}$ denote the number of circles of $\mathcal{C}_0$ that are fully contained in $Z(f_i)$. Put $\mathcal{P}_p^{(1)} = \bigcup_{i=1}^{t} \mathcal{P}_{p,i}^{(1)}$. Notice that (i) $t = O\left(r^{1/3}\right)$, (ii) $n_{p,i} \le q$ for every i, and (iii) $\sum_i n_{p,i} \le n$. Applying (1), we obtain

$$\begin{aligned} I(\mathcal{P}_p^{(1)},\mathcal{C}_0) &= \sum_{i=1}^{t} O\Big(m_{p,i}^{2/3}n_{p,i}^{2/3} \\ &\qquad + m_{p,i}^{6/11}n_{p,i}^{9/11}\log^{2/11}(m_{p,i}^3/n_{p,i}) + m_{p,i} + n_{p,i}\Big) \\ &= \sum_{i=1}^{t} O\Big(m_{p,i}^{2/3}n_{p,i}^{1/3}q^{1/3} \\ &\qquad + m_{p,i}^{6/11}n_{p,i}^{5/11}q^{4/11}\log^{2/11}(m_{p,i}^3) + m_{p,i} + n_{p,i}\Big) \\ &= O\Big(m^{2/3}n^{1/3}q^{1/3} + m^{6/11}n^{5/11}q^{4/11}\log^{2/11} m \\ &\qquad + m + n\Big), \end{aligned} \tag{11}$$

where the last step uses Hölder's inequality.

We would like to combine (9), (10), and (11) to obtain the asserted bound. All the elements in these bounds add up to the bound, with an appropriate sufficiently large choice of $A^{(j)}$, except for the term $O(mr^{2/3})$, which may exceed the bound of the theorem if m is too large. Thus, we restrict m to satisfy

$$mr^{2/3} \le m^{3/7}n^{6/7}, \qquad \text{or} \qquad r \le \frac{n^{9/7}}{m^{6/7}}.$$

Substituting the chosen value of r, we thus require that

$$\frac{m^{3/(3-2\alpha)}}{n^{3\alpha/(3-2\alpha)}} \le \frac{n^{9/7}}{m^{6/7}}.$$

That is, we require

$$m \le n^{\frac{9+\alpha}{13-4\alpha}}.$$

That is, recalling that we write the (upper bound) constraint on m at the j-th stage as $m \le n^{\alpha_j}$, we have the recurrence

$$\alpha_j = \frac{9+\alpha_{j-1}}{13-4\alpha_{j-1}}.$$

To simplify this, we write $\alpha_j = \frac{3}{2} - \frac{1}{x_j}$, and obtain the recurrence

$$x_j = x_{j-1} + \frac{4}{7},$$

with the initial value $x_0 = \frac{6}{7}$ (this gives the initial constraint $m \le n^{1/3}$). In other words, we have $x_j = (4j+6)/7$, and

$$\alpha_j = \frac{3}{2} - \frac{7}{4j+6}.$$

The first few values are $\alpha_0 = 1/3$, $\alpha_1 = 4/5$, $\alpha_2 = 1$, and $\alpha_3 = 10/9$. Note that every $m < n^{3/2}$ is covered by the range of some stage. Specifically, given such an m, it is covered by stage j, where j is the smallest integer satisfying

$$m \le n^{\frac{3}{2} - \frac{7}{4j+6}},$$

and straightforward calculations show that

$$j = \left\lceil \frac{3}{2} \cdot \frac{\log(m/n^{1/3})}{\log(n^{3/2}/m)} \right\rceil.$$

Inspecting the preceding analysis, we see that the bound holds for the j-th stage if we choose $A^{(j)} = A \cdot A^{(j-1)}$, where A is a sufficiently large absolute constant. Hence, for m in the j-th range, the bound on $I(\mathcal{P},\mathcal{C})$ has A^j as the constant of proportionality. This completes the description of the stage, and thus the proof of Theorem 1.2. $\square$

4. THE NUMBER OF POPULAR POINTS IN AN IRREDUCIBLE VARIETY

In this section we provide a sketch of the proof of Lemma 3.1. Most of the statements made in the proof are stated without proof. Complete and detailed proofs can be found in the full version of this paper [37].

Inversion. We consider the three-dimensional *inversion transformation* $I : \mathbb{R}^3 \to \mathbb{R}^3$ about the origin (e.g., see [22, Chapter 37]). The transformation $I(\cdot)$ maps the point $p = (x_1, x_2, x_3) \ne (0,0,0)$ to the point $\bar{p} = I(p) = (\bar{x}_1, \bar{x}_2, \bar{x}_3)$, where

$$\bar{x}_i = \frac{x_i}{x_1^2 + x_2^2 + x_3^2}, \quad i = 1,2,3.$$

The inversion satisfies: (a) Let C be a circle incident to the origin. Then $I(C)$ is a line not passing through the origin. (b) Let C be a circle *not* incident to the origin. Then $I(C)$ is a circle not passing through the origin. (c) The converse statements of both (a) and (b) also hold.

Consider an irreducible surface $Z = Z(g)$ which is neither a plane nor a sphere, and let $E = \deg(g)$. Assume, for contradiction, that there exist three popular points $z_1, z_2, z_3 \in Z$. After a translation, we may assume that z_1 is the origin. We apply the inversion transformation to obtain $\bar{Z} = I(Z)$, which is easily seen to be an irreducible surface, and the zero set of a polynomial $\bar{g}$ of degree at most $2E$. Since each of the $44E^2$ circles that are incident to z_1 is mapped to a line that is fully contained in $\bar{Z}$, Corollary 2.9 implies that $Z(\bar{g})$ is ruled. From this we conclude that for $i = 1,2,3$, every point u in Z is incident to a circle or a line that is also incident to z_i. These three circles or lines are not necessarily distinct, but they can all coincide only when u lies on the unique circle or line γ that passes through z_1, z_2, z_3, and then all the above three circles or lines coincide with γ. We use this property to derive the following claim (which holds up to a permutation of z_1, z_2, z_3): There exists an infinite family of circles $\bar{\mathcal{C}}$ that are fully contained in $\bar{Z}$, such that every circle of $\bar{\mathcal{C}}$ is incident to $\bar{z}_2 = I(z_2)$. We then take an infinite subset $\bar{\mathcal{C}}'$ of $\bar{\mathcal{C}}$ such that no two circles of $\bar{\mathcal{C}}'$ are coplanar.

In the remainder of the proof, we work mainly in the complex projective 3-space, denoted as $\mathbb{C}\mathbf{P}^3$, instead of the real affine space that we have considered so far.

Complexification and projectivization. Given a variety $V \subset \mathbb{R}^3$, the *complexification* $V^* \subset \mathbb{C}^3$ of V is the smallest complex variety that contains V (in the sense that any other complex variety that contains V also contains V^*, e.g., see [34, 42]). As shown in [42, Lemma 6], such a complexification always exists, and V is precisely the set of real points of V^*. According to [42, Lemma 7], there is a bijection between the set of irreducible components of V and the set of irreducible components of V^*, such that each real component is the real part of its corresponding complex component. In particular, the complexification of an irreducible variety is irreducible. Moreover, if follows from Theorems 2.8 and 2.10 that the complexification of a ruled algebraic surface is also ruled.

If $h \in \mathbb{C}[x_1, x_2, x_3]$ is a polynomial of degree E, we write $h = \sum_I a_I x^I$, where each index I is of the form (I_1, I_2, I_3) with $I_1 + I_2 + I_3 \le E$, and $x^I = x_1^{I_1} x_2^{I_2} x_3^{I_3}$. Define

$$h^\dagger = \sum_I a_I x_0^{E-I_1-I_2-I_3} x_1^{I_1} x_2^{I_2} x_3^{I_3}.$$

Then $h^\dagger$ is a homogeneous polynomial of degree E, referred to as the *homogenization* of h. We define the *projectivization* of the complex surface $Z(h)$ to be the zero set of $h^\dagger$ in $\mathbb{C}\mathbf{P}^3$. We define the *complex projectivization* of a real surface S as the projectivization of the complexification S^* of S.

To distinguish between the complex projective space $\mathbb{C}\mathbf{P}^3$ and the real affine space, we denote the homogeneous coordinates in $\mathbb{C}\mathbf{P}^3$ (using a rather standard notation, see [30]) as $[x_0 : x_1 : x_2 : x_3]$. Let $\hat{Z} \subset \mathbb{C}\mathbf{P}^3$ be the complex projectivization of the surface $\bar{Z} = Z(\bar{g})$. The surface $\hat{Z} \subset \mathbb{C}\mathbf{P}^3$ is irreducible and singly ruled. Let

$$\Gamma = \left\{ [x_0 : x_1 : x_2 : x_3] \,\middle|\, x_0 = 0,\ x_1^2 + x_2^2 + x_3^2 = 0 \right\}$$

be the *absolute conic* in $\mathbb{C}\mathbf{P}^3$ (e.g., see [32]).

Using the Plücker representation of lines. The following arguments are based on the machinery in the recent work of Nilov and Skopenkov [32]. Let

$$\Lambda = \left\{ [x_0 : \ldots : x_5] \mid x_0x_5 + x_1x_4 + x_2x_3 = 0 \right\} \subset \mathbb{C}\mathbf{P}^5$$

be the *Plücker quadric.* Given a point $p = [x_0 : \ldots : x_5] \in \Lambda$, at least two of the four "canonical" points $[0 : x_0 : x_1 : x_2]$, $[x_0 : 0 : -x_3 : -x_4]$, $[x_1 : x_3 : 0 : -x_5]$, and $[x_2 : x_4 : x_5 : 0]$ cannot be the undefined point $[0 : 0 : 0 : 0]$ because each of the six coordinates $x_0, \ldots, x_5$ appears as a coordinate of two of these points. Then there exists a unique line ℓ_p in $\mathbb{C}\mathbf{P}^3$ that passes through all nonzero canonical points of p. We refer to the map $p \to \ell_p$ as the *Plücker map*, and observe that it is a bijection between the points $p \in \Lambda$ and the lines $\ell_p \subset \mathbb{C}\mathbf{P}^3$. Further details about the Plücker map and the Plücker quadric can be found in [13, Section 8.6].

Let $\Lambda_{\hat{Z}} = \{p \in \Lambda \mid \ell_p \subset \hat{Z}\}$; that is, $\Lambda_{\hat{Z}}$ is the set of all points in Λ that correspond to lines that are fully contained in $\hat{Z}$. Then $\Lambda_{\hat{Z}}$ is an algebraic variety in $\mathbb{C}\mathbf{P}^5$ that is composed of a single one-dimensional irreducible component, possibly together with an additional pair of isolated points (which correspond to at most two non-generating lines that are contained in the singly ruled surface $\hat{Z}$; see, e.g., [20, Corollary 3.6]).

The set $\Gamma_\Lambda = \{p \in \Lambda \mid \ell_p \cap \Gamma \neq \emptyset\}$ is a variety of codimension 1 in Λ. Since the irreducible one-dimensional curve of $\Lambda_{\hat{Z}}$ is also a variety, either it is fully contained in Γ_Λ, or the intersection $\Lambda_{\hat{Z}} \cap \Gamma_\Lambda$ is a zero-dimensional variety, and therefore finite according to the higher-dimensional variant of Bézout's theorem (Theorem 2.4). If the former case occurs, then at most two lines in $\hat{Z}$ do not intersect Γ. However, since $\hat{Z}$ is the complex projectivization of a real ruled surface, $\hat{Z}$ contains infinitely many real lines (lines whose defining equations involve only real coefficients) that are not contained in the plane $\{x_0 = 0\}$, and if ℓ is such a line then $\ell \cap \{x_0 = 0\}$ is a real point. This is a contradiction since the curve Γ contains no real points. Therefore, the intersection $\Lambda_{\hat{Z}} \cap \Gamma_\Lambda$ is finite.

Every line intersects Γ in at most two points, which implies that $\Gamma \cap \hat{Z}$ is a finite set. Indeed, if this were not the case, then there would exist infinitely many points of Γ that lie in $\hat{Z}$ and each of them is therefore incident to a line contained in $\hat{Z}$. Since every line meets Γ in at most two points, Γ would have intersected infinitely many lines contained in $\hat{Z}$. This is a contradiction since, as argued above, $\Lambda_{\hat{Z}} \cap \Gamma_\Lambda$ is a finite intersection.

Adding the circles to the analysis. Let $\bar{\mathcal{C}}'$ be the collection of circles described above; that is, an infinite set of pairwise non-coplanar circles that are fully contained in $\bar{Z}$ and incident to $\bar{z}_2$. Let $\hat{\mathcal{C}}'$ be the corresponding collection of the complex projectivizations of these circles. As just argued, all of the intersection points between the circles of $\hat{\mathcal{C}}'$ and Γ must lie in the finite intersection $\Gamma \cap \hat{Z}$.

Each circle $\hat{C}$ in $\hat{\mathcal{C}}'$ intersects Γ in precisely two points. Since $\hat{C}'$ contains infinitely many circles and $\Gamma \cap \hat{Z}$ is finite, by the pigeonhole principle there must exist two circles C_1, C_2 in $\hat{\mathcal{C}}'$ such that the sets $C_1 \cap \Gamma$ and $C_2 \cap \Gamma$ are identical (each being a doubleton set). By construction, C_1 and C_2 are contained in two distinct planes Π_1 and Π_2. The line $\ell = \Pi_1 \cap \Pi_2$ contains $C_1 \cap C_2$. Thus, ℓ contains the two common intersection points of C_1, C_2 with Γ. Since these two points are contained in the plane $\{x_0 = 0\}$, ℓ is also contained in this plane. This is impossible, since ℓ also contains $\bar{z}_2$ (common to all circles of $\hat{\mathcal{C}}'$), which is not in the plane $\{x_0 = 0\}$. This contradiction completes the proof of Lemma 3.1. $\square$

5. APPLICATIONS

High-multiplicity points. The following is an easy but interesting consequence of Theorems 1.1 and 1.3.

COROLLARY 5.1. *(a) Let $\mathcal{C}$ be a set of n circles in $\mathbb{R}^3$, and let $q < n$ be an integer so that no sphere or plane contains more than q circles of $\mathcal{C}$. Then there exists a constant k_0 (independent of $\mathcal{C}$) such that for any $k \ge k_0$, the number of points incident to at least k circles of $\mathcal{C}$ is*

$$O^*\left(\frac{n^{3/2}}{k^{7/4}} + \frac{n^{3/2}q^{1/2}}{k^3} + \frac{n^{3/2}q^{3/10}}{k^{11/5}} + \frac{n}{k}\right). \qquad (12)$$

In particular, if $q = O(1)$, the number of such points is

$$O^*\left(\frac{n^{3/2}}{k^{7/4}} + \frac{n}{k}\right).$$

(b) If the circles of $\mathcal{C}$ are all congruent the bound improves to

$$O^*\left(\frac{n^{3/2}}{k^{11/6}} + \frac{n^{3/2}q^{1/2}}{k^3} + \frac{n}{k}\right). \qquad (13)$$

In particular, if $q = O(1)$, the number of such points is

$$O^*\left(\frac{n^{3/2}}{k^{11/6}} + \frac{n}{k}\right).$$

Proof. Let m be the number of points incident to at least k circles of $\mathcal{C}$, and observe that these points determine at least mk incidences with the circles of $\mathcal{C}$. Comparing this lower bound with the upper bound in Theorem 1.1 (for (a)), or in Theorem 1.3 (for (b)), the claims follow. □

Remarks. (1) It is interesting to compare the bounds in (12) and (13) with the various recent bounds on incidences between points and lines in three dimensions [14, 19, 20]. In all of them the threshold value $m = \Theta(n^{3/2})$ plays a significant role. Specifically: (i) The number of joints in a set of n lines in $\mathbb{R}^3$ is $O(n^{3/2})$, a bound tight in the worst case [19]. (ii) If no plane contains more than $\sqrt{n}$ lines, the number of points incident to at least $k \geq 3$ lines is $O(n^{3/2}/k^2)$ [20]. (iii) A related bound where $m = n^{3/2}$ is a threshold value, under different assumptions, is given in [14]. The bounds in (12) and (13) are somewhat weaker (because of the extra small factors hidden in the $O^*(\cdot)$ notation, the rather restrictive constraints on q, and the constraint $k \geq k_0$) but they belong to the same class of results. It would be interesting to understand how general this phenomenon is; for example, does it also show up in incidences with other classes of curves in $\mathbb{R}^3$? We tend to conjecture that this is the case, under reasonable assumptions concerning those curves.
(2) The bounds can be slightly tightened by using Theorem 1.2 (or a similar theorem for unit circles, established in the full version [37]) instead of Theorem 1.1 or Theorem 1.3, respectively, but we leave these slight improvements to the interested reader.

Similar triangles. Another application of Theorem 1.1 (or rather of Theorem 1.2) is an improved bound on the number of triangles spanned by a set $\mathcal{P}$ of t points in $\mathbb{R}^3$ and similar to a given triangle Δ. Let $F(\mathcal{P}, \Delta)$ be the number of triangles spanned by $\mathcal{P}$ that are similar to Δ, and let $F(t)$ be the maximum of $F(\mathcal{P}, \Delta)$ as $\mathcal{P}$ ranges over all sets of t points and Δ ranges over all triangles. We then have:

THEOREM 5.2.

$$F(t) = O(t^{15/7}) = O(t^{2.143}).$$

Proof. Let $\mathcal{P}$ be a set of t points in $\mathbb{R}^3$ and let $\Delta = uvw$ be a given triangle. Suppose that pqr is a similar copy of Δ, where $p, q, r \in \mathcal{P}$. If p corresponds to u and q to v, then r has to lie on a circle c_{pq} that is orthogonal to the segment pq, whose center lies at a fixed point on this segment, and whose radius is proportional to $|pq|$. Thus, the number of possible candidates for the point r, for p, q fixed, is exactly the number of incidences between $\mathcal{P}$ and c_{pq}. There are $2\binom{t}{2} = t(t-1)$ such circles, and no circle arises more than twice in this manner. It follows that $F(t)$ is bounded by twice the number of incidences between the t points of $\mathcal{P}$ and the $t(t-1)$ circles c_{pq}. We now apply Theorem 1.2 with $m = t$ and $n = t(t-1)$. (The theorem applies for these values, which satisfy $m \approx n^{1/2}$, much smaller than the threshold $n^{3/2}$; in fact, m lies in the second range $[n^{1/3}, n^{4/5}]$.) It remains to show that the expression (2) is $O(t^{15/7})$.

The first term of (2) is $O(t^{15/7})$. To control the remaining terms, it suffices to show that at most $O\left(\left(\frac{n^3}{m^2}\right)^{3/7}\right) = O(t^{12/7})$ of the circles lie on a common plane or sphere. In fact, we claim that at most $O(t)$ circles can lie on a common plane or sphere. Indeed, let Π be a plane. Then for any circle c_{pq} contained in Π, pq must be orthogonal to Π, pass through the center of c_{pq}, and each of p and q must lie at a fixed distance from Π (the distances are determined by the triangle Δ and by the radius of c_{pq}). This implies that each point of $\mathcal{P}$ can generate at most two circles on Π. The argument for cosphericality is essentially the same. The only difference is that one point of $\mathcal{P}$ may lie at the center of the given sphere σ, and then it can determine up to $2(t-1)$ distinct circles on σ. Still, the number of circles on σ is $O(t)$. As noted above, this completes the proof of the theorem. □

As already mentioned in the introduction, this slightly improves a previous bound in [6] (see also [1]).

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