Sharp Bounds on Geometric Permutations of Pairwise Disjoint Balls in $\mathbb{R}^d$

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1 Geometric Permutations of Pairwise Disjoint Balls in $\mathbb{R}^d$

1.1 Upper Bounds

Let S be a given set of n pairwise disjoint (closed) balls in $\mathbb{R}^d$. We prove that $g_d(S) = O(n^{d-1})$. The main step of the proof is to show that S admits a separation set of size $O(n)$. As a matter of fact, we prove the stronger result that there exists a set H of $O(n)$ hyperplanes such that each pair of balls in S is separated by a hyperplane in H, rather than a hyperplane parallel to one in H.

Let $S = \{B_1, \ldots, B_n\}$ be a set of n pairwise-disjoint balls in $\mathbb{R}^d$; ball B_i has radius r_i and center b_i. We assume, without loss of generality, that $r_1 > r_2 > \cdots > r_n$. (If several balls have the same radius, we slightly increase their radii, making them all distinct and keeping the balls disjoint. This can only increase $g_d(S)$.)

Let $\mathcal{S}_{d-1}$ be the unit sphere of directions. Let $\mathcal{C} = \{C_1, \ldots, C_K\}$ be a covering of $\mathcal{S}_{d-1}$ by a set of K spherical patches of diameter δ, where δ is chosen so that the angle θ between any pair of unit vectors $\hat{u}, \hat{v} \in C_k$ is at most $\sin^{-1}((\sqrt{3}-1)/2) \approx XXX$ (or about XXX degrees). Each set C_k determines a convex cone $C_k(p)$ with respect to any given apex point

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Figure 1: The construction of $h_{i,k}$

p; this is the union of all rays emanating from p and having orientations in C_k. Note that we can always cover $\mathcal{S}_{d-1}$ with a *constant* number (depending on dimension) of sets C_k; i.e., K is a constant, depending (exponentially) on d.

We construct a set H of $O(n)$ hyperplanes as follows. Consider a ball B_i and a set C_k of directions, which define a cone, $C_k(b_i)$, with apex at b_i. If $C_k(b_i)$ contains the center of at least one ball that is larger than B_i, then we let B_j $(j < i)$ be that ball with center $b_j \in C_k(b_i)$ closest to b_i, and we define $h_{i,k}$ to be the hyperplane supporting B_i, orthogonal to the vector $b_j - b_i$ and separating b_i and b_j; see Figure 7. Clearly, $h_{i,k}$ separates B_i from B_j. We let H be the set of all such hyperplanes $h_{i,k}$; since K is a constant depending on dimension, $|H| = O(n)$, for any fixed dimension d.

Theorem 1.1 *H is a separating set for S.*

Proof: We must show that for every choice of B_i, and $j < i$, there is a hyperplane in H that separates B_i from B_j.

Our proof is by induction on i. The base of the induction is the trivial claim that H contains hyperplanes separating B_1 from each ball that has larger radius (there are none). We now make the following induction hypothesis (on i): *H contains a hyperplane separating B_i from each B_j with $j < i$.*

Suppose the hypothesis holds for all $i' \leq i$, and consider ball $B = B_{i+1}$. Without loss of generality, we can assume that $r_{i+1} = 1$ and b_{i+1} is the origin, O. Consider an arbitrary $B' = B_j$, with $j < i+1$, radius $r' = r_j > 1$, and center $v = b_j$ lying in a cone $C = C_k(b_{i+1})$, for some $k \in \{1, \ldots, K\}$.

By the construction of H, since C contains the center of a larger ball, we know that there exists a hyperplane $h = h_{i+1,k} \in H$ separating B from some ball, B'', with radius $r'' > 1$ and center $u \in C$. (In fact, by construction, h is supporting B and is orthogonal to u.) Our goal is to show that H contains a hyperplane separating B from B'. If $B' = B''$, we are done. So, we assume that B' and B'' are distinct.

By the induction hypothesis, there exists a hyperplane $h' \in H$ that separates B' from B'' (since each has radius larger than that of B). If h already separates B' from B, then we are done. So we assume that it does not, which means that B' intersects h.

We let θ be the angle between u and v. We let ρ denote the ray containing u with endpoint at the origin. We let $p = h \cap B$ denote the point on ρ where h supports B, and we let p' denote the point on ρ, further from p, at distance $|v - p|$ from p. Finally, we let θ' denote the angle between vector $v - p$ and ρ. See Figure 8 for an illustration.

We will need the following technical lemma:

Lemma 1.2 $2 \sin \frac{\theta'}{2} \leq \cos \theta'$.

Proof: Referring to Figure 8, we need to show that $|vp'| \leq |pp''|$, where p'' is the foot of

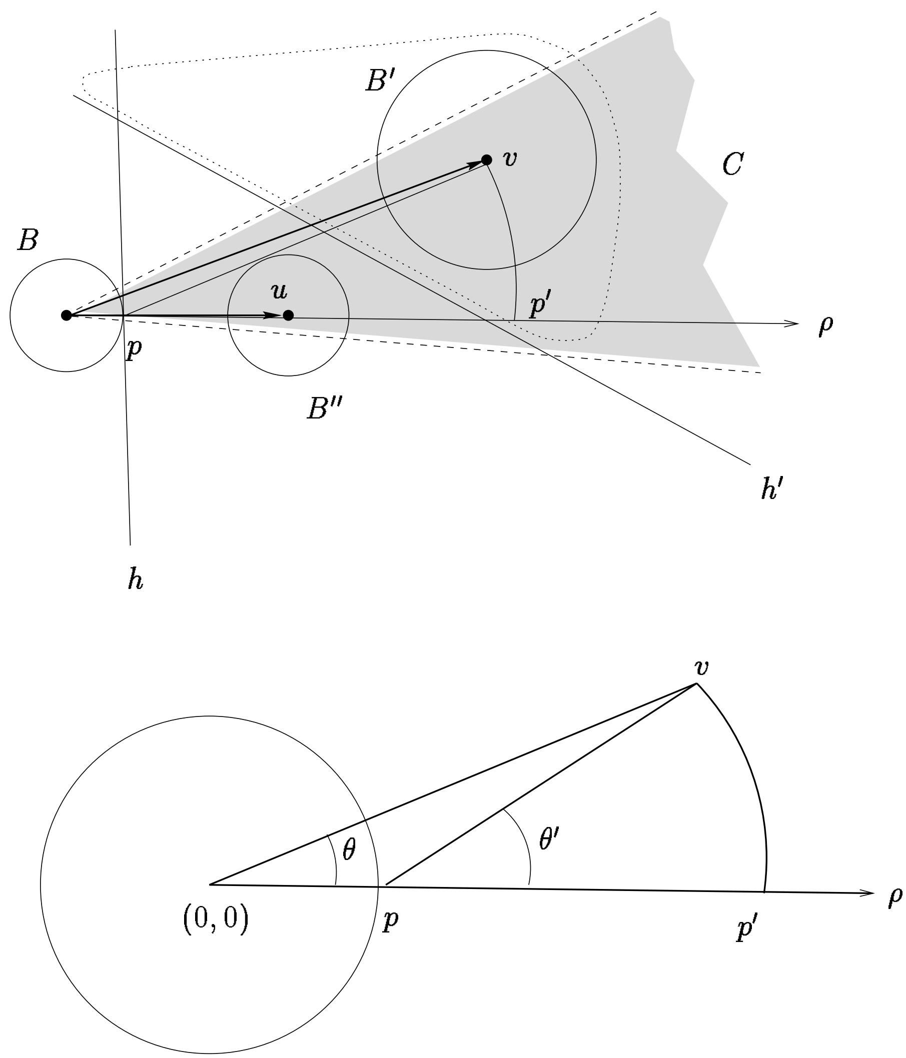

Figure 2: Illustration of the notation in the proof of Theorem 2.1. (The dotted loop surrounding B' is meant to convey the fact that B' is assumed to cross h, even though, for clarity, we have not drawn it large enough to do so.)

the perpendicular from v to ρ. It is easily seen that this can be rewritten as

$$\frac{|v|\sin\theta}{\cos\frac{\theta'}{2}} \leq |v|\cos\theta - 1.$$

Since θ is acute and $|v| > 2$, it follows that $\angle Ovp < \theta$ and hence $\theta' < 2\theta$. We thus have

$$\frac{|v|\sin\theta}{\cos\frac{\theta'}{2}} \leq |v|\tan\theta,$$

so it suffices to show that $|v|\tan\theta \leq |v|\cos\theta - 1$; since $|v| > 2$, it suffices to show that $\cos\theta - \tan\theta > 1/2$, or that $1 - \sin^2\theta - \sin\theta \geq \frac{1}{2}\cos\theta$. By construction, we have $\sin\theta \leq \frac{\sqrt{3}-1}{2}$, which implies that $1 - \sin^2\theta - \sin\theta \geq \frac{1}{2}$, thus completing the proof of the lemma. □

Note that Lemma 2.5 trivially implies that $\theta' \leq \pi/4$.

First, we claim that B' intersects ρ in an interval that lies after u (i.e., an interval of points that are farther from the origin than is the point u); thus, h' separates the origin (and B'') from B'. We argue as follows. Since $\theta' \leq \pi/4$, we know that point v is at least as close to ray ρ as it is to hyperplane h; thus, B' intersects ray ρ. By Lemma 2.5, v is in fact closer to point p' than to any point on h; thus, B' contains point p'. Now, by construction of H, $|u| \leq |v|$, which implies that $|u - p| = |u| - 1 \leq |v| - 1 \leq |v - p| = |p' - p|$. Thus, ray ρ intersects B' after B''. Since h' separates B' and B'', ray ρ must intersect B before B'' before h' before B'.

Second, we claim that h' does not intersect B; thus, h' separates B from B'. To see this claim, consider for each $q \in B$ the ray ρ_q that is parallel to ρ, with apex q. Since B'' is larger than B, each ray ρ_q must intersect B''. Now ray ρ intersects B before B'' before h', so, by continuity, each ray ρ_q must also intersect B before B'' before h'. This shows that h' cannot intersect B, since every point $q \in B$ is the apex of a ray that intersects h' only after passing through B'' (which is disjoint from h').

Since we have shown that h' separates B and B', this completes the induction step and thus concludes the proof of the theorem. □

As a result of Lemma **??** and Theorem 2.1 we have:

Theorem 1.3 *The number of geometric permutations of a set of n pairwise disjoint balls in $\mathbb{R}^d$ is $O(n^{d-1})$.*

Remark 1.4 For general pairwise disjoint convex sets in $\mathbb{R}^3$, the size of a separating set can be $\Theta(n^2)$. For example, in the standard construction of a Voronoi diagram in $\mathbb{R}^3$ with $\Theta(n^2)$ complexity, one needs $\Theta(n^2)$ different plane orientations to separate all pairs of cells. Hence the current proof of Theorem 2.2 does not extend to families of general convex sets.