Computational Game Theory **Spring Semester, 2009/10**

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1.1 Introduction

Several fields in computer science and economics are focused on the analysis of Game theory. Usually they observe Game Theory as a way to solve optimization problems in systems where the participants act independently and their decisions affect the whole system.
Following is a list of research fields that utilize Game Theory:

- Artificial Intelligence (AI) - There has been a shift from interest in Machine Learning, where one builds a hypothesis based on data, to Reinforcement Learning, which deals with a single agent (robot), to Multi Agent learning.
- Communication Networks - The Internet can be used to model environment with multiple players where each player wishes to maximize his own objectives. The goal has been to suggest mechanisms, where it will be both in the user's interest to follow them, and the outcome will be efficient.
- Computer Science Theory - There are several subfields that use Game Theory:
  - Algorithm Mechanism Design.
  - Complexity Issues.
  - Quality of Nash Equilibrium (compared to optimal solution).
- Industry - Sponsored search advertisment is based on multi-participant biddings, for which game theory is relevant.

1.2 Course Syllabus

1. Introduction
2. Quality of an Equilibrium
   (a) Worst case - Price of Anarchy
   (b) Best case - Price of Stability

(c) Test Cases:

i. Job Scheduling (Price of Anarchy)
ii. Routing (Price of Anarchy)
iii. Network Creation (Price of Anarchy and Price of Stability)
iv. Resource Allocation (Price of Stability)

3. Equilibrium Existence

(a) Two players zero sum game
(b) Congestion and potential games.
(c) Nash Equilibrium (existence)
(d) Correlated Equilibrium

4. Repeated Games

(a) Internal and External Regret
(b) Dynamics of reaching equilibrium

5. Mechanism Design

(a) Social choice
(b) Maximizing social welfare
(c) Auctions (combinatorial)
(d) Auctions (digital goods)

1.3 What is Game Theory

Game Theory analyzes the way rational players behave in a competitive environment. A rational player is a player with priorities (or utility) that tries to maximize the utility (or minimize cost) while considering that other players are also rational. A competitive environment is an environment with multiple rational players.

1.4 Basic Game Theory Examples

1.4.1 The Prisoner's Dilemma

There are two prisoners that committed a crime. If they both do not confess, they get a low punishment. If they both confess, they get a more severe punishment. If one confesses and the other does not, then the one that confesses gets a low punishment and the other gets a very severe punishment. The game can be formalized in the following matrix, each entry includes a pair, the first is the cost to the first player and the second is the cost to the second player.

Table 1.1: Prisoner Dilemma Cost Matrix

	Confess	Silent
Confess	$(4, 4)$	$(1, 5)$
Silent	$(5, 1)$	$(2, 2)$

Game theory predicts the case where both prisoners confess (4,4): player i doesn't know what the other player chooses. Should the other player confess, then player i can either confess (4 years imprisonment) or not confess (5 years). Should the other player choose to remain silent, then player i can confess (1 year) or keep silent (2 years). Thus, in both cases it is better to *confess*. This is an example of a strong dominant strategy (to be formally defined later).

1.4.2 Internet service provider - ISP

ISP's often agree to mutually route each other's traffic for free. Consider the following example where each ISP is individually motivated to choose the shortest path in its AS (autonomous system).

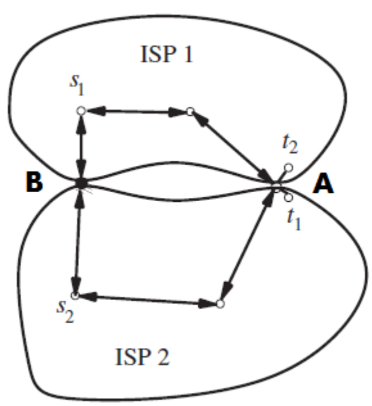

Figure 1.1: ISP example

From the figure we can easily see that each ISP has an internal shorter path if it uses the longest route via the other ISP. This is formalized in the following table.

Table 1.2: Route Distances for both ISPs

	A	B
A	$(4,4)$	$(1,5)$
B	$(5,1)$	$(2,2)$

The outcomes are identical to the priosner's dilema game. Because each player strives to minimize cost arising from internal traffic, the strategy chosen is to route traffic to the nearest gateway (and from there to the other ISP). Because all players are rational, this strategy is ubiquitous and thus each player ends up paying more than if they had cooperated. The current example, demonstrates again how each player chooses a strategy that maximizes his individual benefit (B,B). This is paradoxical because by choosing (A,A) each player would get a larger utility (lower cost).

1.5 Strategic Games

A strategic game is a model for decision making where there are N players, each choosing an action. A player's action is chosen once and cannot be changed afterwards.

Each player i can choose an action a_i from a set of actions A_i. let A be the set of all possible action vectors $\times_{j \in N} A_j$. Each player has either a utility function $u_i : A \rightarrow \mathbb{R}$ which is to be maximized or alternatively, a cost function $c_i : A \rightarrow \mathbb{R}$ which should be minimized.

Definition A **Strategic Game** is a triplet $\langle N, (A_i)_{i=1}^N, (u_i)_{i=1}^N \rangle$ where $N = \{1...n\}$ is the set of n players, A_i is the finite set of actions for player i, and u_i is the utility function of player i.

Several player behaviors are assumed in a strategic game:

- The game is played only once.
- Each player "knows" the game (each player knows all the actions and the possible outcomes of the game).
- The players are rational. A rational player is a player that plays selfishly, wanting to maximize her own benefit of the game (the utility function).
- All the players choose their actions simultaneously (but do not know the other players current choices).

We now define a dominating action. An action is dominating if it is better than any other action, regardless of the action of the other players.

Definition Action a_i is a **Weak Dominant Strategy** for player i if

$$\forall b_{-i} \in A_{-i}.\forall b_i \in A_i : u_i(b_{-i}, b_i) \leq u_i(b_{-i}, a_i)$$

Action a_i is a **Strong Dominant Strategy** for player i if

$$\forall b_{-i} \in A_{-i}.\forall b_i \in A_i : u_i(b_{-i}, b_i) < u_i(b_{-i}, a_i)$$

Where $(a_{-i}) = (a_1, a_2, ..., a_{i-1}, a_{i+1}, ..., a_n)$.

In the Prisoner's Dilemma game the action 'Confess' is a strong dominant strategy for both players.

1.6 Pareto Optimality

An outcome $\vec{a} \in A$ of a game $\langle N, (A_i)_{i=1}^N, (u_i)_{i=1}^N \rangle$ is Pareto Optimal if there is no other outcome $\vec{b} \in A$ that makes each player at least as well off and at least one player strictly better off. That is, a Pareto Optimal outcome cannot be improved upon without hurting at least one player.

Definition An outcome $\vec{a}$ is **Pareto Optimal** if

$$\forall b \in A \exists i \in N \; u_i(\vec{b}) < u_i(\vec{a}).$$

1.7 More Game Theory Examples

1.7.1 Picking up after your dog

Let us consider the following game: A group of N players live in a neighborhood where each player has a dog. Each neighbor can choose whether to pick up after his dog or not. Cleaning after your dog has a cost ($c = 3$) and thus is not personally desirable, where as simply ignoring the sights and smells only costs 1. Walking in a polluted street is also costly in proportion to the number of droppings. Thus the total cost for each player is the number of players that did not clean after their dogs and the outcome of his personal actions ($c = 1$ or 3).

Analyzing a player's actions: If k players choose "Leave" and $n - k - 1$ choose "Pick" our player can either choose to "Leave" and pay a total cost of $k + 1$ or "Pick" and pay $k + 3$.

Thus the dominant action is "Leave". This is paradoxical because if all players chose "Pick" then the individual costs would be only 3.

If the cost function were to be changed such that "Leave" would be additionally penalized (fines etc.), the players would be motivated to "pick" instead.

1.7.2 Tragedy of the commons

Consider a shared resource (e.g. network bandwidth or centralized servers) and N players. Each player can choose a value $x_i \in [0, 1]$ (ratio of resource utilized). The outcome is as follows:
if $\sum x_i > 1$ then $\forall j : u_j = 0$.
if $\sum x_i < 1$ then $\forall j : u_j = x_j \cdot (1 - \sum x_i)$
Analyzing player i's actions: Let us assume that $\sum_{j \neq i} x_j = t$. Player i will try to maximize his utility by maximizing the following function: $f = x_i \cdot (1 - \sum x_j) = x_i \cdot (1 - t - x_i)$
Let us derive: $\frac{df}{dx_i} = 1 - t - 2x_i \implies x_i = \frac{1-t}{2}$. This action is the *best response* (BR) for player i given x_{-i}.
Definition $BR_i(x_{-i}) = argmax_{x_i}(u_i(x_i, x_{-i}))$

In order to maximize f, we need that $\frac{df}{dx_i} = 0$ and $\frac{d^2 f}{dx_i^2} < 0$. Setting $\frac{df}{dx_i} = 0$ we have $2x_i = 1 - \sum_{j \neq i} x_j$, which implies $x_i = 1 - \sum x_j$
Now, summing over all players: $(\sum x_i) = N - N \cdot (\sum x_j)$
Denote $S = \sum x_i$ then $S \cdot (N + 1) = N$, hence, $S = \frac{N}{N+1}$. The action of player i is $x_i = 1 - S = \frac{1}{N+1}$. This is an equilibrium. That means that no player can gain from changing his own action.
The utility of each player is: $u_i = \frac{1}{N+1} \cdot (1 - \frac{N}{N+1}) = \frac{1}{(N+1)^2}$. Although this is an equilibrium, it is not pareto-optimal as can be shown by the fact that if all players would have chosen $x_i = \frac{1}{2N}$ instead, then the individual utilities would be much higher; namely,
$u_i = \frac{1}{2N} \cdot \sum x_i = \frac{1}{4N}$.

1.8 Equilibrium

1.8.1 Definition of Equilibrium

An equilibrium is a joint action $x \in A$ such that:
$\forall i \in N, \forall y_i \in A_i : u_i(x_i, x_{-i}) \geq u_i(y_i, x_{-i})$ or alternatively,
$\forall i \in N : x_i \in BR(x_{-i})$.
Namely, no player can unilateraly improve his payoff.

1.8.2 Example: Battle of the Sexes

In this game, two players (of different gender) need to coordinate on an event (sports or opera). They both prefer to go to the same event together(gaining a value of 2 each if they go to the same event, or 0 if not), but they have a different preference between the events (value 2 for preferred event and 1 for the other).

Table 1.3: Different Outcomes for the Battle of the Sexes

	Sports	Opera
Sports	$(4, 3)$	$(2, 2)$
Opera	$(1, 1)$	$(3, 4)$

There are two Equilibrium points: (Sports, Sports) and (Opera, Opera). Both are also Pareto Optimal.

1.8.3 Example: Routing Game

In this game two players need to send information in the shortest path from a point of origin. They can choose to either use a short or long path (cost 1 or 2, respectively). Unlike the battle of the sexes, they prefer to select different routes (actions). Should both of them choose the same path it would congest, thus resulting in a cost of 2.

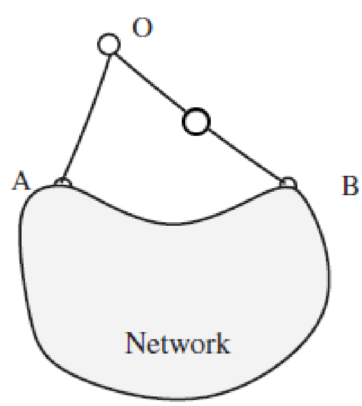

Figure 1.2: Routing Example

There are two Equilibrium points: (A, B) and (B, A).

Table 1.4: Routing Costs

	A	B
A	$(3,3)$	$(1,2)$
B	$(2,1)$	$(4,4)$

1.9 Mixed Strategy

1.9.1 Matching Pennies

In this game each player select Head or Tails. The row player wins if they match, and the column player wins if they mismatch (Matching Pennies).

	Head	Tail
Head	$(1,-1)$	$(-1,1)$
Tail	$(-1,1)$	$(1,-1)$

In this game there is no Deterministic Equilibrium point. Also, this is a zero sum game (the sum of the profits of the player for each possible outcome is 0).
If we allow each player to randomly choose an outcome with a pre-defined probability the game will be an example of a mixed strategy game: Let us assume that player 1 chooses "Head" with probability p and "Tail" with probability $1-p$ and that player 2 chooses "Head" with probability q and "Tail" with probability $1-q$. In this case each player wants to maximize its expected utility: $maxE[u_i,(x)]$.
Let us analyze the best response of player 1: If player 1 plays "Head" the outcome will be -1 with probability q and 1 with probability $1-q$ thus the expected utility is $2q-1$. If player 1 plays "Tail" the outcome will be -1 with probability $1-q$ and 1 with probability q thus the expected utility is $1-2q$. It follows that player 1 should:
Play "Head" if $2q-1>1-2q \Longrightarrow q>\frac{1}{2}$
Play "Tail" if $2q-1<1-2q \Longrightarrow q<\frac{1}{2}$
Thus, for $q=\frac{1}{2}$ player 1 is indifferent to the different outcomes. The same holds for player 2. We can see that there is an equilibrium if both players play with strategy $(\frac{1}{2}:\frac{1}{2})$.

1.9.2 Definit ion of Mixed Nash Equilibrium

Each player i has a distribution $p_i \in \Delta(A_i)$, where Δ is the set of possible distributions defined over A. A vector of such distributions $\vec{P}=\langle p_1,....p_N\rangle$ is a *Mixed Nash Equilibrium* if for every player $i \in N$, $\forall q_i \in \Delta$, $q_i \neq p_i$: $E_{x_i \sim p_i, x_j \sim p_j}[u_i(x_1..x_n)] \geq E_{x_i \sim q_i, x_j \sim p_j}[u_i(x_1..x_n)]$

holds.
An alternative definition: $\forall i \in N, \forall a_i \in A_i$: $E_{x_j \sim p_j}[u_i(x_1..x_n)] \geq E_{x_{-i} \sim p_{-i}}[u_i(a_i, x_{-i})]$. This definition is equivalent, because if the inequality holds for any single action it also holds for any distribution.

Definition $support(P_i) = \{a | P_i(a) > 0\}$. Thus if an action is contained within a certain distribution with positive probability it belongs to the support of the distribution.

Definition $BR_i(p_{-i}) = maxarg_{a_i}\{u(a_i, p_{-i})\}$ Thus given a distribution of the other players, the *best response* for player i are the actions that maximize its expected utility.

A distribution p is a *mixed nash equilibrium* if: $\forall i : support(p_i) \subseteq BR_i(p_{-i})$. That is, given that p_{-i} is the distribution played by the other players, whatever action player i chooses according to p_i, will lead to a *best result*, and this holds for all players.

1.9.3 Example: Rock Paper Scissors

Based on the familiar children's game. This is a zero sum game where there is no pure equilibrium only a mixed nash equilibrium where all players have uniform distributions $(\frac{1}{3} : \frac{1}{3} : \frac{1}{3})$.

Table 1.5: Rock Paper Scissors utility table

	R	P	S
R	$(0,0)$	$(-1,1)$	$(1,-1)$
P	$(1,-1)$	$(0,0)$	$(-1,1)$
S	$(-1,1)$	$(1,-1)$	$(0,0)$

1.9.4 Example: Location Game

- Assume a set of N ice cream vendors at a beach. The beach is shaped as a line spanning $[0,1]$.

- Each vendor must choose a location along this line. The location chosen by the player and the other players affects the sales volume (i.e. utility). Assuming players are spatially sorted in ascending order, the utility function is defined as: $u_i(x_i) = \frac{x_{i+1} - x_{i-1}}{2}$. Where $x_0 = 0$ and $x_{n+1} = 1$.

When $N = 2$, there is an equilibrium when one of the players is in the middle, and the other player is as close as possible to him (from one of his sides).

When $N = 3$ there is no pure equilibrium. No player wants to be in the middle, since the other players will be as close as possible to the middle player, either from the left or the right.

For a circle there is always an equilibrium, where the players are placed at equal distances.

1.9.5 Example: Cournot Competition

Given N competing companies producing and selling a similar product. Every company i selects a certain production volume $x_i \geq 0$. The market price is determined by total production volume: $S = \sum x_i$. The utility for company i includes the production cost and the price paid for the produced products: $u_i(x) = price(S) \cdot x_i - cost(x_i)$. Where price and cost vary depending on the model defined.

- Case I: Linear price, no production cost:
  $price(S) = 1 - S$
  $cost(x_i) = 0$
  The utility $u_i(x) = price(S) \cdot x_i - cost(x_i) = x_i \cdot (1 - \sum x_j)$
  In equilibrium $x_i = \frac{1}{N+1}$, $u_i = \frac{1}{(N+1)^2}$ (As in the computation in the Tragedy of the Commons) and a pure equilibrium is reached.

- Case II: Harmonic price, no production cost:
  $price(S) = \frac{1}{S}$
  $cost(x_i) = 0$
  Let us denote $t = \sum_{j \neq i} x_j$
  The utility $u_i(x) = price(S) \cdot x_i - cost(x_i) = x_i \cdot \frac{1}{\sum x_j} = \frac{x_i}{t + x_i}$
  It follows that each player has the incentive to create an unbounded amount of products and there is no equilibrium (pure or mixed) in this game.

1.9.6 Example: Auction

There are N players, each one wants to buy an object which is for sale.

- Player i's valuation of the object is v_i, and, without loss of generality, $v_1 \geq v_2 \geq ... \geq v_n \geq 0$.

- In a *first price auction* the players simultaneously submit bids - $b_i \in [0, \infty)$. The player who submit the highest bid - b_i wins and the payment of the winner is the price of his bid. That is, the payoff of player i is $u_i = \begin{cases} v_i - b_i, \text{ if player } i \text{ won} \\ 0, \; otherwise \end{cases}$.

  One equilibrium point is $b_1 = v_2 + \epsilon, b_2 = v_2, ..., b_n = v_n$. In fact one can see that $b_3, \ldots, b_n$ have no influence. Unfortunately, the first player needs to "know" the second

highest bid, which in practice is not available to him.
Another equilibrium is $b_1 = v_1, b2 = v_1 - \epsilon, b_3 = v_3, \ldots$. in fact, the winner will always be player 1 but the price in any equilibrium can be any value in $[v_1, v_2]$

- In a *second price auction* the payment of the winner is the highest bid among those submitted by the players who do not win. Player i's payoff when he bids v_i is at least as high as his payoff when he submits any other bid, regardless of the other players' actions.
  Therefore, for each player i, the bid $b_i = v_i$ is a weakly dominant strategy. This suggests that it is in best interest of all players to bid their own value. Also, when all players bid their own value, it is an equilibrium. This strategy causes the player to bid truthfully.

  For example, consider the two players case. We show that $\forall b_1, b_2 : u_1(v_1, b_2) \geq u_1(b_1, b_2)$.
  If $v_1 > b_2$, then $u_1(v_1, b_2) = v_1 - b_2 \geq u_1(b_1, b_2) = \begin{cases} v_1 - b_2, \ b_1 > b_2 \\ 0, \ b_1 < b_2 \end{cases}$
  If $v_1 < b_2$, then $u_1(v_1, b_2) = 0 \geq u_1(b_1, b_2) = \begin{cases} v_1 - b_2(< 0), \ b_1 > b_2 \\ 0, \ b_1 < b_2 \end{cases}$