Computational Models
Instructor: Benny Chor
T.A.: Gadi Kimmel
Winter Semester, 2003/4

Assignment 4

Handed: 31/12/03, Due: 13/1/04

1. For each of the following languages, determine whether they belong to R, and whether they belong to RE.
   (a) L_1 = { B | B is a Turing machine and $\exists x.B(x) \neq \perp$}
   (b) L_2 = { x | There is a Turing machine B, s.t. $B(x) \neq \perp$}
   (c) L_3 = { B_1, B_2 | B_1, B_2 are Turing machine and $\exists x.B_1(x) \neq \perp \wedge B_2(x) \neq \perp$}
   (d) L_4 = { B_1, B_2 | B_1, B_2 are Turing machine and $\forall x.B_1(x) = B_2(x)$}
   (e) L_5 = { B | B is a Turing machine and B accepts more than one input}
   (f) L_6 = { B | B is a Turing machine and B accepts at most one input}
   (g) L_7 = { B | B is a Turing machine and $B(n) = T \Leftrightarrow n$ is even}

2. Prove that every infinite and recursively-enumerable language L, contains an infinite and recursive language $L' \subseteq L$.

3. A state q of a TM M is called *important*, if there is at least one input x, such that M is in state q at least twice while running on x. Otherwise (that is for every x, M is in q at most once while running on x) q is called *not important*. Prove or disprove: There is a constant C such that every TM, M, is equivalent to a TM, M', with at most C important states. Note that C should be fixed, that is, independent of the TM M.

4. For each of the following problems, argue if is it decidable or not.
   (a) Input: TM M, and input x for M.
   Output: When M runs on x, is it in the same state two consecutive steps?
   (b) Input: TM M.
   Output: When M runs on the empty string, does its head ever move to the left?

5. Given a TM, M, define F_M to be the number of steps that M performs when running on the empty string. If M does not halt on the empty input, define F_M to be 0.

   Let $S : \mathbb{N} \to \mathbb{N}$ be a function such that: $S(0) = 0$, and for every $k \geq 1$, $S(k)$ is the maximum of F_M over all k states, one tape deterministic TMs, whose tape alphabet is $\{0, 1\}$ plus the blank symbol.

   (a) Prove that $S(k)$ is monotonically increasing (that is, for all $k \geq 0$, we have $S(k+1) > S(k)$).
   (b) Prove that $S(k)$ is not computable (that is, there does not exist a TM, M such that for all $k \geq 0$, $f_M(k) = S(k)$).

6. (a) Show the existence of a TM, M_0, for which the function $K_{M_0}(x)$ is computable. Recall that

$$K_{M_0} = \min_k \{k : \exists y \, |y| = k \wedge f_{M_0}(y) = x\}$$

(the Kolmogorov complexity of the string x with respect to M_0).

   (b) Prove or disprove: the two argument function $f(< M >, x) = K_M(x)$ is computable.

7. A language L_0 is called *RE-complete* if $L_0 \in RE$, and for each $L \in RE$, $L \leq_m L_0$. The notion of *R-completeness* is defined in similar manner.

   (a) Is there a RE-complete language, which belongs to R?

   (b) Is there a language L_0, which is not in RE, such that for every $L \in RE$, $L \leq_m L_0$?

   (c) Prove that if $L_0 \in R$, $L_0 \neq \Sigma^*$, and $L_0 \neq \emptyset$, then L_0 is R-complete.

   (d) Give an example for a language L_0 which belongs to R, and is *not* R-complete. Prove your claim.