Exercise 3 - Computational Models - Spring 2009

Version 1.1: Mostly minor changes, except (1b) and (5b)
Submission date and time: 7/5/2009, 18:00

1. A Turing machine (TM) computes a function $f : \Sigma^* \to \Sigma^*$ if it halts with $f(x)$ on its tape, when given x as input. Write a 1-tape TM that computes the following functions. Give a full formal description, including the transition function.
   (a) $f(w) = 01w$ (concatenating $w \in \{0,1\}^*$ to the string $'01'$)
   (b) $f(w) = w^R$ (the reverse of $w \in \{0,1\}^*$)
2. Write a a 1-tape TM that decides (i.e., halts for any input with the correct answer) the following languages. Describe the way the TM works. There is no need to specify the transition function.
   (a) $L_1 = \{w \in \{0,1\}^* : w$ does not contain the substring '0011' $\}$
   (b) $L_2 = \{w \in \{0,1\}^* : w$ does not contain twice as many '0's as '1's $\}$
3. Prove that the class $\mathcal{R}$ (the class of languages decidable by a TM) is closed under the following operations:
   (a) Union
   (b) Intersection
   (c) Concatenation
   (d) Kleene star
4. For the following variants of TMs, determine whether the class of languages they decide is a proper subset of $\mathcal{R}$, a proper superset of $\mathcal{R}$ or is equal to $\mathcal{R}$. Prove your claims.

(a) TMs that can move up to 2009 cells to the right and to the left at each step of the computation. i.e., $\delta(q_{19}, a) = (q_5, b, 117R)$ means that if the machine is in state q_{19} and the current cell contains the character a, then we move to state q_5, write b on the tape, and move 117 steps to the right.

(b) TMs with a tape that is infinite both to the right and to the left.

(c) TMs where $|Q| < 2009$ and $|\Gamma| < 2009$ (Q is the set of states and Γ the tape alphabet).

5. (a) Prove that a PDA with two stacks can simulate any TM. A PDA with two stacks is identical to a PDA with one stack, only it reads, pops and pushes from both of the stacks at each step of the computation. We make the following assumptions about computation with a 2-stack PDA:

   i. The bottom of the stacks is marked with the symbol $.

   ii. The end of the input is marked with the symbol #.

   iii. $\delta : Q \times \Sigma_\epsilon \times \Gamma_\epsilon \times \Gamma_\epsilon \to Q \times \Gamma^* \times \Gamma^*$. That is, the 2-stack PDA can push more than one symbol at a time onto each of the stacks.

   Hint: Use one stack to hold what is to the left of the head of the tape, and the other stack to hold what is to the right of the head of the tape.

   (b) Would your answer change if the transition function has the form $\delta : Q \times \Sigma_\epsilon \times \Gamma_\epsilon \times \Gamma_\epsilon \to Q \times \Gamma \times \Gamma$? That is, the 2-stack PDA can push just than one symbol at a time onto each of the stacks. Explain your answer – it suffices to give a verbal sketch.

6. We have seen in the recitation that a multi-tape TM can simulate the RAM model. Let BRAM be the RAM model with bounded memory cells, that is, each integer held in a memory cell is in the range $\{-C, ..., C\}$ for some constant C that is fixed per machine. The number of memory cells is still infinite. Show that the language of binary palindromes is not decidable by any BRAM machine.