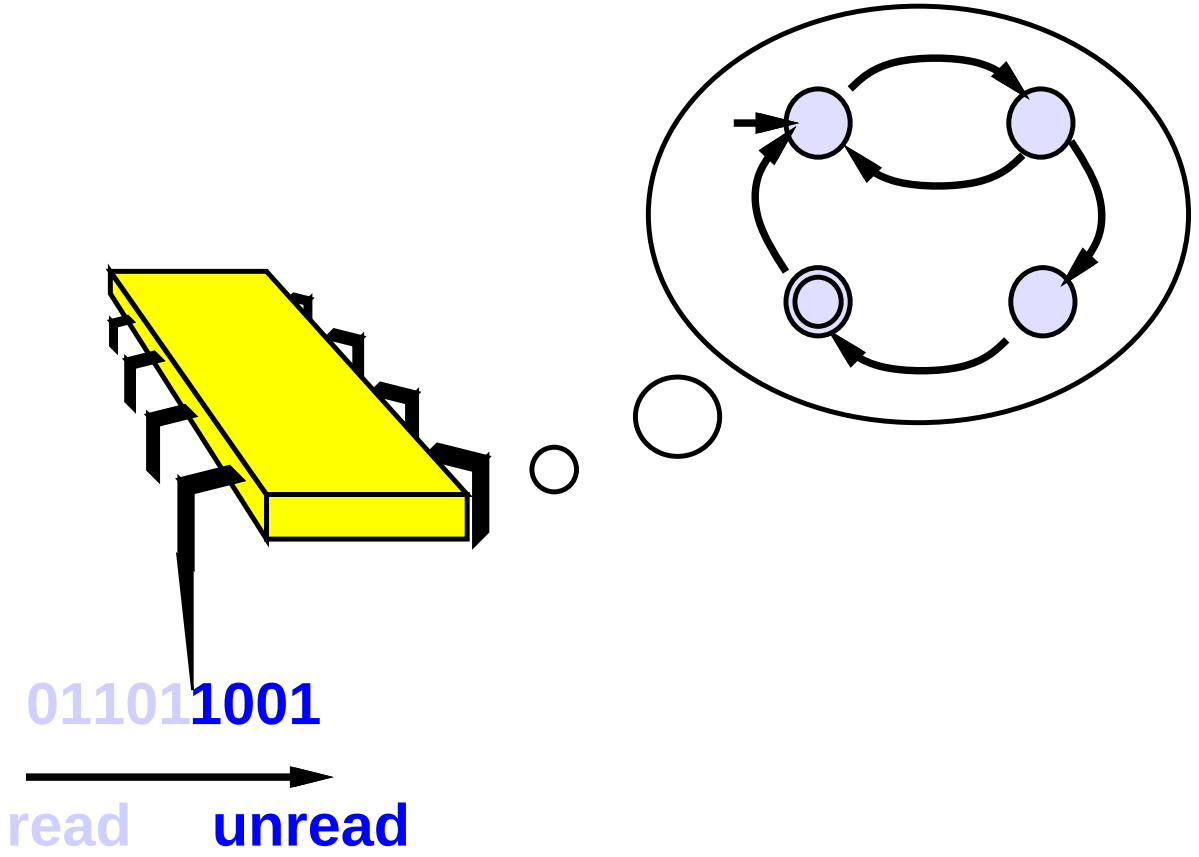
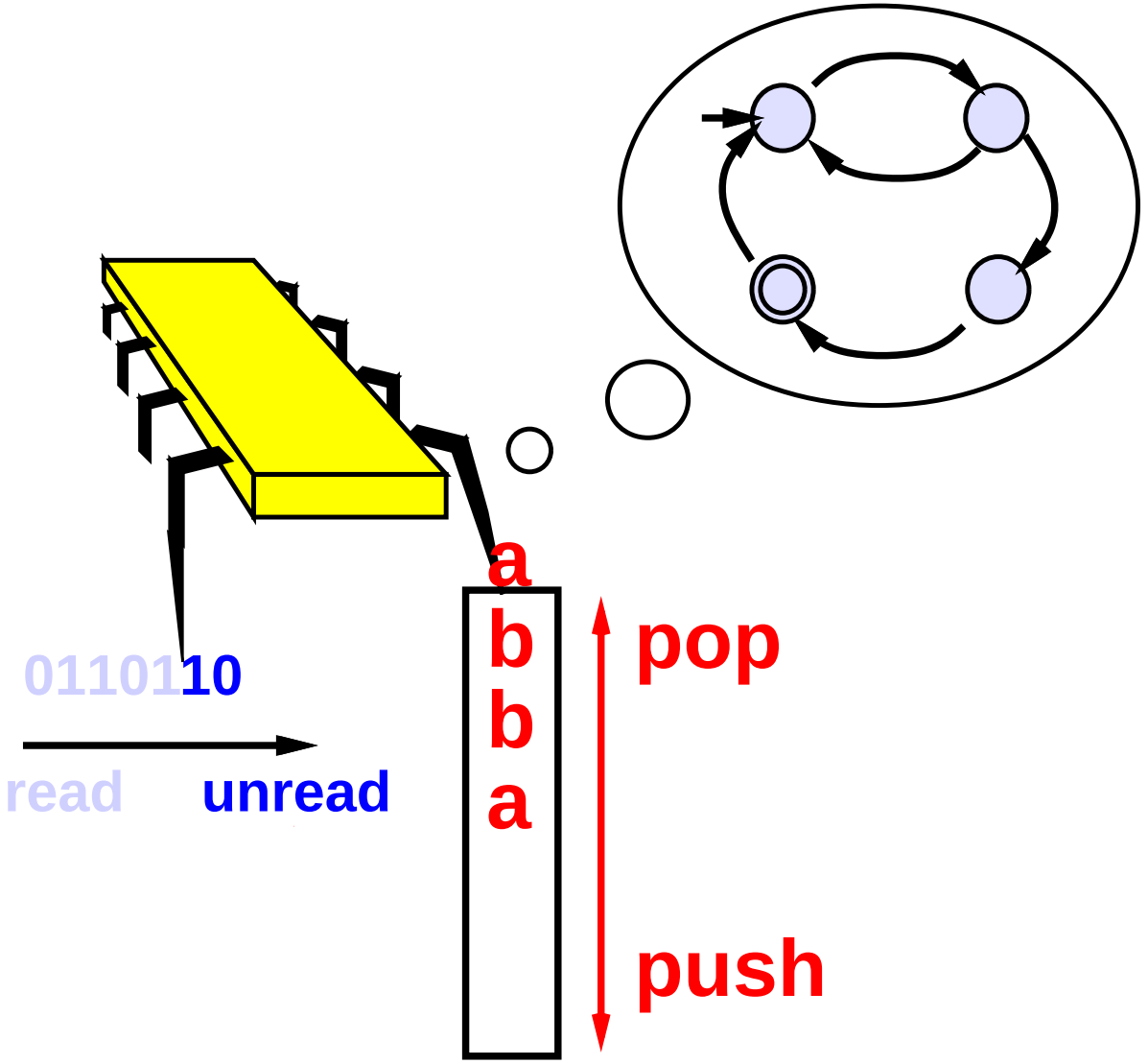
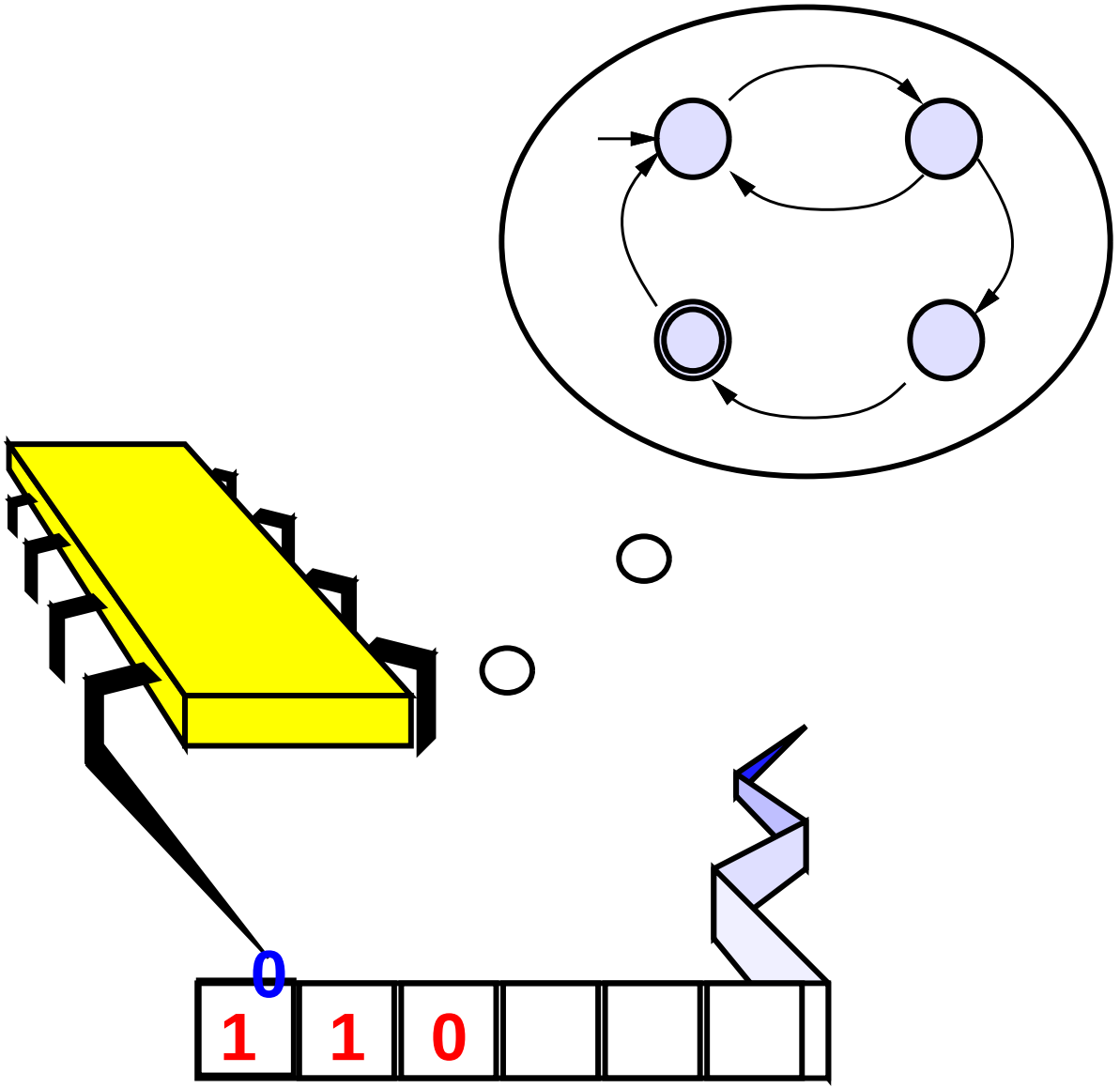
A Finite Automaton



A Pushdown Automaton



A Turing Machine



Alan Turing (1912–1954)

http://www.turing.org.uk/turing/index.html

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- machine can also run forever, never halting.

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- special accepting/rejecting states take immediate effect

Example

A machine that tests for membership in the language

$$A = \{w\#w \mid w \in \{0,1\}^*\}$$

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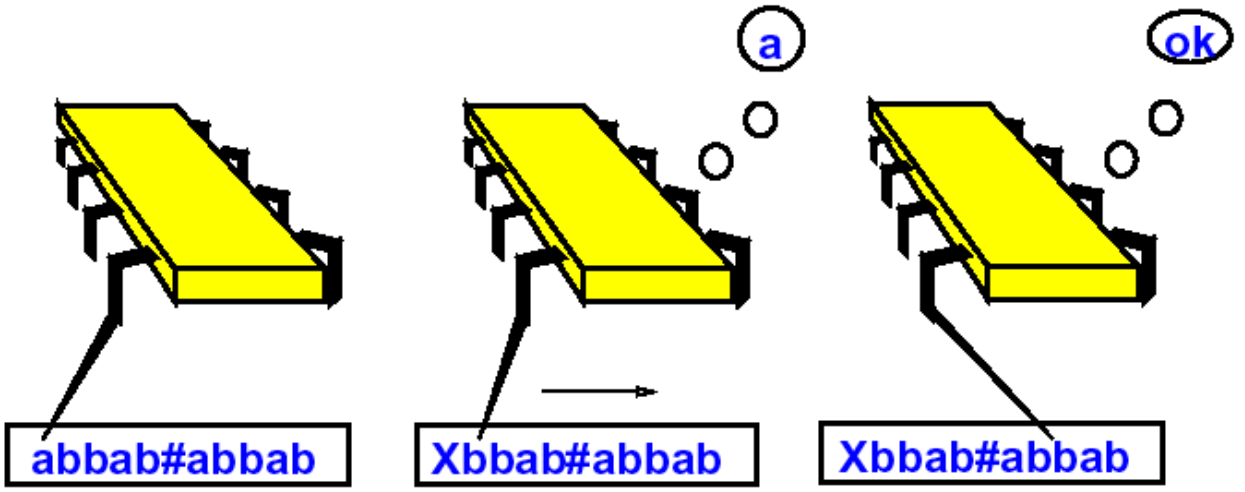
Zig-zags across tape, crossing off matching symbols.

Example (2)



- tape head starts over leftmost symbol

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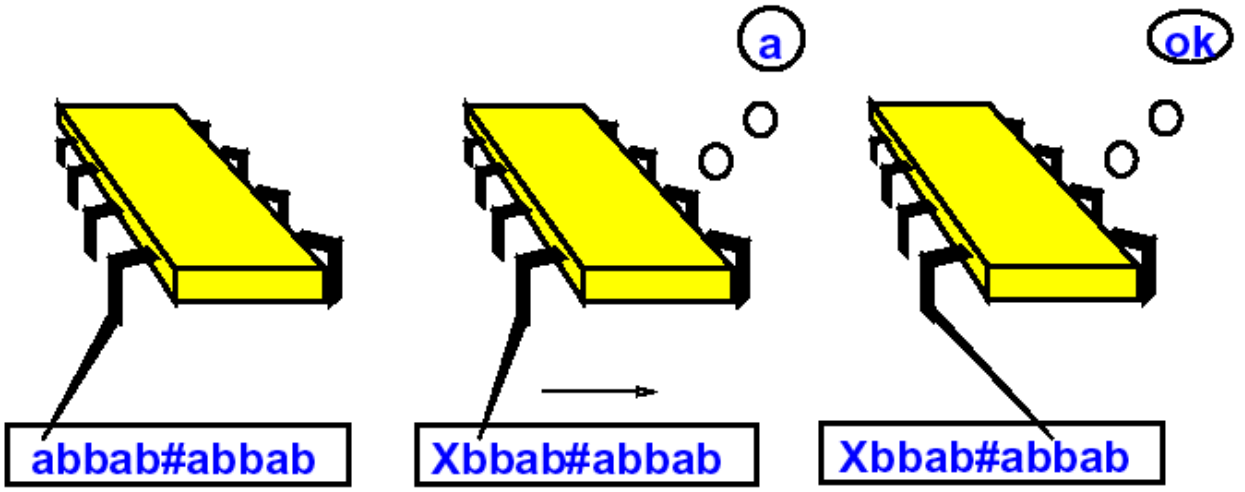

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- record symbol in control and overwrite X on tape

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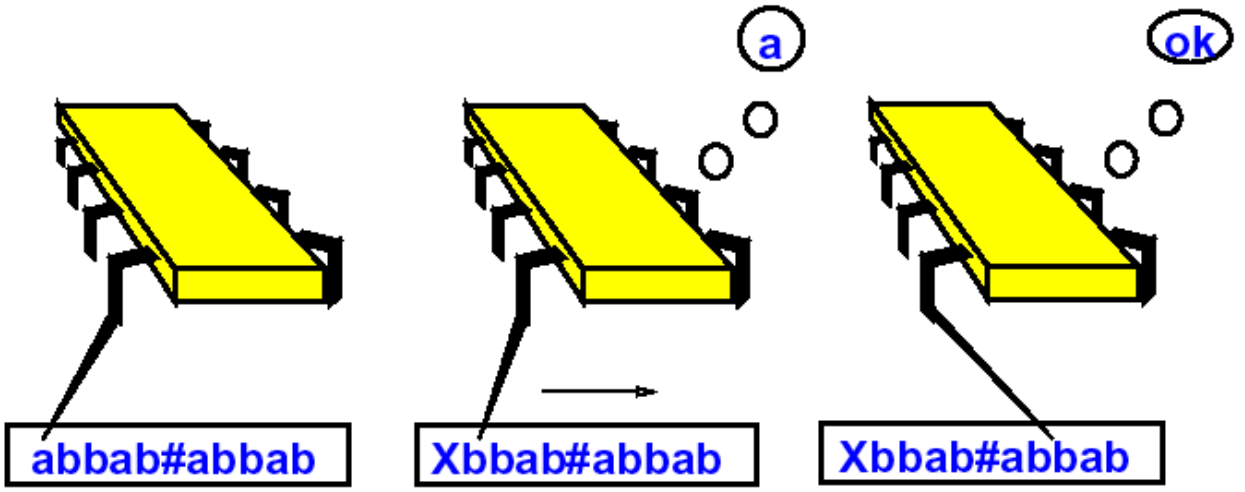

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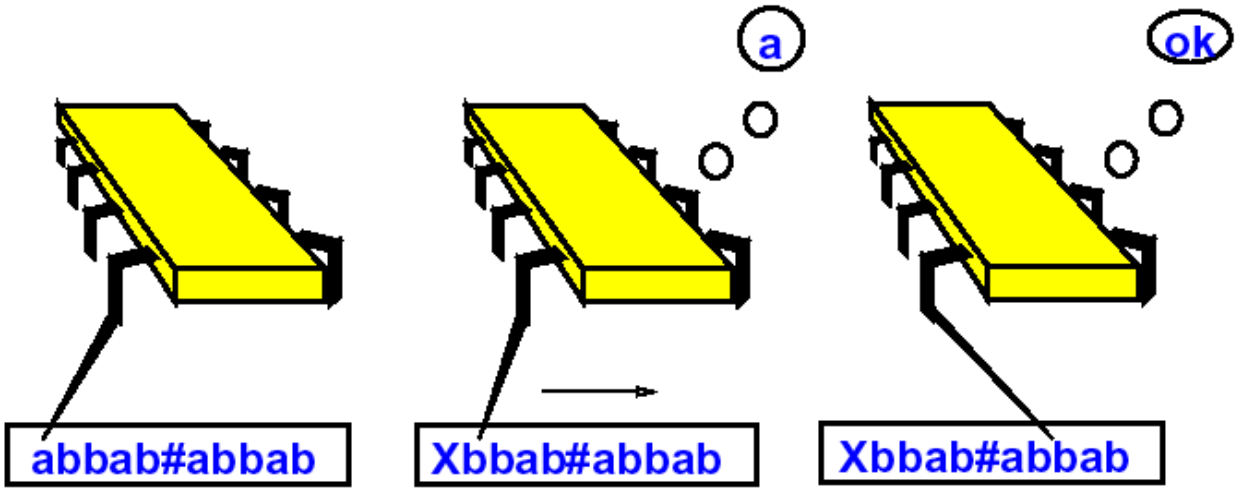

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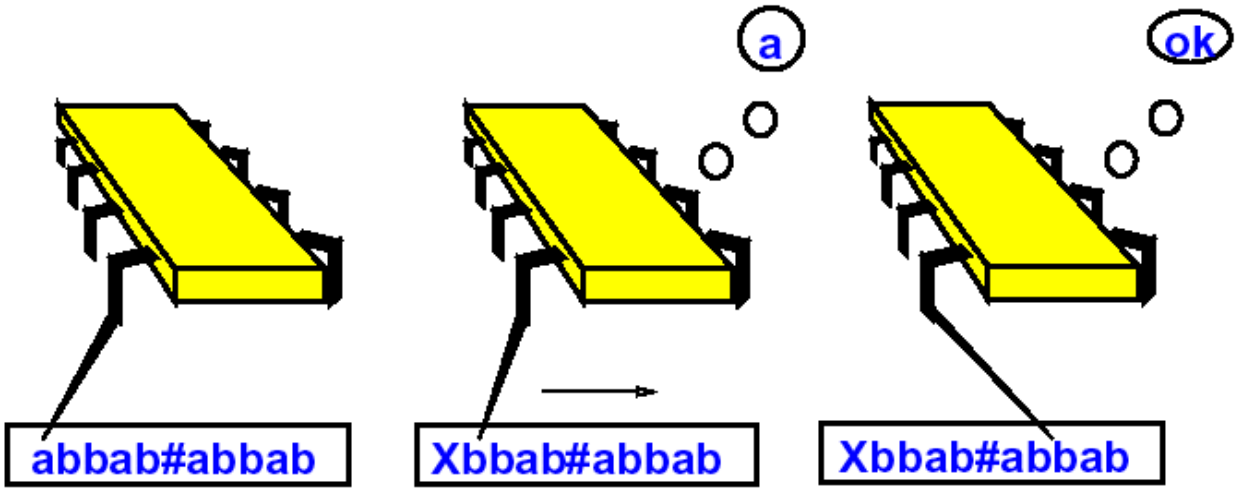

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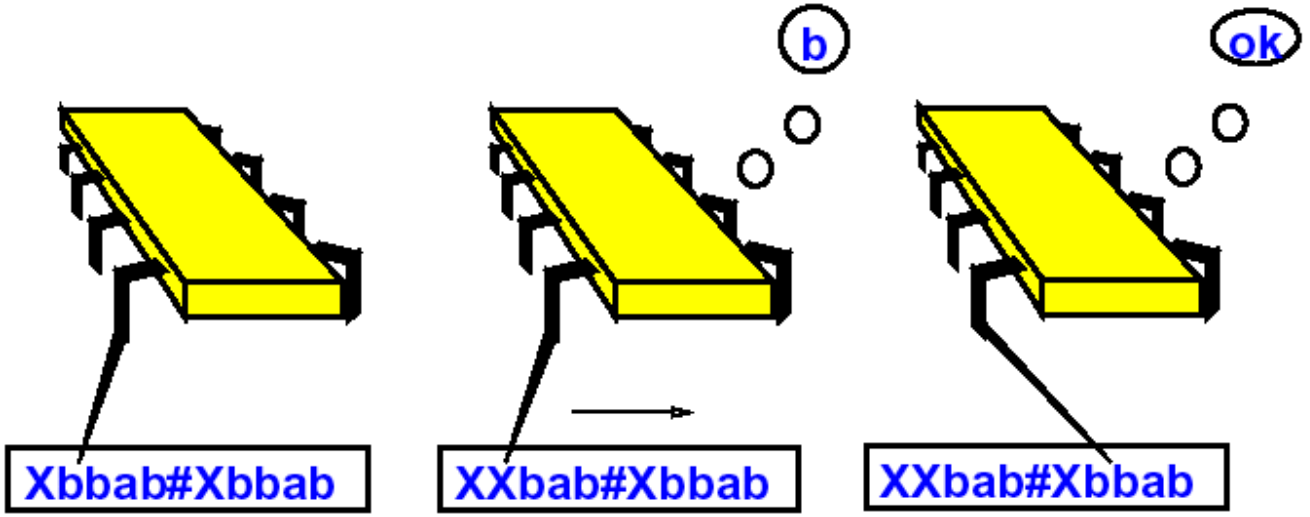

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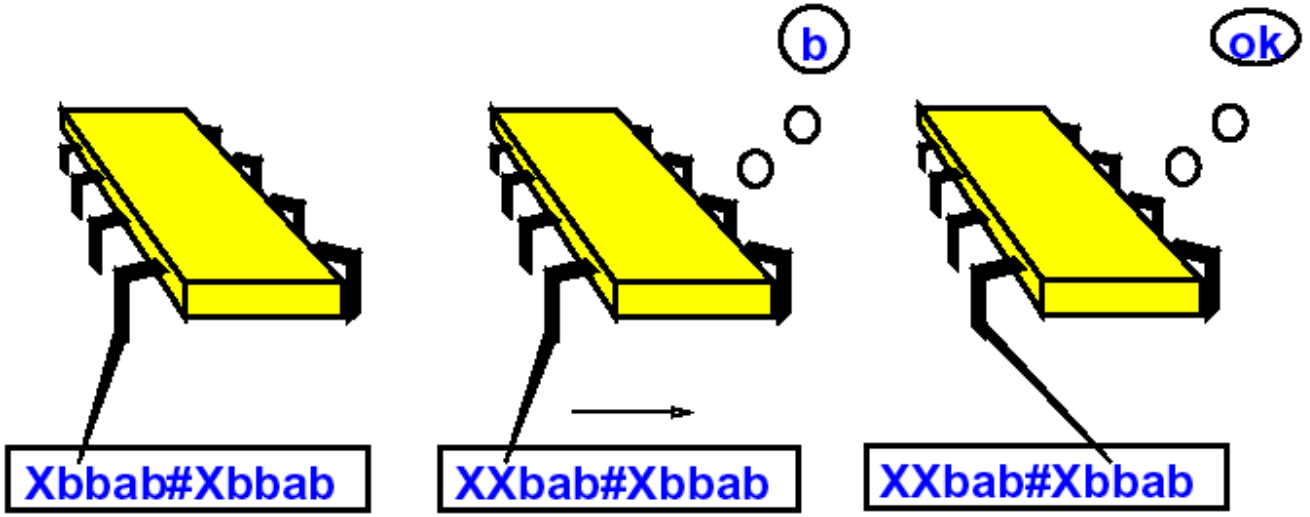

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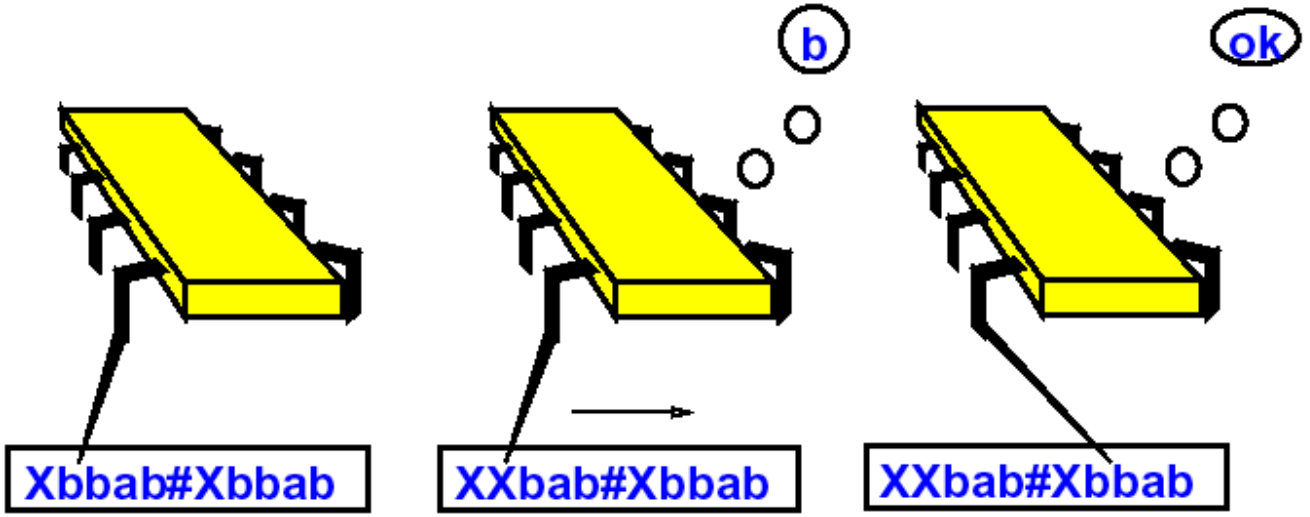

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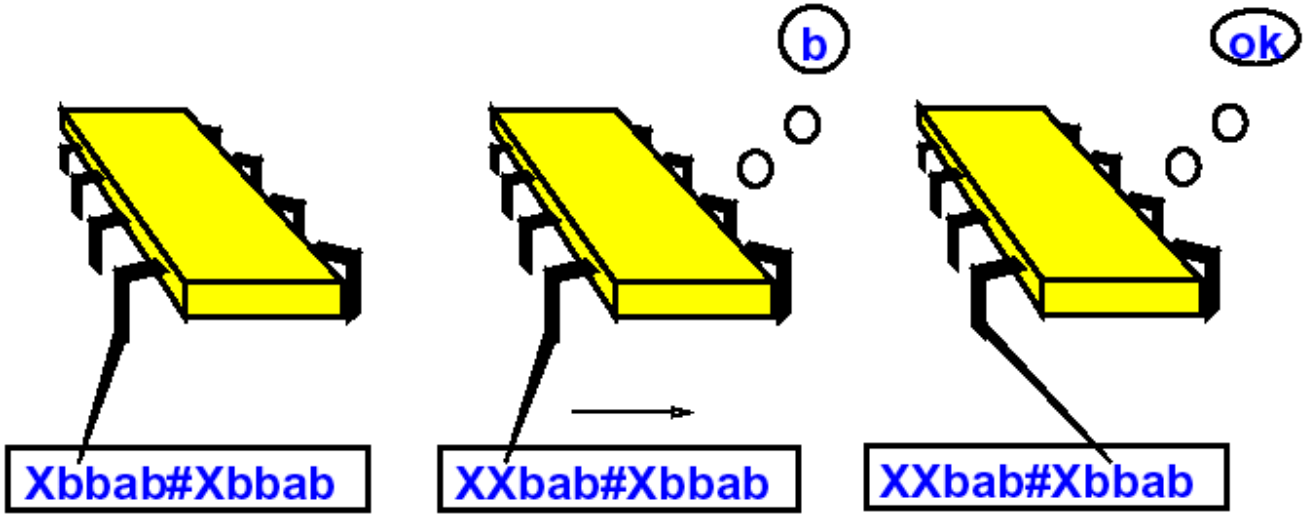

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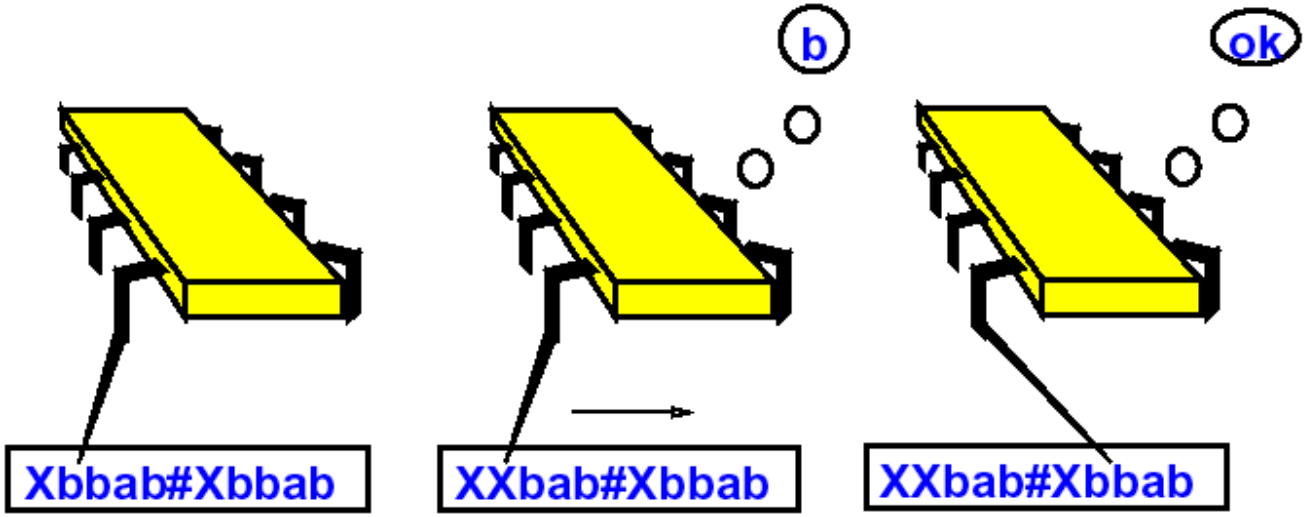

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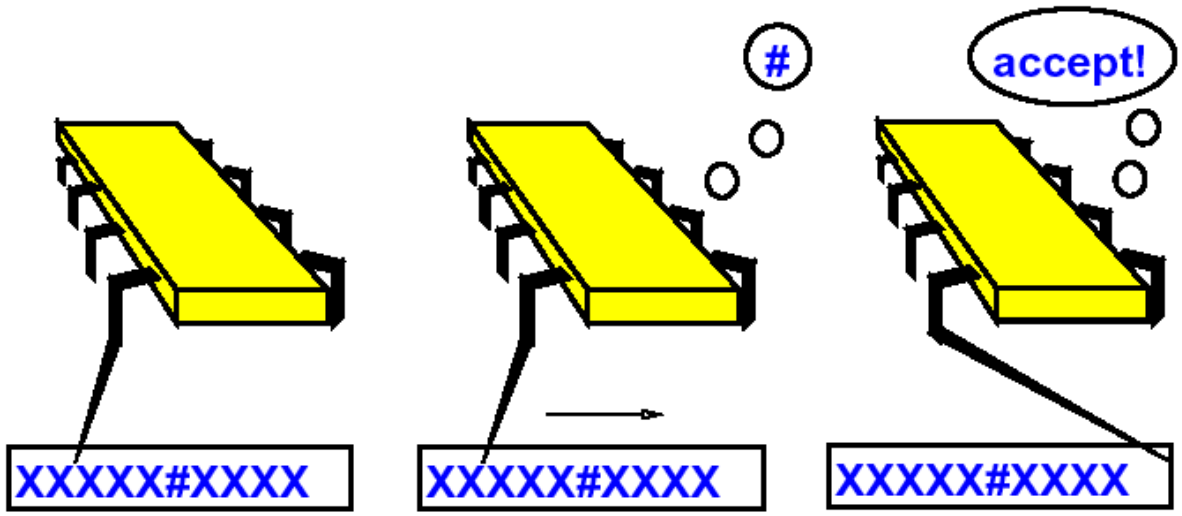

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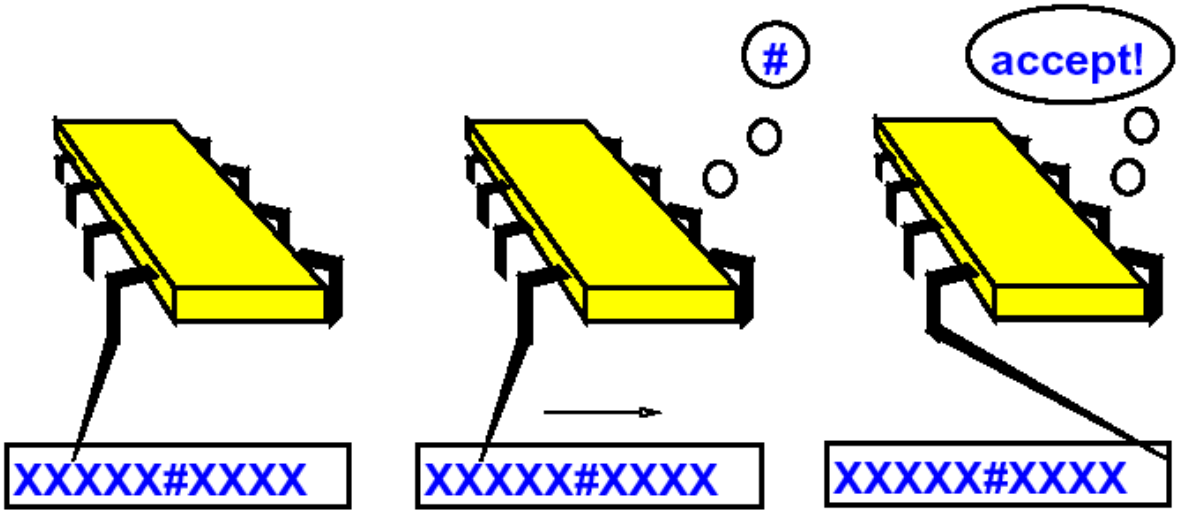

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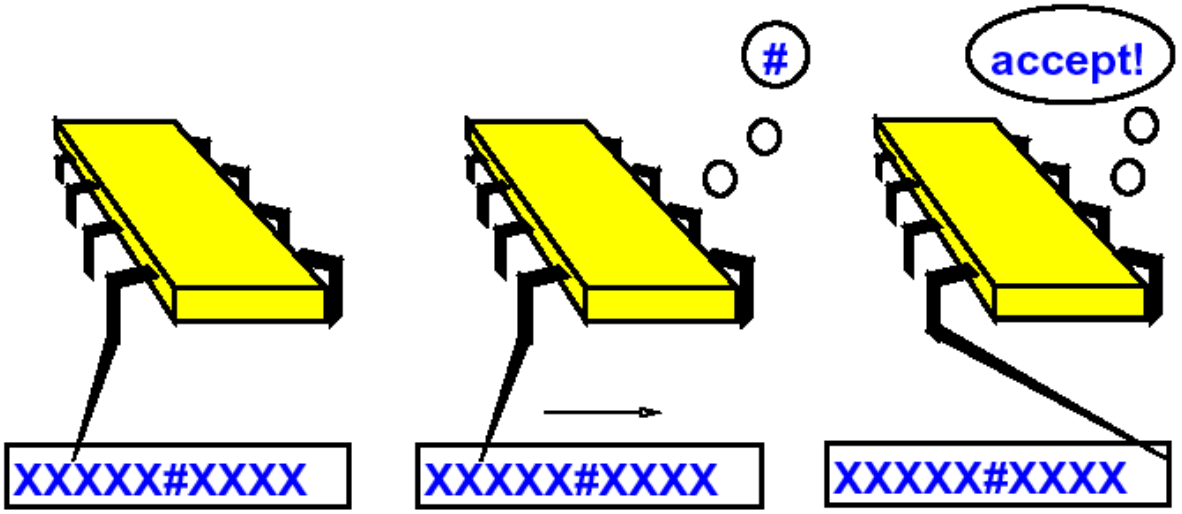

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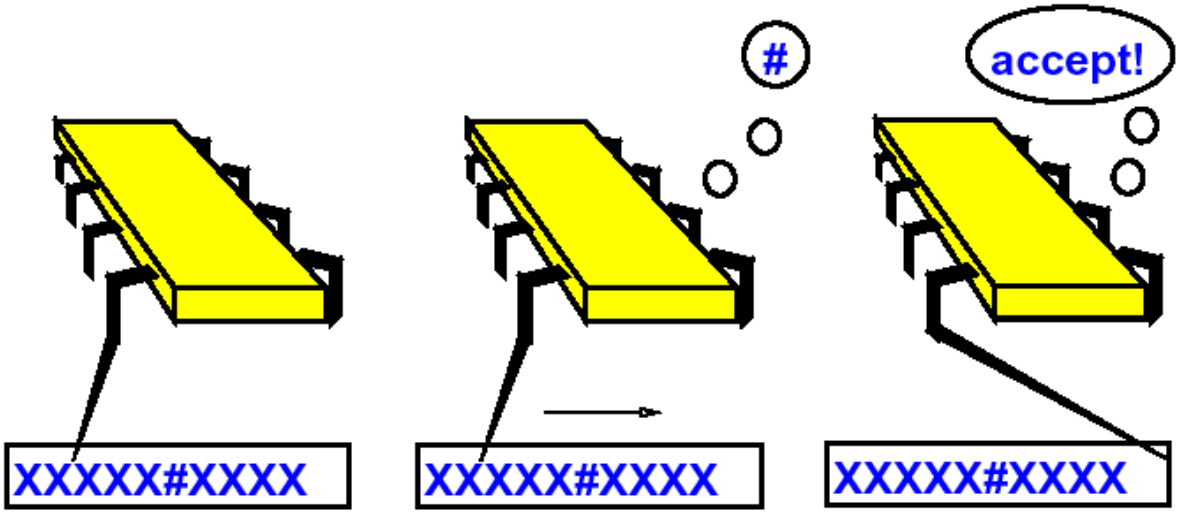

- finally, scan left
- if X encountered, keep going left
- if 0 or 1 encountered, reject
- when blank ($\sqcup$) encountered, accept

Formal Definition

We focus on the transition function

$$\delta : Q \times \Gamma \rightarrow Q \times \Gamma \times \{L, R\} \ .$$

$\delta(q, a) = (r, b, L)$ means:

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  (this is what the L stands for).

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- since $\sqcup \notin \Sigma$, leftmost blank indicates end of input.

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- Special case (2): What happens when head is at right end? We let wq_i and $wq_i\sqcup$ denotes the same configuration, so moves to the right can now be accomodated.

More Configurations

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- starting configuration $q_0 w$

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A Turing machine M accepts input an w if there is a sequence of configurations $C_1, C_2, \ldots, C_k$ such that

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The collection of strings accepted by M is called the language of M, and is denoted $L(M)$.

Enumerable Languages

Definition: A language is (recursively) enumerable if some Turing machine accepts it.

Enumerable Languages (2)

On an input, w, a TM may

- accept

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Major concern: In general, we never know if TM will halt.

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Definition: A TM decides a language if for every input $w \in \Sigma^*$, the TM halts.

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Example

Here is a TM that accepts

$$\{a^i b^j c^k | i \times j = k \text{ where } i, j, k \geq 1\}$$

- scan from left to right to check that input is $a^* b^* c^*$

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- cross off one a and scan right until b occurs. Shuttle between b's and c's, crossing off one of each, until all b's are gone.

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- cross off one a and scan right until b occurs. Shuttle between b's and c's, crossing off one of each, until all b's are gone.
- Restore the crossed-off b's and repeat previous step if more a's exist. If all a's crossed off, check if all c's crossed off. If yes, accept, otherwise reject.

Example (cont.)

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Answer: Mark it with a special symbol.

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- if special symbol still there, head is at start

Example (cont.)

Question: To implement algorithm, should be able to tell when a TM is at the left end of the tape.

Answer: Mark it with a special symbol.

An alternative, trickier way:

- remember current symbol
- replace current symbol with special symbol
- move left
- if special symbol still there, head is at start
- otherwise restore previous symbol and move left

Example

Consider the element distinctness problem

$$E = \{\#x_1\#x_2\#\ldots\#x_\ell \mid \text{each } x_i \in \{0,1\}^* \text{ and for each } i \neq j, x_i \neq x_j\} .$$

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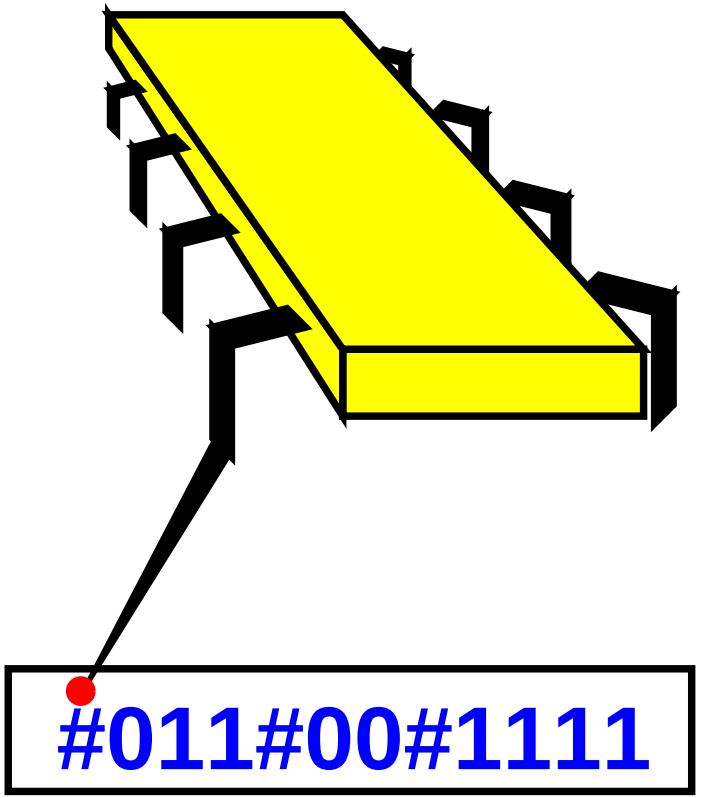
Verbally,

- List of strings in $\{0,1\}^*$ separated by $\#$'s.
- List is in language (& machine should accept) if all strings are different.

Element Distinctness – Solution

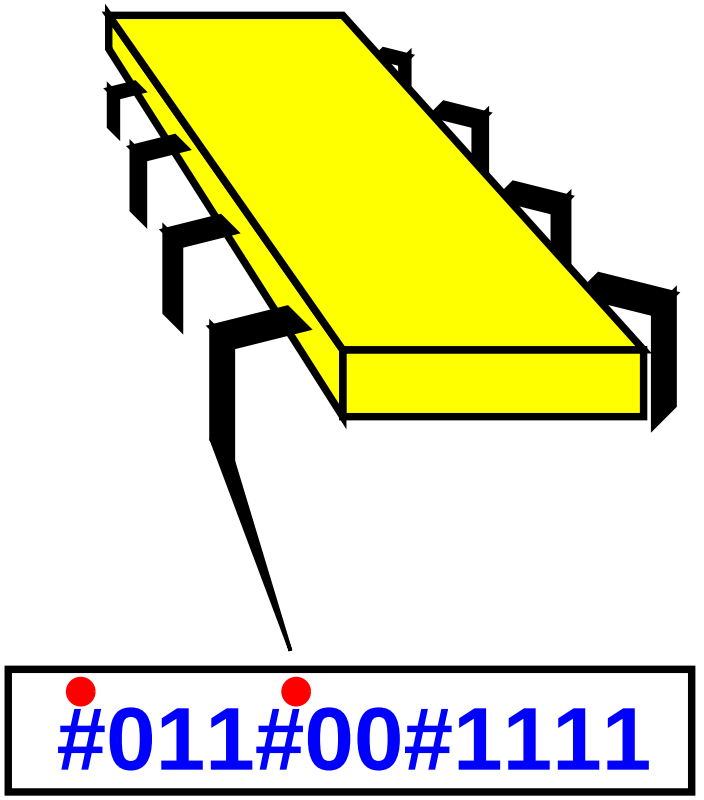
On input w

- place a mark on leftmost tape symbol. If symbol not $\#$, reject.



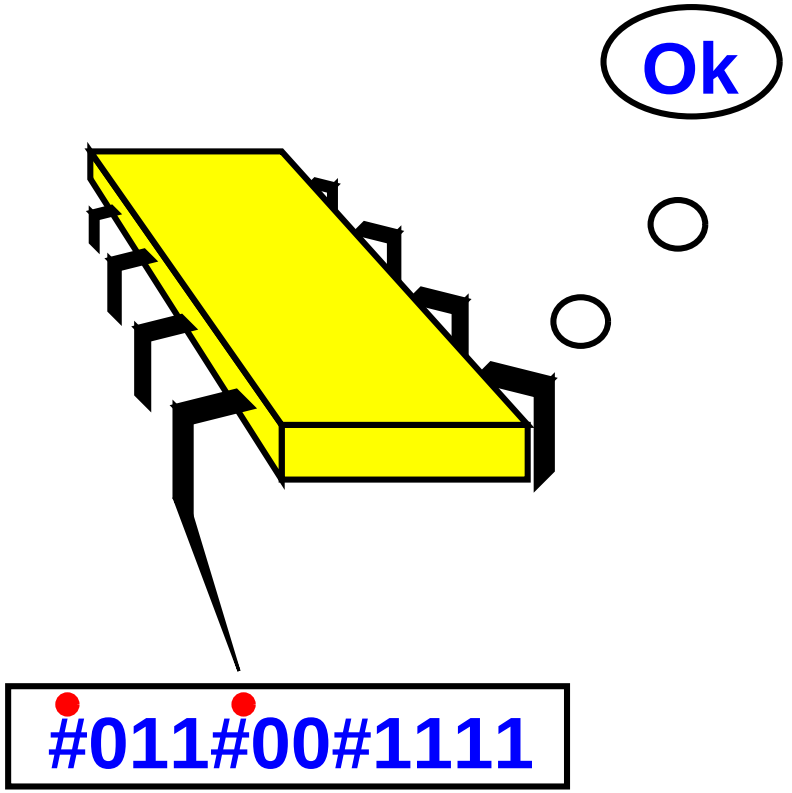
Element Distinctness – Solution (2)

- scan right to next # and place mark on top. If none encountered, reject



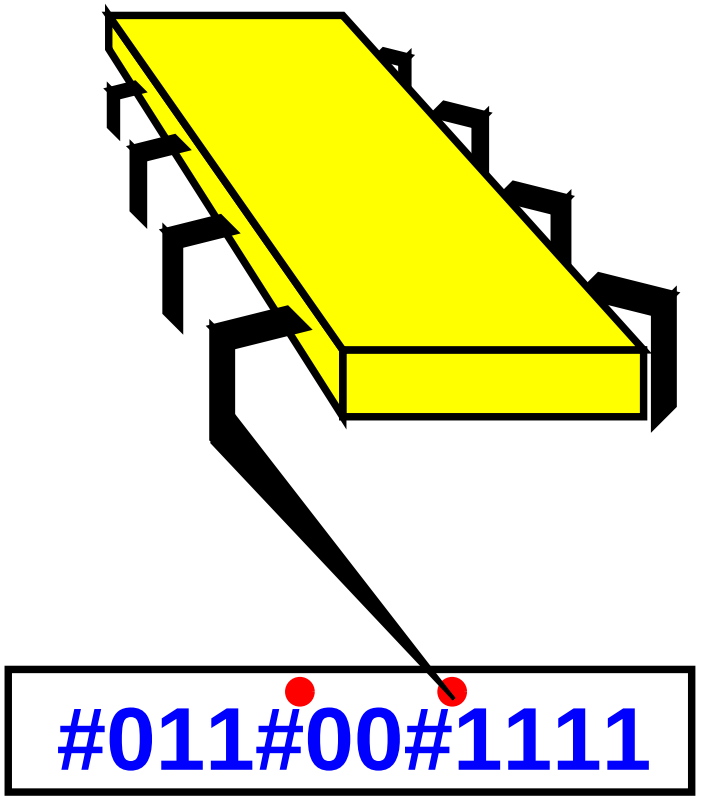
Element Distinctness – Solution (3)

- By zig-zagging, compare the two strings to the right of the marked #'s. If equal, reject.



Element Distinctness – Solution (4)

- Move rightmost mark to next # on right, if any.
- Otherwise move leftmost mark to next # on right and rightmost mark to # after that.
- If not possible, accept.



Element Distinctness – Solution (end)

Question: How do we "mark" a symbol?

Answer: For each tape symbol $\#$, add tape symbol $\bullet$ $\#$ to the tape alphabet Γ.

TMs Variants

Alternative Turing machine definitions abound.

For example, suppose the Turing machine head is allowed to stay put.

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Important notion here: Two-way simulation (model A capable of simulating model B; model B capable of simulating model A).

Multitape Turing Machines

- each tape has its own head
- initially, input string on tape 1 and rest blank

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- and moves each head right or left as specified.

Equivalence

Theorem: A language is enumerable if and only if there is some multitape Turing machine that accepts it.

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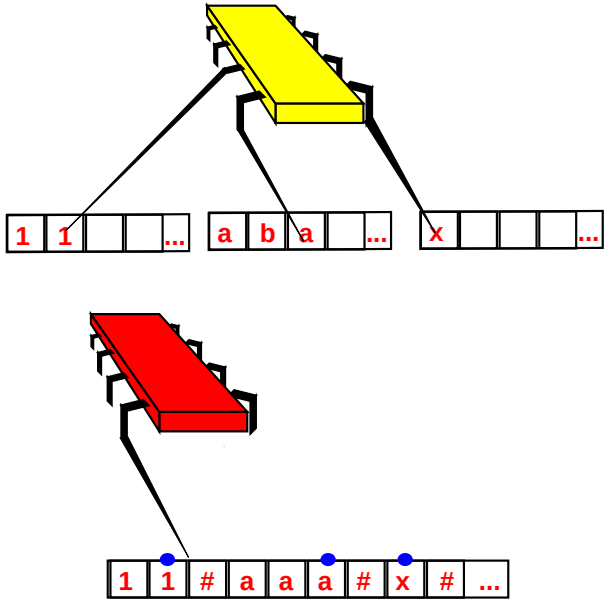
To prove the other direction, we will show how to convert a multitape TM, M, into an equivalent single-tape TM, S.

Simulation



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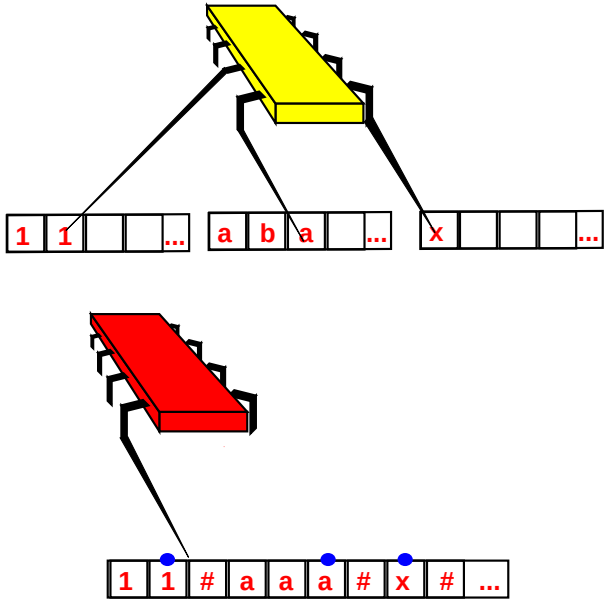

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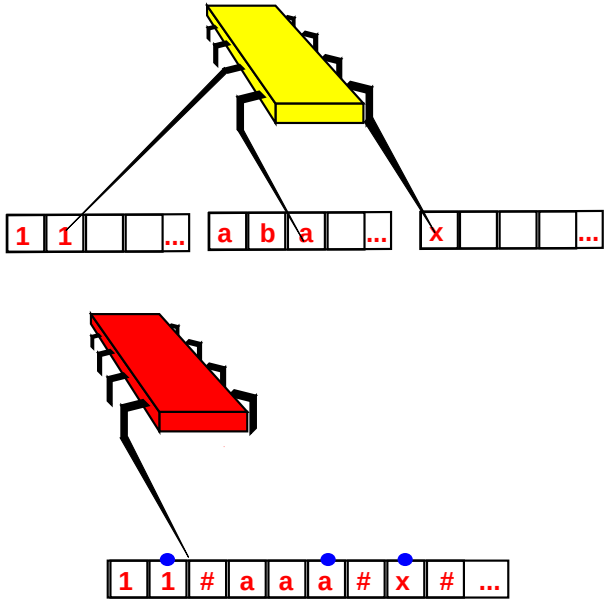
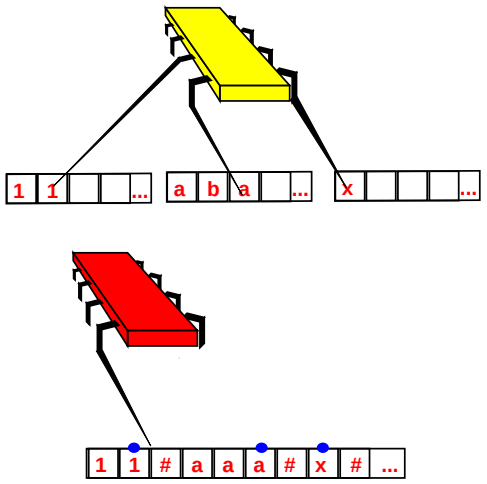


- S simulates k tapes of M by storing them all on a single tape with delimiter $\#$.
- S marks the current positions of the k heads by placing $\bullet$ "above" the letters in current positions. It "knows" which tape the mark belongs to by counting (up to k) from the $\#$'s to the left.

Simulation (2)



On input $w = w_1 \cdots w_n$, S:

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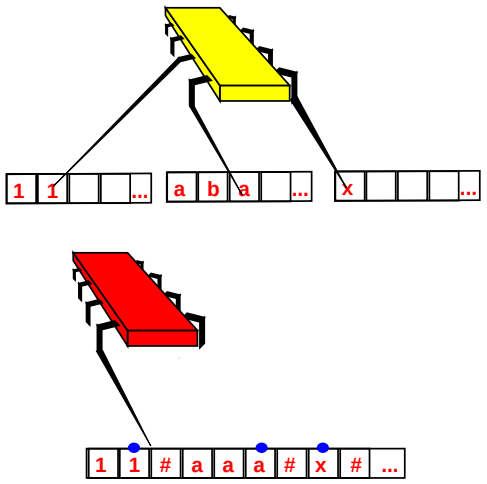

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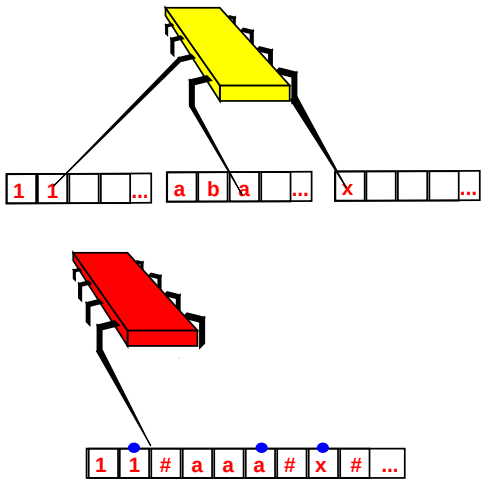

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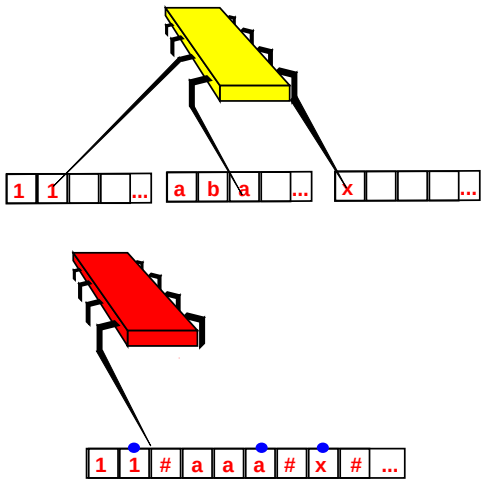

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- S tries to move virtual head onto $\#$ when M is trying to move head onto unused blank square. S writes blank $\sqcup$ on tape, and shifts rest of the tape one square to the right.

Non-Deterministic Turing Machines

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Simulating Non-Determinism

Basic idea:

- D tries all possible branches

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N's computation is a tree.

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- breadth-first search does work, as we'll show.

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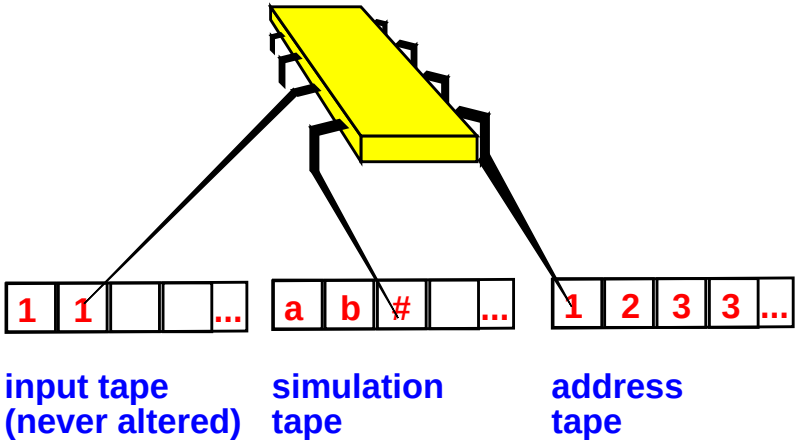

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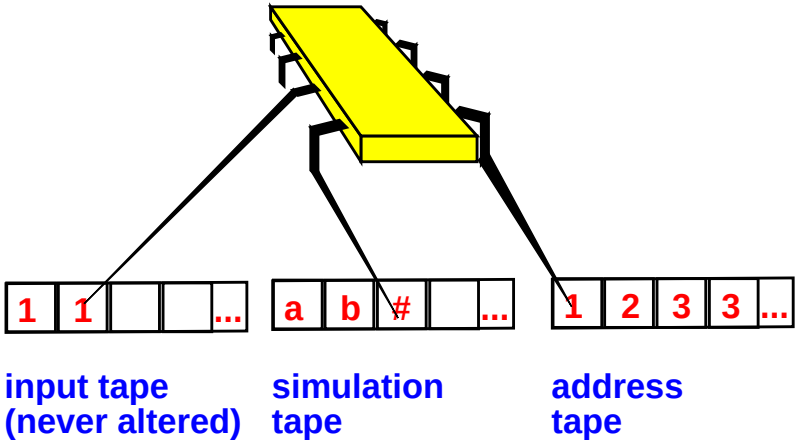

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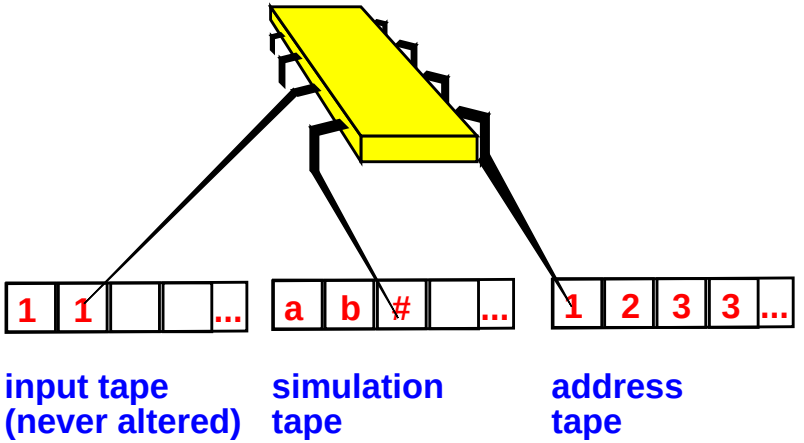

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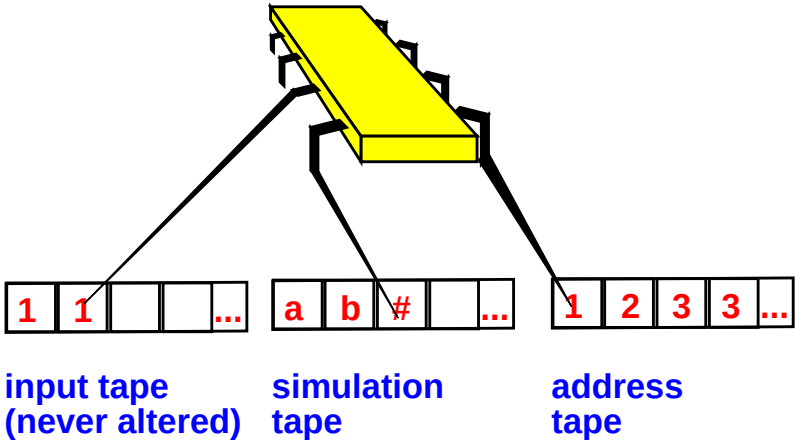

D has three tapes

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- the address tape keeps track of D's location in N's computation tree.

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- ignore meaningless addresses (choices not available for configuration along branch)

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- Replace string on address tape with the lexicographically next string. Jump to Step 2 to simulate this branch of N's computation.

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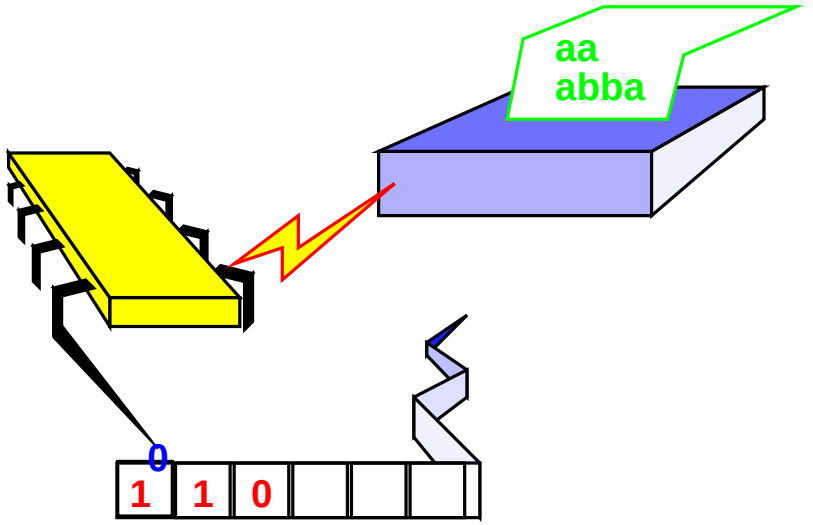
Definition (reminder): A language is decidable if some deterministic Turing machine decides it.

Enumerators

A language is enumerable if it is accepted by some Turing machine.
But why *enumerable*?



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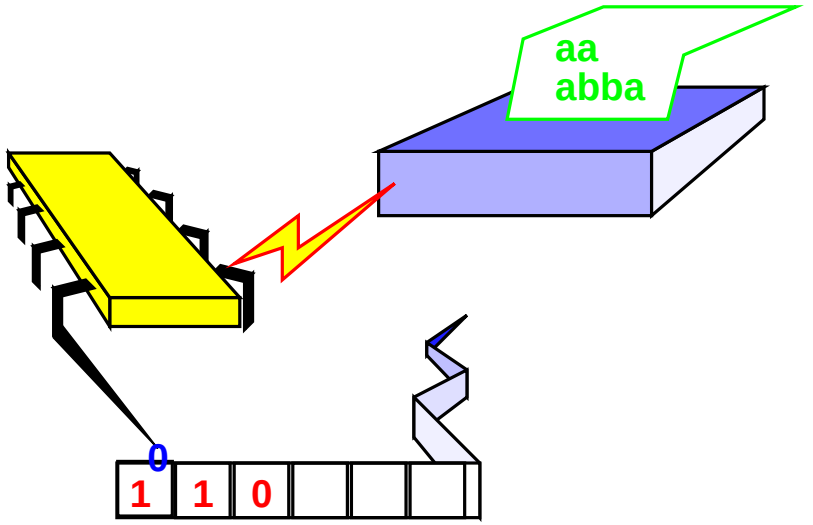

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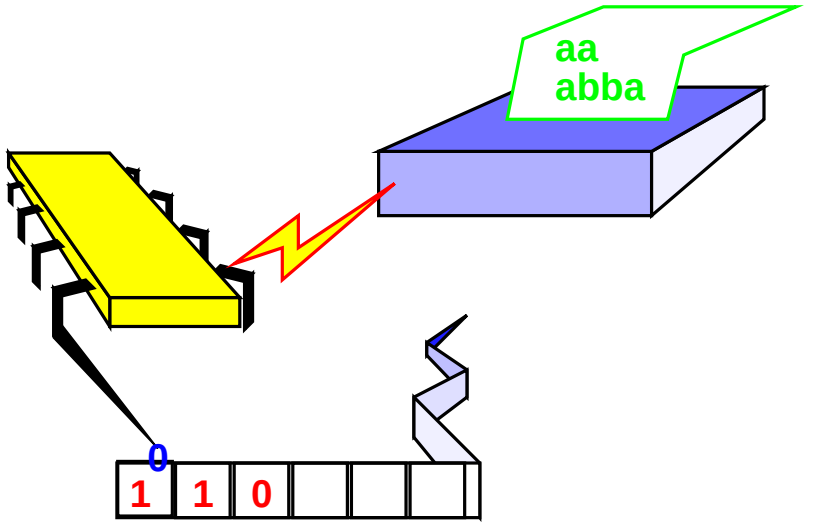

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Will show

- If E enumerates language A, then some TM M accepts A.
- If M accepts A, then some enumerator E enumerates it.

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- if any computation accepts, print out the corresponding s.

♣

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Let $s_1, s_2, s_3, \ldots$ is a list of all strings in Σ^* (*e.g.* strings in lexicographic order).
The enumerator, E

- repeat the following for $i = 1, 2, 3, \ldots$
- run M for i steps on each input $s_1, s_2, \ldots, s_i$.
- if any computation accepts, print out the corresponding s.

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Note that with this procedure, each output is duplicated infinitely often.

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- All “reasonable” programming languages (*e.g.* Java, Pascal, C, Scheme, Mathematica, Maple, Cobol,. . .) are equivalent.
- The notion of an algorithm is model-independent!
- We don’t really care about Turing machines *per se*, we care about understanding computation.

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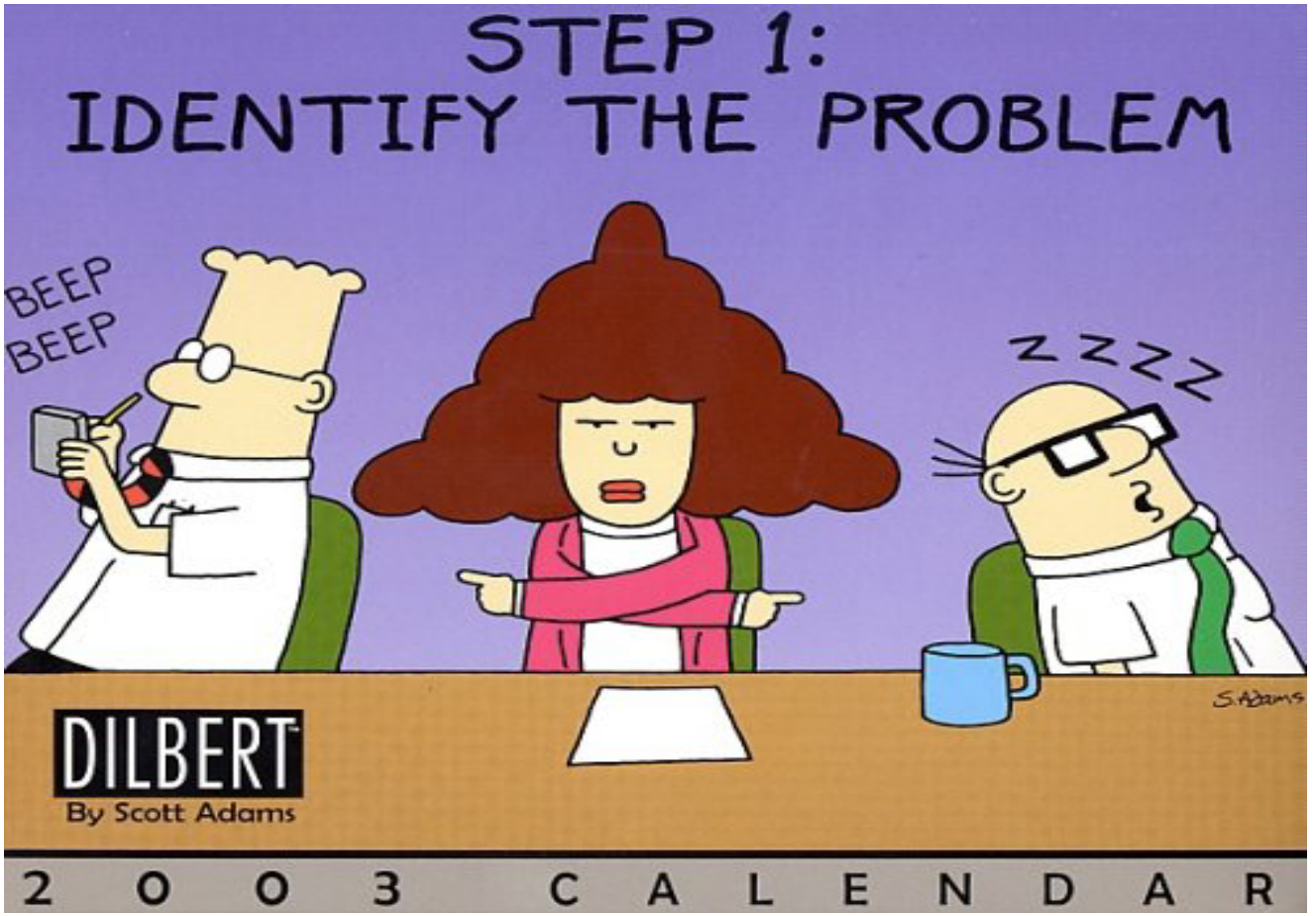
Dilbert's Problems



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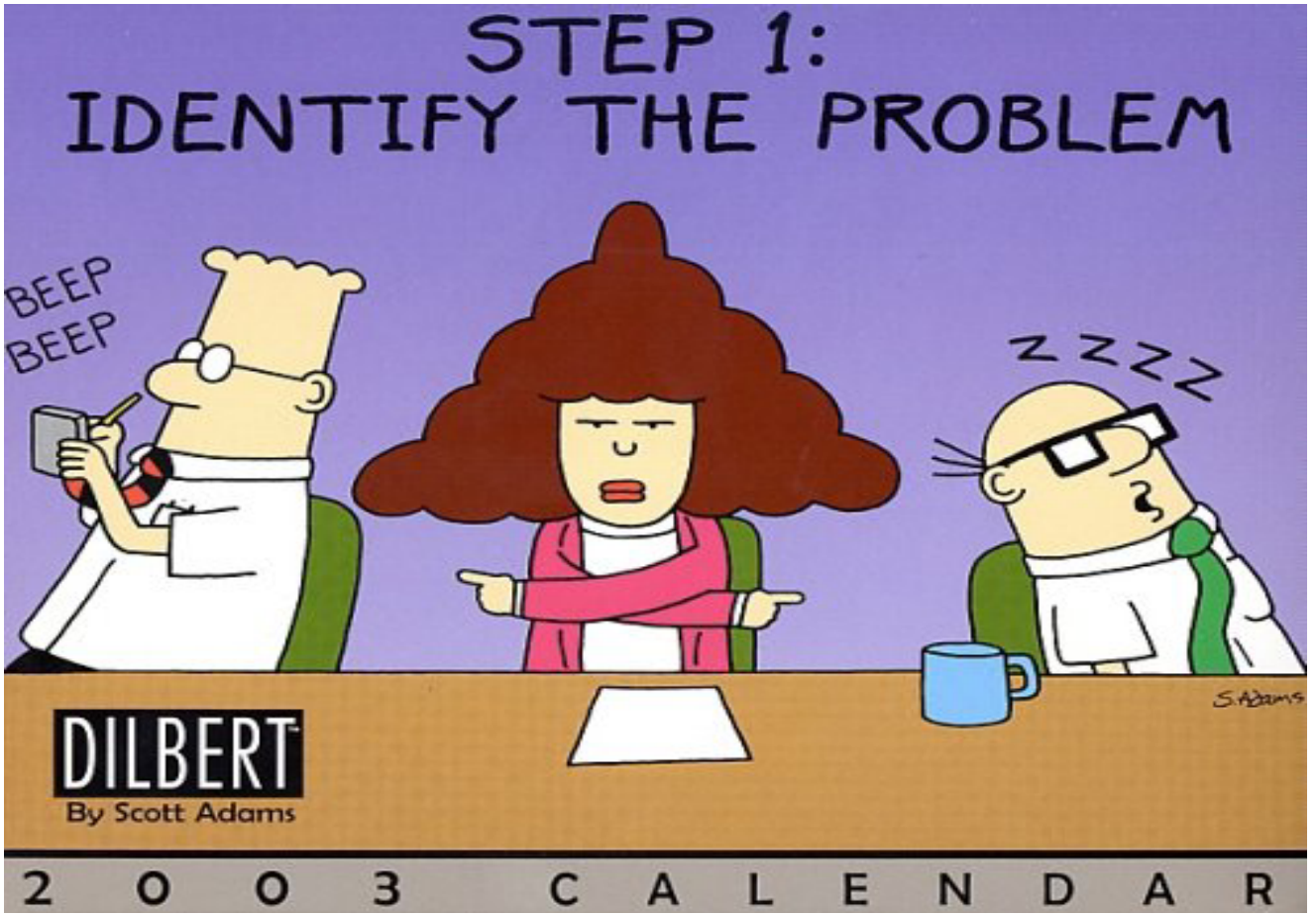

Too much beer last night? These are D. Hilbert's problems, not Dilbert's problems, we are supposed to talk about...

Hilbert's 10th Problem

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But for us, start with some background on the 10th ...

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- For example, $x = 5, y = 3$, and $z = 0$ is a root of the polynomial above.
- Here, we are interested in integral roots, namely an assignment of integers to all variables.
- Some polynomials have integral roots, some don't (*e.g.* $x^2 - 2$).

Hilbert's Tenth Problem

The Problem: Devise an algorithm that tests whether a polynomial has an integral root.

Actually, what he said (translated from German) was

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Note that

- Hilbert explicitly asks that algorithm be "devised"
- apparently Hilbert assumes that such an algorithm must exist, and someone "only" need find it.

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- Mathematicians of 1900 could not have proved this, because they didn't have a formal notion of an algorithm.
- Intuitive notions work fine for constructing algorithms (we know one when we see it).
- Formal notions are required to show that no algorithm exists.

Church-Turing Thesis

Formal notions appeared in 1936:

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The Church-Turing Thesis:

> "The intuitive notion of algorithms equals Turing machine algorithms".

Hilbert's Tenth Problem

In 1970, 23 years old Yuri Matijasevič, building on work of Martin Davis, Hilary Putnam, and Julia Robinson, proved that no algorithm exists for testing whether a polynomial has integral roots

(a survey of the proof)

Reformulating Hilbert's Tenth Problem

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We now know it is not decidable, but it is enumerable!

Univariate Polynomials

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Here is a Turing machine that accepts D_1.
On input p,

- evaluate p with x set successively to $0, 1, -1, 2, -2, \ldots$.
- if p evaluates to zero, accept.

Univariate Polynomials (2)

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Note that

- If p has an integral root, the machine accepts.
- If not, M_1 loops.
- M_1 is an acceptor, but not a decider.

Univariate Polynomials (3)

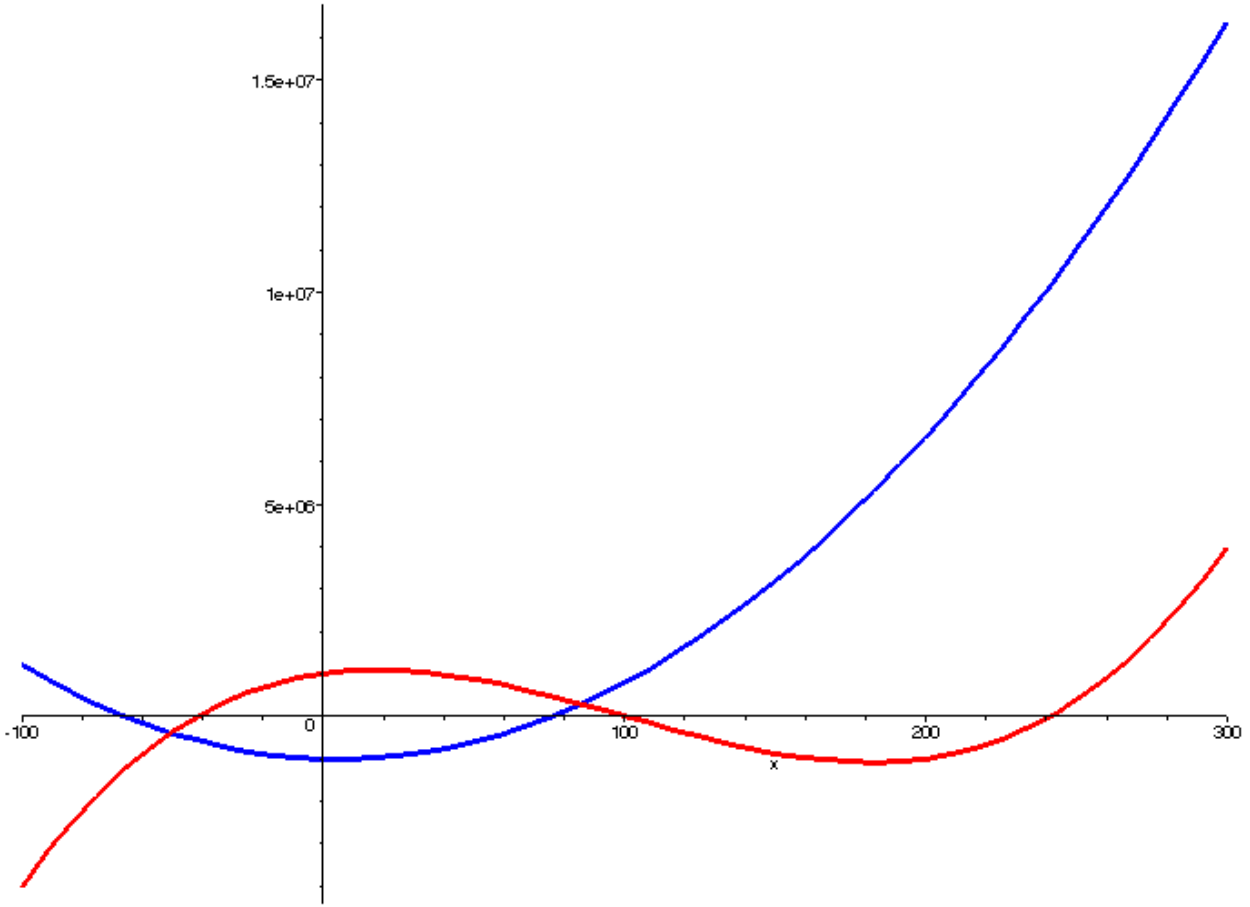
```
> f:=x->x^3-300*x^2+10000*x+1000000;
```

$$f := x \to x^3 - 300\,x^2 + 10000\,x + 1000000$$

```
> g:=x->200*x^2-2000*x-1000000;
```

$$g := x \to 200\,x^2 - 2000\,x - 1000000$$

```
> plot([f(x),g(x)],x=-100..300,color=[red,blue],thickness=3);
```



Univariate Polynomials (4)

In fact, D_1 is decidable.

Can show that all real roots of $p[x]$ lie inside interval

$$(-|kc_{max}/c_1|, |kc_{max}/c_1|) \ ,$$

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By Matijasevič theorem, such effective bounds on range of real roots cannot be computed for multivariable polynomials.

Wild Models

What about “unreasonable” models of computation?

Consider MUntel’s ℵ-AXP© processor (to be released XMAS 2003).

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Question: Does the $\aleph$-AXP$^{\copyright}$ invalidate the Church-Turing Thesis?

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- But we want algorithms that work on graphs, matrices, polynomials, Turing machines, etc.
- Need to choose an encoding for objects.
- Can often be done in many reasonable ways.
- Sometimes distinguish between X, the object, and $\langle X \rangle$, its encoding.

Encoding

Consider strings representing undirected graphs.

A graph is connected if every node can be reached from any other node by traveling along edges.

Define the language:

$$A = \{\langle G \rangle \mid G \text{ is a connected undirected graph}\}$$

High-Level Description

High-level description of a machine that decides

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On input $\langle G \rangle$, encoding of graph G

- select first node of G and mark it.

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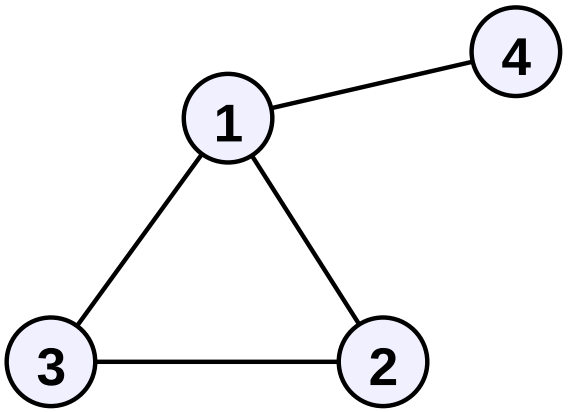
- select first node of G and mark it.
- repeat until no new nodes marked:
- For each node in G, mark if attached by an edge to a node already marked.
- scan nodes of G to determine whether they are all marked. If so, accept, otherwise reject.

Some Details

Question: How is G encoded?



vertexes edges

(1,2,3,4)((1,2),(2,3),(3,1),(1,4))

G <G>

Answer: List of nodes, followed by list of edges.

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On input M checks that input is valid graph encoding

- two lists

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Now ready to start “step one”.

Detailed Algorithm

On input $\langle G \rangle$, encoding of graph G

1. mark first node with a dot on leftmost digit.
2. loop:
   - Scans list and "underlines" undotted node n_1.

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